A Microscopic approach to Souslin-tree constructions. Part I

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Abstract

We propose a parameterized proxy principle from which κ-Souslin trees with various additional features can be constructed, regardless of the identity of κ. We then introduce *the microscopic approach*, which is a simple method for deriving trees from instances of the proxy principle. As a demonstration, we give a construction of a coherent κ-Souslin tree that applies also for κ inaccessible.

We then carry out a systematic study of the consistency of instances of the proxy principle, distinguished by the vector of parameters serving as its input. Among other things, it will be shown that all known $\diamondsuit$-based constructions of κ-Souslin trees may be redirected through this new proxy principle.

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1. Introduction

Recall that the real line is the unique separable, dense linear ordering with no endpoints in which every bounded set has a least upper bound. A problem posed by Mikhail Souslin around the year 1920 [Sou20] asks whether the term *separable* in the above characterization may be weakened to *ccc*. (A linear order is said to be *separable* if it has a countable dense subset. It is *ccc* — short for satisfying the *countable chain condition* — if every pairwise-disjoint family of open intervals is countable.) The affirmative answer to Souslin's problem is known as Souslin's Hypothesis, and abbreviated SH.[1]

Amazingly enough, the resolution of this single problem led to key discoveries in set theory: the notions of Aronszajn, Souslin and Kurepa trees [Kur35], forcing axioms and the method of iterated forcing [ST71], the diamond and square principles $\diamondsuit(S)$, $\square_\kappa$ [Jen72], and the theory of iteration without adding reals [DJ74].

Kurepa [Kur35] proved that SH is equivalent to the assertion that every tree of size $\aleph_1$ contains either an uncountable chain or an uncountable antichain. A counterexample tree is

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[1]For more on the history of Souslin's problem, see the surveys [Rud69], [Alv99] and [Kan11].

said to be a *Souslin tree*. This concept admits a natural generalization to higher cardinals, in the form of κ-Souslin trees for regular uncountable cardinals κ. There is a zoo of consistent constructions of κ-Souslin trees, and these constructions often depend on whether κ is a successor of a regular cardinal, a successor of a singular cardinal of countable cofinality, a successor of a singular cardinal of uncountable cofinality, or an inaccessible cardinal.[2] This poses challenges, since, in applications, it is often needed that the constructed Souslin trees have additional features (such as rigidity, homogeneity, or admitting/omitting an *ascent path*), and this necessitates revisiting each of the relevant constructions and tailoring it to the new need. Let us exemplify.

Example 1.1. The behavior of κ-Aronszajn and κ-Souslin trees depends heavily on the identity of κ:[3]

- (Aronszajn, see [Kur35, §9.5, Theorem 6 and footnote 3, p. 96]) There exists a κ-Aronszajn tree, for $\kappa = \aleph_1$;
- (Specker, [Spe49]) If GCH holds, then for every regular cardinal λ, there exists a special λ^+-Aronszajn tree;
- (Magidor-Shelah, [MS96]) GCH is consistent with the nonexistence of any λ^+-Aronszajn tree at some singular cardinal λ (modulo large cardinals);
- (Erdős-Tarski, [ET61]) If κ is a weakly compact cardinal, then there exists no κ-Aronszajn tree;
- (Jensen, see [DJ74]) The existence of an ω_1-Souslin tree is independent of GCH;
- (Baumgarner-Malitz-Reinhardt, [BMR70]) Any ω_1-Aronszajn tree can be made special in some cofinality-preserving extension;
- (Devlin, [Dev83]) If $V = L$, then for every uncountable cardinal λ, there exists a λ^+-Souslin tree that remains non-special in any cofinality-preserving extension.

Example 1.2. Consistent constructions of κ-Souslin trees that depend on the identity of κ:[4]

- (Jensen, [Jen68]) Suppose that λ is a regular cardinal.

  If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then there exists a λ^+-Souslin tree;[5]

[2]Most of the standard references in set theory, including [Dra74],[Kun80],[Tod84],[Roi90],[JW97],[HJ99], [Lev02],[Jec03],[Kun11], provide a construction of a κ-Souslin tree only for the case $\kappa = \aleph_1$. For Jensen's general construction [Jen72], see [Dev84, Theorem IV.2.4] or [Sch14, Lemma 11.68].

[3]All tree-related terminology will be defined in Section 2 below.

[4]See the Appendix below for a list of combinatorial principles and some of their known interactions.

[5]Here, E^{β}_{α} denotes the set of all ordinals below β whose cofinality is equal to α. The sets $E^{\beta}_{\geq\alpha}$, $E^{\beta}_{<\alpha}$ and $E^{\beta}_{\neq\alpha}$ are defined in a similar fashion.

- (Jensen, [Jen72]) Suppose that λ is a singular cardinal.

  If $\mathsf{GCH} + \square_\lambda$ holds, then there exists a λ^+-Souslin tree;

- (Baumgartner, see [Dev83], building on Laver [LS81]) $\mathsf{GCH} + \square_{\aleph_1}$ implies the existence of an $\aleph_2$-Souslin tree that remains non-special in any cofinality-preserving extension;

- (Cummings, [Cum97]) Suppose that λ is a regular uncountable cardinal.

  If $\boxdot_\lambda$ holds and $\lambda^{<\lambda} = \lambda$, then there exists a λ-complete λ^+-Souslin tree that remains non-special in any cofinality-preserving extension;

- (Cummings, [Cum97]) Suppose that λ is a singular cardinal of countable cofinality.

  If $\square_\lambda + \mathrm{CH}_\lambda$ holds and $\mu^{\aleph_1} < \lambda$ for all $\mu < \lambda$, then there exists a λ^+-Souslin tree that remains non-special in any cofinality-preserving extension;[6]

- (Cummings, [Cum97]) Suppose that λ is a singular cardinal of uncountable cofinality.

  If $\square_\lambda + \mathrm{CH}_\lambda$ holds and $\mu^{\aleph_0} < \lambda$ for all $\mu < \lambda$, then there exists a λ^+-Souslin tree that remains non-special in any cofinality-preserving extension;

- (Jensen, see [DJ74]) $\diamondsuit(\omega_1)$ entails the existence of a homogeneous ω_1-Souslin tree;

- (Rinot, [Rin14b]) Suppose that λ is a singular cardinal.

  If $\square_\lambda + \mathrm{CH}_\lambda$ holds, then there exists a homogeneous λ^+-Souslin tree.

The focus of the present research project, of which this paper is a core component, is on developing new foundations for constructing κ-Souslin trees. Specifically, we propose a single parameterized proxy principle from which κ-Souslin trees with various additional features can be constructed, regardless of the identity of κ.

In this paper and in the next one [BR17b] (being Part I and Part II, respectively) we establish, among other things, that all known $\diamondsuit$-based constructions of κ-Souslin trees may be redirected through this new proxy principle. This means that any κ-Souslin tree with additional features that will be shown to follow from the proxy principle will automatically be known to hold in many unrelated models.

But the parameterized principle gives us more:

► It suggests a way of calibrating how fine is a particular class of Souslin trees, by pinpointing the weakest vector of parameters sufficient for the proxy principle to enable construction of a member of this class.

For instance, the existence of a *uniformly coherent* Souslin tree entails the existence of a *free* one (see [Lar98],[SZ99],[SF10]), while it is consistent that there exists a free κ-Souslin tree, but not a uniformly coherent one, for $\kappa = \aleph_1$ [Lar98]. This is also consistently true for $\kappa = \aleph_2$, as can be verified by the model of [Tod81, Theorem 4.4]. And, indeed, the vector

[6]We write CH_λ for the assertion that $2^\lambda = \lambda^+$.

of parameters sufficient to construct a free κ-Souslin tree is weaker than the corresponding one for a coherent κ-Souslin tree.

▶ It allows comparison and amplification of previous results.

Recall that it is a longstanding open problem whether GCH entails the existence of an $\aleph_2$-Souslin tree, and a similar problem is open concerning λ^+-Souslin trees for λ singular (see [Sch05] and [Rin11a, Question 14]). A milestone result in this vein is the result from [Gre76] and its improvement [KS93]. In recent years, new weak forms of $\diamondsuit$ at the level of λ^+ for $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ were proposed and shown to entail the existence of λ^+-Souslin trees. This includes the club-guessing principle $\curlywedge^*(\chi, S)$ from [KLY07] and the *reflected-diamond* principle $\langle T \rangle_S$ from [Rin11b].[7] In this paper, we put all of these principles under a single umbrella by computing the corresponding vector of parameters that holds in each of the previously studied configurations. From this and the constructions we present in a future paper, it follows, for example, that the Gregory configuration [Gre76] suffices for the construction of a *specializable* λ^+-Souslin tree, and the König-Larson-Yoshinobu configuration [KLY07] suffices for the construction of a free λ^+-Souslin tree.

▶ It allows the construction of various types of trees at a broader class of cardinals. To give two examples:

A combinatorial construction of a free κ-Souslin tree for $\kappa = \aleph_1$ may be found in [DJ74],[Tod84]. In [BR17c, §6], we give another combinatorial construction of a free κ-Souslin tree, this time using the proxy principle, and therefore it automatically applies to all regular uncountable cardinals κ, including successors of singular cardinals.

A combinatorial construction of a uniformly coherent κ-Souslin tree for a successor cardinal κ may be found in [DJ74],[Lar99],[Vel86]. In Section 2 below, we give a proxy-based construction of a uniformly coherent κ-Souslin tree, and therefore it automatically applies to all regular uncountable cardinals κ, including inaccessible cardinals.

▶ It allows obtaining completely new types of Souslin trees.

Here, and mostly in the other papers of this project, we develop a very simple method for deriving Souslin trees from the proxy principle — the *microscopic approach*. This approach involves devising a library of miniature actions and an apparatus for recursively invoking them at the right timing against a witness to the proxy principle. The outcome will always be a tree, but its features depend on which actions were invoked along the way, and to which vector of parameters of the proxy principle the witness was given. Once the construction of Souslin trees becomes so simple, it is then easier to carry out considerably more complex constructions, and this has already been demonstrated in [BR17c], where we gave the first example of a Souslin tree whose reduced powers behave independently of each other, proving, e.g., that $V = L$ entails the existence of an $\aleph_6$-Souslin tree whose reduced $\aleph_n$-power is $\aleph_6$-Aronszajn iff $n < 6$ is not a prime number.

▶ It allows detection of completely new scenarios in which Souslin trees must exist.

This is done by finding new configurations in which an instance of the proxy principle holds. To give three examples: In [BR16], we prove that Prikry forcing over a measurable

[7] See the Appendix.

cardinal λ validates a strong instance of the proxy principle at λ^+. In [Rin17a], the second author proved that for every uncountable cardinal λ, $\mathsf{GCH} + \square(\lambda^+)$ entails an instance of the proxy principle at λ^+, implying, in particular, that if GCH holds and there are no $\aleph_2$-Souslin trees, then $\aleph_2$ is a weakly compact cardinal in L.[8] In [BR17b], we shall extend Gregory's theorem [Gre76] from dealing with successors of regular cardinals to dealing also with successors of singular cardinals.

In the next subsection, we define the proxy principle in its full generality, but before that, we would like to give two simplified versions of it, $\boxtimes^-(S)$ and $\boxtimes(S)$, which may be thought of as generic versions of the principle $\square(\kappa)$. For this, let us set up some notation, as follows. Suppose that D is a set of ordinals. Write $\operatorname{acc}(D) := \{\alpha \in D \mid \sup(D \cap \alpha) = \alpha > 0\}$, $\operatorname{nacc}(D) := D \setminus \operatorname{acc}(D)$, and $\operatorname{acc}^+(D) := \{\alpha < \sup(D) \mid \sup(D \cap \alpha) = \alpha > 0\}$. For any $j < \operatorname{otp}(D)$, denote by $D(j)$ the unique $\delta \in D$ for which $\operatorname{otp}(D \cap \delta) = j$, e.g., $D(0) = \min(D)$. For any ordinal σ, write

$$\begin{aligned}\operatorname{succ}_\sigma(D) &:= \{\delta \in D \mid \operatorname{otp}(D \cap \delta) = j + 1 \text{ for some } j < \sigma\}\\ &= \{D(j+1) \mid j < \sigma \ \&\ j + 1 < \operatorname{otp}(D)\}.\end{aligned}$$

Notice that $\operatorname{succ}_\sigma(D) \subseteq \operatorname{nacc}(D) \setminus \{\min(D)\}$ provided $D \neq \emptyset$, and $\operatorname{succ}_\sigma(D) = \operatorname{nacc}(D) \setminus \{\min(D)\}$ whenever $\operatorname{acc}^+(D) \subseteq D$ and $\sigma + 1 \geq \operatorname{otp}(D) > 0$.

Definition 1.3. For any regular uncountable cardinal κ and any stationary $S \subseteq \kappa$, $\boxtimes^-(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for every ordinal $\alpha < \kappa$ and every $\bar\alpha \in \operatorname{acc}(C_\alpha)$;
- for every cofinal $A \subseteq \kappa$, there exist stationarily many $\alpha \in S$ such that $\sup(\operatorname{nacc}(C_\alpha) \cap A) = \alpha$.

In Section 2, we shall present a construction of a *slim* κ-Souslin tree from $\boxtimes^-(\kappa) + \diamondsuit(\kappa)$. We hope that the reader would find this construction appealing for its simplicity and uniform nature, allowing, e.g., a single construction of a κ-Souslin tree that is relevant in L to any regular uncountable cardinal that is not weakly compact. As it is trivial to prove that $\diamondsuit(\omega_1) \Rightarrow \clubsuit(\omega_1) \Rightarrow \boxtimes^-(\omega_1)$, we think that this exposition is altogether preferable even at the level of ω_1.

We shall also prove that modulo a standard arithmetic hypothesis, $\boxtimes^-(E^\kappa_{\geq\chi}) + \diamondsuit(\kappa)$ entails the existence of a χ-complete κ-Souslin tree. Note that unlike the classical approach that derived a χ-complete κ-Souslin tree from $\diamondsuit(E^\kappa_{\geq\chi})$, here we settle for $\diamondsuit(\kappa)$. To appreciate this difference, we mention that the model of Example 1.30 below witnesses that $\boxtimes(E^{\aleph_2}_{\aleph_1}) + \diamondsuit(\omega_2)$ is consistent together with the failure of $\diamondsuit(E^{\aleph_2}_{\aleph_1})$.

But we haven't yet defined the stronger principle $\boxtimes(S)$. We do so now:

[8]This was open for 40 years; see [KM78, p. 261], [KS93, Problem 2], and [Rin11a, Problem 9].

Definition 1.4. For any regular uncountable cardinal κ and any stationary $S \subseteq \kappa$, $\boxtimes(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for every ordinal $\alpha < \kappa$ and every $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every sequence $\langle A_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that $\sup\{\beta < \alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq A_i\} = \alpha$ for all $i < \alpha$.

Furthermore, in Section 2, we shall present a construction of a uniformly coherent κ-Souslin tree from $\boxtimes(\kappa) + \diamondsuit(\kappa)$. In Section 3, we shall prove that $\square_\lambda + \mathrm{CH}_\lambda$ entails $\boxtimes(\lambda^+)$ for every singular cardinal λ, and that $\diamondsuit(\omega_1)$ entails $\boxtimes(\omega_1)$. Note that neither of the two implications is trivial.

In [BR16], we introduce the considerably weaker principle $\boxtimes^*(S)$ (a relative of Jensen's weak square principle $\square^*$), which still suffices to entail the existence of a (plain) κ-Souslin tree, in the presence of $\diamondsuit(\kappa)$.

1.1. The proxy principle

Definition 1.5 (Proxy principle)**.** Suppose that:

- κ is any regular uncountable cardinal;
- ν and μ are cardinals such that $2 \leq \nu \leq \mu \leq \kappa$;
- $\mathcal{R}$ is a binary relation over $[\kappa]^{<\kappa}$;
- θ is a cardinal such that $1 \leq \theta \leq \kappa$;
- $\mathcal{S}$ is a nonempty collection of stationary subsets of κ;
- σ is an ordinal $\leq \kappa$; and
- $\mathcal{E}$ is an equivalence relation over a subset of $\mathcal{P}(\kappa)$.

The principle $\mathrm{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every limit $\alpha < \kappa$, $\mathcal{C}_\alpha$ is a collection of club subsets of α;
- for every ordinal $\alpha < \kappa$, $0 < |\mathcal{C}_\alpha| < \mu$, and $C \mathrel{\mathcal{E}} D$ for all $C, D \in \mathcal{C}_\alpha$;
- for every ordinal $\alpha < \kappa$, every $C \in \mathcal{C}_\alpha$, and every $\bar\alpha \in \mathrm{acc}(C)$, there exists $D \in \mathcal{C}_{\bar\alpha}$ such that $D \mathrel{\mathcal{R}} C$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there exist stationarily many $\alpha \in S$ for which:

– $|\mathcal{C}_\alpha| < \nu$; and
– for every $C \in \mathcal{C}_\alpha$ and $i < \min\{\alpha, \theta\}$:

$$\sup\{\beta \in C \mid \mathrm{succ}_\sigma(C \setminus \beta) \subseteq A_i\} = \alpha.$$

We shall sometimes say that *the sequence has width* $< \mu$, and refer to the $\mathcal{R}$*-coherence* of the sequence.

Definition 1.6. $\mathrm{P}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts that both $\mathrm{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ and $\diamondsuit(\kappa)$ hold.

We will often shorten the statement of the proxy principle (both $\mathrm{P}^-(\kappa, \dots)$ and $\mathrm{P}(\kappa, \dots)$) when a final segment of the parameters take on their weakest useful values, as follows:

- If we omit $\mathcal{E}$, then $\mathcal{E} = (\mathcal{P}(\kappa))^2$;
- If we omit $\mathcal{E}$ and σ, then $\sigma = 1$;
- If we omit $\mathcal{E}$, σ and ν, then $\nu = \mu$;
- If we omit $\mathcal{E}$, σ, ν and $\mathcal{S}$, then $\mathcal{S} = \{\kappa\}$;
- If we omit $\mathcal{E}$, σ, ν, $\mathcal{S}$ and θ, then $\theta = 1$.

Definition 1.7. The binary relations $\mathcal{R}$ over $[\kappa]^{<\kappa}$ used as the third parameter in the proxy principle include $\sqsubseteq$, $\sqsubseteq^*$, ${}_\chi{\sqsubseteq}$, ${}_\chi{\sqsubseteq^*}$, and $\sqsubseteq_\chi$, which we define as follows, where $\chi \leq \kappa$ can be any ordinal:

- $D \sqsubseteq C$ iff there exists some ordinal $\beta < \kappa$ such that $D = C \cap \beta$, that is, C *end-extends* D;
- $D \sqsubseteq^* C$ iff there exists $\gamma < \sup(D)$ such that $D \setminus \gamma \sqsubseteq C \setminus \gamma$;
- $D \ {}_\chi{\sqsubseteq}\ C$ iff $((D \sqsubseteq C)$ or $(\mathrm{cf}(\sup(D)) < \chi))$;
- $D \ {}_\chi{\sqsubseteq^*}\ C$ iff $((D \sqsubseteq^* C)$ or $(\mathrm{cf}(\sup(D)) < \chi))$;
- $D \sqsubseteq_\chi C$ iff $((D \sqsubseteq C)$ or $(\mathrm{otp}(C) < \chi$ and $\mathrm{nacc}(C)$ consists only of successor ordinals$))$.

Modulo arithmetic hypotheses, each of the coherence relations defined above suffices, when used as the third parameter of the proxy principle, for the construction of a κ-Souslin tree. Of course, coherence relations weaker than $\sqsubseteq$ tend to produce trees satisfying inferior additional properties, but as demonstrated by Corollary 1.20 and Examples 1.26 and 1.27 below, the study of weaker coherence relations is necessary.

Definition 1.8. The equivalence relations $\mathcal{E}$ over a subset of $\mathcal{P}(\kappa)$ used as the eighth parameter in the proxy principle include the default $(\mathcal{P}(\kappa))^2$, as well as $=^*$ and $\mathcal{E}_\chi$, which we define as follows, where $\chi \leq \kappa$ can be any ordinal:

- $D =^* C$ iff ($D = C$ or there exists some $\gamma < \sup(D)$ such that $D \setminus \gamma = C \setminus \gamma$);
- $D \ \mathcal{E}_\chi \ C$ iff (($\mathrm{otp}(D) \leq \chi$) and ($\mathrm{otp}(C) \leq \chi$)).

We shall establish below that by an appropriate choice of a vector of parameters, the proxy principles P^- and P capture many of the combinatorial principles considered in the Appendix, including $\clubsuit_w(S)$, $\diamondsuit(S)$, $\langle\lambda\rangle^-_S$, $\square_\lambda$, $\boxminus_{\lambda,\geq\chi}$, $\boxtimes_\lambda$, and $\square_\lambda + \mathrm{CH}_\lambda$.

At some point, we realized that our paper is growing large and decided to split it into two parts. This paper, being Part I, concentrates solely on the case where the second parameter of the proxy principle takes its strongest possible value, that is, $\mu = 2$. Part II concentrates on the case $2 < \mu \leq \kappa$.

What is so nice about the case $\mu = 2$, is that for every ordinal $\alpha < \kappa$, $\mathcal{C}_\alpha$ is a singleton, say $\{C_\alpha\}$, and the parameter ν conveys no additional information. Therefore, in this case, it makes sense to simplify the notation and say that a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ witnesses $\mathrm{P}^-(\kappa, 2, \mathcal{R}, \theta, \mathcal{S}, 2, \sigma, \mathcal{E})$ whenever:

- for every limit $\alpha < \kappa$, C_α is a club subset of α, and $C_\alpha \in \mathrm{dom}(\mathcal{E})$;
- for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$, we have $C_{\bar\alpha} \ \mathcal{R} \ C_\alpha$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there exist stationarily many $\alpha \in S$ satisfying, for every $i < \min\{\alpha, \theta\}$:

$$\sup\{\beta \in C_\alpha \mid \mathrm{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_i\} = \alpha.$$

To conclude this subsection, let us point out how the simplified axioms from the previous subsection may be defined using the proxy principle P^-.

Proposition 1.9. *For a regular uncountable cardinal κ and a stationary subset $S \subseteq \kappa$:*

- *$\boxtimes(S)$ is equivalent to $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{S\}, 2, \omega)$;*
- *$\boxtimes^-(S)$ is equivalent to $\mathrm{P}^-(\kappa, 2, \sqsubseteq, 1, \{S\}, 2, 1)$.* $\square$

We also define $\boxtimes^*(S)$ to assert that $\mathrm{P}^-(\kappa, \kappa, {}_\chi{\sqsubseteq^*}, 1, \{S\}, \kappa, 1)$ holds for $\chi := \min\{\mathrm{cf}(\alpha) \mid \alpha \in S \text{ limit}\}$, and similarly define $\boxtimes_\lambda(S)$, $\boxtimes^-_\lambda(S)$, and $\boxtimes^*_\lambda(S)$ by letting the eighth parameter be $\mathcal{E}_\lambda$ in each respective part of the preceding.

1.2. Sample corollaries

To give an idea of the flavor of consequences the results of this paper entail, we state here a few sample corollaries. We remind the reader that the definitions of all relevant combinatorial principles may be found in the Appendix.

Our first corollary lists sufficient conditions for the proxy principle to hold with its parameters taking on their strongest useful values:

Corollary 1.10. $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa, \mathcal{S}, 2, \omega)$ *holds, assuming any of the following:*

1. *$\kappa = \aleph_1, \mathcal{S} = \{S\}, S \subseteq \omega_1$ and $\diamondsuit(S)$ holds;*
2. *$\kappa = \lambda^+$, λ is a singular cardinal, $\mathcal{S} = \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}$ and $\square_\lambda + \mathrm{CH}_\lambda$ holds;*
3. *$\kappa = \lambda^+$, λ is a regular uncountable cardinal, $\mathcal{S} = \{E^{\lambda^+}_{\lambda}\}$, and $\boxminus_\lambda$ holds;*
4. *$\kappa = \lambda^+$, λ is not subcompact, $\mathcal{S} = \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}$ and V is a Jensen-type extender model of the form $L[E]$;*[9]
5. *κ is a regular uncountable cardinal that is not weakly compact, $\mathcal{S} = \{E^\kappa_{\geq\chi} \mid \chi < \kappa \;\&\; \forall\lambda < \kappa(\lambda^{<\chi} < \kappa)\}$ and $V = L$;*
6. *$\kappa = \lambda^+$, λ is regular uncountable, $\mathcal{S} = \{S \subseteq E^{\lambda^+}_{\lambda} \mid S$ is stationary$\}$ and $V = W^{\mathrm{Add}(\lambda,1)}$,*[10] *where*
$$W \models \mathsf{ZFC} + \square_\lambda + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda.$$

Proof. 1. By Theorem 3.6.
2. By Corollary 3.10.
3. By Theorem 3.6.
4. For $\kappa = \aleph_1$, appeal to Clause (1); For $\kappa = \lambda^+$ where λ is singular, appeal to Clause (2) and [SZ04, Theorem 0.1]. Finally, if κ is a successor of a regular uncountable cardinal λ that is not subcompact, then appeal to Clause (3) and [Kyp09, Theorem 2.3], taking $D = E^{\lambda^+}_{\lambda}$ there.
5. For κ successor, this follows from Clause (4) and the fact that no cardinal in L is subcompact. For κ inaccessible not weakly compact, appeal to Fact 3.13.
6. By Theorem 4.2(2) for $\chi = \aleph_0$, using Lemma 4.1. □

Remark 1.11. The instance of the proxy principle obtained in the preceding corollary allows the construction of κ-Souslin trees of the most complicated nature:

- In [BR17c], $\mathrm{P}(\aleph_3, 2, \sqsubseteq, \aleph_3, \{E^{\aleph_3}_{\aleph_2}\}, 2, \omega) + \mathsf{GCH}$ is shown to yield four $\aleph_3$-trees, T_0, T_1, T_2, and T_3, along with ultrafilters $\mathcal{U}_0 \subseteq [\aleph_0]^{\aleph_0}$, $\mathcal{U}_1 \subseteq [\aleph_1]^{\aleph_1}$, such that the reduced powers of the corresponding trees behave as follows:

	T	$T^\omega/\mathcal{U}_0$	$T^{\omega_1}/\mathcal{U}_1$
T_0	$\aleph_3$-Souslin	$\aleph_3$-Aronszajn	$\aleph_3$-Aronszajn
T_1	$\aleph_3$-Souslin	$\neg\aleph_3$-Aronszajn	$\neg\aleph_3$-Aronszajn
T_2	$\aleph_3$-Souslin	$\aleph_3$-Aronszajn	$\neg\aleph_3$-Aronszajn
T_3	$\aleph_3$-Souslin	$\neg\aleph_3$-Aronszajn	$\aleph_3$-Aronszajn

- In Section 2 below, we prove that $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa)$ entails the existence of a uniformly coherent κ-Souslin tree.

Corollary 1.12. *Suppose that CH_λ holds for a regular uncountable cardinal λ, and $S \subseteq E^{\lambda^+}_{\lambda}$ is stationary. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Leftarrow$ (6)*:*

[9]See [Zem10] for a study of □ in these models.
[10]Here, $\mathrm{Add}(\lambda, 1)$ stands for the notion of forcing for adding a Cohen subset to λ.

1. $\diamondsuit(S)$;
2. $\curlywedge^-(\chi, S)$ *for all uncountable cardinals* $\chi \leq \lambda$;
3. $\curlywedge^-(\chi, S)$ *for some uncountable cardinal* $\chi \leq \lambda$;
4. $\langle\lambda\rangle^-_S$;
5. $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \omega, \mathcal{E}_\lambda)$;
6. $V = W^{\mathrm{Add}(\lambda,1)}$, *where* $W \models \mathsf{ZFC} + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$.

Proof. (1) $\Rightarrow$ (2) By Fact 8.9.
(3) $\Rightarrow$ (4) By Theorem 5.3.
(4) $\Leftrightarrow$ (5) By Corollary 5.5.
(6) $\Rightarrow$ (5) By Theorem 5.7. □

Remark 1.13. By [BR17c, §6], the principle $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\})$ augmented with the arithmetic hypothesis $\lambda^{<\lambda} = \lambda$ entails the existence of a λ-free λ^+-Souslin tree.

Corollary 1.14. *Suppose that* $\boxminus_{\lambda,\geq\chi} + \mathrm{CH}_\lambda$ *holds for given infinite cardinals* $\mathrm{cf}(\chi) = \chi \leq \theta < \lambda$. *Then:*

1. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds;*
2. *If* λ *is singular, then* $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds;*
3. *If* λ *is regular, then* $\mathrm{P}(\lambda^+, 2, {}_\chi\sqsubseteq^*, \chi, \{E^{\lambda^+}_\lambda\})$ *holds.*

Proof. (1) By Theorem 4.3(1) and Theorem 4.4(1).
(2) By Theorem 4.3(3).
(3) By Corollary 6.2(2). □

Remark 1.15. By [BR17c, §5], $\mathrm{P}(\kappa, 2, \sqsubseteq_\chi, \theta, \{\kappa\}, 2, \omega)$ entails the existence of a κ-Souslin tree T with an *injective* $\mathcal{F}^\chi_\theta$*-ascent path.*

Note that if $\chi = \aleph_0$, then $\boxminus_{\lambda,\geq\chi}$, $\sqsubseteq_\chi$ and ${}_\chi\sqsubseteq^*$ are respectively equivalent to $\square_\lambda$, $\sqsubseteq$ and $\sqsubseteq^*$. In particular, Lemma 3 of [SS88] implies that the above-mentioned λ^+-Souslin tree T (obtained from $\square_\lambda + \mathrm{CH}_\lambda$) remains non-special in any cofinality-preserving extension. More generally, Corollary 1.7 of [Luc17] implies that for $\kappa > \aleph_1$, the κ-Souslin tree obtained from $\mathrm{P}(\kappa, 2, \sqsubseteq, \aleph_1, \{\kappa\}, 2, \omega)$ via [BR17c, Theorem 5.1] remains non-special in any cofinality-preserving extension.

Corollary 1.16. *Suppose that* $\square_\lambda + \mathrm{CH}_\lambda$ *holds for a given uncountable cardinal* λ.
Then $\mathrm{P}(\lambda^+, 2, \sqsubseteq^*, 1, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega)$ *holds.*

Proof. By Corollary 6.2(1) with $\chi = \aleph_0$ for λ regular, and Corollary 3.10 for λ singular. □

For completeness, we mention that in a more recent work, the second author proved the following:

Fact 1.17 ([Rin17a])**.** *Suppose that* $\square(\lambda^+) + \mathsf{GCH}$ *holds for a given uncountable cardinal* λ.
Then $\mathrm{P}(\lambda^+, 2, \sqsubseteq^*, 1, \{S\})$ *holds for every stationary* $S \subseteq \lambda^+$.

Remark 1.18. By [BR16], if $\mathrm{P}(\lambda^+, 2, \sqsubseteq^*, 1, \{E^{\lambda^+}_{\geq\chi}\})$ holds and $\lambda^{<\chi} = \lambda$, then there exists a χ-complete λ^+-Souslin tree.

Corollary 1.19. *If CH_λ holds for a given regular uncountable cardinal λ, and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq^*, \theta, \{E^{\lambda^+}_\lambda\}, 2, \omega)$ holds for all regular cardinals $\theta < \lambda$.*

Proof. By Theorem 6.3. □

So far, it seems like all of our hypotheses are in the spirit of "$V = L$". The next model shows that the proxy principle is also consistent with strong forcing axioms.

Corollary 1.20. *Assuming the consistency of a supercompact cardinal, there exists a model of* ZFC *that satisfies simultaneously:*

1. *Martin's Maximum holds, and hence:*
   (a) $\square^*_\lambda$ *fails for every singular cardinal λ of countable cofinality;*
   (b) $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal λ;*
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal λ;*
3. $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal λ.*

Proof. The definitions of $\square^*_\lambda$ and $\square_{\lambda,\aleph_1}$ may be found in [CM11], and the model we construct is a slight variation of the model V_3 from Section 3 of that paper. Specifically, we start by working in the model V_1 from [CM11, §3] so that κ is a supercompact cardinal indestructible under $(<\kappa)$-directed-closed notions of forcing, and CH_λ holds for all cardinals $\lambda \geq \kappa$. Now we do an iteration of length ON with Easton support; for each singular cardinal $\lambda > \kappa$, we force with Baumgartner's poset $Q(\kappa, \lambda)$,[11] and for each regular cardinal $\lambda \geq \kappa$, we force with Cohen's poset $\mathrm{Add}(\lambda^+, 1)$. The resulting model V_2 satisfies:

- Cardinals and cofinalities are preserved;
- κ is supercompact;
- CH_λ holds for all cardinals $\lambda \geq \kappa$;
- $\boxminus_{\lambda,\geq\kappa}$ holds for every singular cardinal $\lambda > \kappa$;
- $\diamondsuit(E^{\lambda^+}_\lambda)$ holds for every regular cardinal $\lambda \geq \kappa$.

Now we force over V_2 with the standard forcing for the consistency of MM. The forcing poset is semi-proper and κ-*cc* with cardinality κ. After forcing we get a model V_3 in which:

- $\aleph_1$ is preserved, and κ is the new $\aleph_2$;
- MM holds. In particular, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$;

[11] See [CM11, §3] or the proof of Corollary 4.7 below.

- CH_λ holds for all cardinals $\lambda \geq \aleph_2$;
- $\boxminus_{\lambda,\geq\aleph_2}$ holds for every singular cardinal λ;
- $\diamondsuit(E^{\lambda^+}_\lambda)$ holds for every regular cardinal $\lambda \geq \aleph_2$.

By MM and [CM11], Clauses (a) and (b) hold. Since MM implies PFA, we get from [Bau84, §7] that $\diamondsuit(E^{\lambda^+}_\lambda)$ holds also for $\lambda = \aleph_1$.

▶ By Theorem 5.6, we infer that $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ holds for every regular uncountable cardinal λ.

▶ By Theorem 4.3, we infer that $\mathrm{P}(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ holds for every singular cardinal λ. □

Remark 1.21. By [BR17c, §6], in the preceding model, for every uncountable cardinal λ, there exists a free λ^+-Souslin tree.

Here is another anti-"$V = L$" scenario.

Corollary 1.22. *If λ is a successor of a regular cardinal θ, and $\mathrm{NS} \restriction E^\lambda_\theta$ is saturated, then CH_λ entails $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq^*, \theta, \{E^{\lambda^+}_\lambda\}, 2, \theta)$.*

Proof. This is Theorem 6.4. □

Remark 1.23. In [BR17b], it is proved that $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq^*, 1, \{E^{\lambda^+}_\lambda\})$ together with $\lambda^{<\lambda} = \lambda$ entails the existence of a specializable λ^+-Souslin tree.

And now we consider a scenario which is orthogonal to the two most important scenarios for successors of regular cardinals: Jensen's [Jen72] and Gregory's [Gre76].

Corollary 1.24. *Assuming the consistency of a supercompact cardinal, there exists a model of* ZFC *with a cardinal λ satisfying:*

1. *$\diamondsuit(E^{\lambda^+}_\lambda)$ fails;*
2. *λ is supercompact, and hence $\lambda^{<\lambda} = \lambda$ and every stationary subset of $E^{\lambda^+}_{<\lambda}$ reflects;*
3. *$\mathrm{P}(\lambda^+, 2, \sqsubseteq_\lambda, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ holds.*

Proof. Using the same techniques of the proof of Corollary 1.20, or of [AC00, Fact 2.7], we may assume that λ is a supercompact cardinal indestructible under $(<\lambda)$-directed-closed notions of forcing, and $\mathrm{CH}_\lambda + \boxminus_{\lambda,\geq\lambda}$ holds. By [SK80], there exists a $(<\lambda)$-directed-closed, λ-distributive notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models \neg\diamondsuit(E^{\lambda^+}_\lambda)$. By the λ-distributivity of $\mathbb{P}$, we have $V^{\mathbb{P}} \models \lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \boxminus_{\lambda,\geq\lambda}$, and then $V^{\mathbb{P}*\mathrm{Add}(\lambda,1)} \models \mathrm{P}(\lambda^+, 2, \sqsubseteq_\lambda, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ follows from Theorem 4.2. Since $\mathrm{Add}(\lambda, 1)$ has the λ^+-*cc*, a well-known argument of Kunen entails that $\diamondsuit(E^{\lambda^+}_\lambda)$ remains failing in $V^{\mathbb{P}*\mathrm{Add}(\lambda,1)}$. Since $\mathbb{P}*\mathrm{Add}(\lambda, 1)$ is $(<\lambda)$-directed-closed, λ remains supercompact in $V^{\mathbb{P}*\mathrm{Add}(\lambda,1)}$. □

1.3. Distinguishing the various parameters

To justify the various parameters of the proxy principle, for each parameter (except for the second and sixth, since we promised that their value be 2 throughout this paper), we give an example showing how changing the value of that parameter can alter the validity of the principle. Note that the techniques given here give rise to many more models distinguishing the validity of the principle with various vectors of parameters. We also remark that a thorough study of implications between various vectors will be carried out in Part II.

Example 1.25 (Distinguishing the 1st parameter)**.** The conjunction of the following statements is consistent:

- $\mathrm{P}(\aleph_1, 2, \sqsubseteq)$ fails;
- $\mathrm{P}(\aleph_2, 2, \sqsubseteq)$ holds.

Proof. Force over L with a *ccc* poset to get a model in which Martin's axiom holds and $2^{\aleph_0} = \aleph_2$. As Martin's axiom holds, there are no $\aleph_1$-Souslin trees, and hence $\mathrm{P}(\aleph_1, 2, \sqsubseteq)$, which is the same as $\boxtimes^-(\omega_1) + \diamondsuit(\omega_1)$, fails by Proposition 2.3. In the extension, as we have $\square_{\aleph_1} + \mathrm{CH}_{\aleph_1}$, by Corollary 3.9, $\mathrm{P}(\aleph_2, 2, \sqsubseteq, 1, \{\aleph_2\}, 2, \omega_2, \mathcal{E}_{\omega_1})$ holds, and hence so does $\mathrm{P}(\aleph_2, 2, \sqsubseteq)$. □

Example 1.26 (Distinguishing the 3rd parameter)**.** Relative to a weakly compact cardinal, the conjunction of the following statements is consistent:

- $\mathrm{P}(\aleph_2, 2, {}_{\aleph_0}\!\sqsubseteq)$ fails;
- $\mathrm{P}(\aleph_2, 2, {}_{\aleph_1}\!\sqsubseteq)$ holds.

Proof. Suppose that in V, κ is weakly compact, and let $\mathbb{P} := \mathrm{Col}(\aleph_1, <\kappa)$ be Levy's notion of forcing for collapsing κ to ω_2. Note that as κ is strongly inaccessible, for every $\alpha < \kappa$, the collection $\mathcal{N}_\alpha := \{\tau \in V_{\alpha+1} \mid \tau \text{ is a } \mathbb{P}\text{-name}\}$ has size $< \kappa$.

Let G be $\mathbb{P}$-generic over V, and work in $V[G]$. For all $\alpha < \kappa$, let $\mathcal{A}_\alpha := \{\tau_G \mid \tau \in \mathcal{N}_\alpha\} \cap \mathcal{P}(\alpha)$. Since $\mathbb{P}$ has the κ-*cc*, $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$ forms a $\diamondsuit^+(\aleph_2)$-sequence, and in particular, $\diamondsuit(E^{\aleph_2}_{\aleph_1})$ holds.[12] So, by Corollary 1.12, $\mathrm{P}(\aleph_2, 2, {}_{\aleph_1}\!\sqsubseteq, \aleph_2, \{E^{\aleph_2}_{\aleph_1}\}, 2, \omega, \mathcal{E}_{\omega_1})$ holds, and hence so does $\mathrm{P}(\aleph_2, 2, {}_{\aleph_1}\!\sqsubseteq)$.

As for the first bullet, by [Vel86, Theorem 5], $\square(\aleph_2)$ fails. And then, by Lemma 3.2, $\mathrm{P}(\aleph_2, 2, \sqsubseteq)$ fails as well. □

Example 1.27 (Distinguishing the 3rd parameter)**.** Relative to a supercompact cardinal, it is consistent that there exist uncountable limit cardinals $\kappa < \lambda$ satisfying both of the following statements:

- $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\theta)$ fails for all $\theta < \kappa$;

[12] See [Kun80] for the definition of $\diamondsuit^+(\kappa)$ and the proof that it implies $\diamondsuit(S)$ for every stationary $S \subseteq \kappa$.

- $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\kappa)$ holds.

Proof. Work in the model from Corollary 4.7. Then κ is supercompact and $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\kappa, \lambda^+, \{\lambda^+\}, 2, 1, \mathcal{E}_\lambda)$ holds, for $\lambda = \kappa^{+\omega}$. In particular, $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\kappa)$ holds, establishing the second bullet.

As for the first bullet, as κ is supercompact, we have that any pair of stationary subsets of $E^{\lambda^+}_{<\kappa}$ reflect simultaneously. It now follows from Lemma 4.8 that $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\theta)$ must fail for all $\theta < \kappa$. $\square$

Example 1.28 (Distinguishing the 4th parameter)**.** The conjunction of the following statements is consistent:

- $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, \aleph_1, \{\aleph_2\}, 2, 2, \mathcal{E}_{\omega_1})$ fails;
- $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, \aleph_0, \{\aleph_2\}, 2, 2, \mathcal{E}_{\omega_1})$ holds.

Proof. Let $\mathbb{P}$ be, in L, the forcing notion from [Asp14]. Then $\mathbb{P}$ is a σ-closed, cofinality-preserving notion of forcing, and in $L^{\mathbb{P}}$, for every sequence $\langle C_\delta \mid \delta \in E^{\aleph_2}_{\aleph_1}\rangle$ of local clubs of order-type ω_1, there exists some club $D \subseteq \omega_2$ for which $\sup\{\beta < \delta \mid \mathrm{succ}_2(C_\delta \setminus \beta) \subseteq D\} < \delta$ for all $\delta \in E^{\aleph_2}_{\aleph_1}$.

Work in the extension. Towards a contradiction, suppose that $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, \aleph_1, \{\aleph_2\}, 2, 2, \mathcal{E}_{\omega_1})$ holds, as witnessed by $\vec{C} = \langle C_\delta \mid \delta < \omega_2\rangle$. Let D be a club such that $\sup\{\beta < \delta \mid \mathrm{succ}_2(C_\delta \setminus \beta) \subseteq D\} < \delta$ for all $\delta \in E^{\aleph_2}_{\aleph_1}$. Let $\langle A_i \mid i < \omega_1\rangle$ be some partition of D into stationary sets. Then, by the choice of $\vec{C}$, there must exist some limit ordinal $\delta < \omega_2$ such that $\sup\{\beta < \delta \mid \mathrm{succ}_2(C_\delta \setminus \beta) \subseteq A_i\} = \delta$ for all $i < \omega_1$. As $A_i \cap A_j = \emptyset$ for all $i < j < \omega_1$, we infer that $|C_\delta| \geq \aleph_1$. Recalling that the eighth parameter is $\mathcal{E}_{\omega_1}$, we conclude that $\mathrm{cf}(\delta) = \omega_1$, thus, yielding a contradiction, and establishing the first bullet.

As $L^{\mathbb{P}}$ is a σ-closed, cofinality-preserving extension of L, we have that $\square_{\aleph_1} + \diamondsuit(\omega_1)$ holds in the extension, and hence so does $\square_{\aleph_1} + \clubsuit(\omega_1) + (\aleph_2)^{\aleph_0} = \aleph_2$. By $\square_{\aleph_1} + \clubsuit(\omega_1)$ and the main result of [JNSS92], there exists a sequence $\langle S_\alpha \mid \alpha \in E^{\aleph_2}_{\aleph_0}\rangle$ such that for every $\alpha \in E^{\aleph_2}_{\aleph_0}$, S_α is a cofinal subset of α of order-type ω, and such that for every uncountable $X \subseteq \omega_2$, there exists some $\alpha \in E^{\aleph_2}_{\aleph_0}$ for which $S_\alpha \subseteq X$. Using the fact that $(\aleph_2)^{\aleph_0} = \aleph_2$, it is now easy to build a witness to $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, \aleph_0, \{E^{\aleph_2}_{\aleph_0}\}, 2, 2, \mathcal{E}_{\omega_1})$, establishing the second bullet. $\square$

Example 1.29 (Distinguishing the 5th parameter)**.** The conjunction of the following statements is consistent:

- $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, 1, \{E^{\aleph_2}_{\aleph_1}\}, 2, 2, \mathcal{E}_{\omega_1})$ fails;
- $\mathrm{P}^-(\aleph_2, 2, {}_{\aleph_1}{\sqsubseteq}, 1, \{E^{\aleph_2}_{\aleph_0}\}, 2, 2, \mathcal{E}_{\omega_1})$ holds.

Proof. This is the same model and virtually the same proof as of Example 1.28. $\square$

Example 1.30 (Distinguishing the 7th parameter)**.** The conjunction of the following statements is consistent:

- $\mathrm{P}(\aleph_2, 2, \sqsubseteq, 1, \{E^{\aleph_2}_{\aleph_1}\}, 2, \omega_1, \mathcal{E}_{\omega_1})$ fails;
- $\mathrm{P}(\aleph_2, 2, \sqsubseteq, 1, \{E^{\aleph_2}_{\aleph_1}\}, 2, \omega, \mathcal{E}_{\omega_1})$ holds.

Proof. Let $\mathbb{P}$ be, in L, the forcing notion from [SK80], so that $L^{\mathbb{P}} \models \neg\diamondsuit(E^{\aleph_2}_{\aleph_1})$. Then $\mathbb{P}$ is σ-closed, ω_1-distributive, and has the $\aleph_3$-*cc*. So L and $L^{\mathbb{P}}$ share the same cardinals structure, GCH holds in the extension, and so does $\square_{\aleph_1}$. Now, force with $\mathrm{Add}(\omega_1, 1)$ over $L^{\mathbb{P}}$. By Theorem 4.2(2) with $\chi := \aleph_0$, $\mathrm{P}(\aleph_2, 2, \sqsubseteq, \aleph_2, \{E^{\aleph_2}_{\aleph_1}\}, 2, \omega, \mathcal{E}_{\omega_1})$ holds in the extension. Since $\mathrm{Add}(\omega_1, 1)$ has the $\aleph_2$-*cc*, a well-known argument of Kunen entails that $\diamondsuit(E^{\aleph_2}_{\aleph_1})$ remains failing in the extension, and then Theorem 5.1 finishes the proof. $\square$

Example 1.31 (Distinguishing the 8th parameter)**.** Relative to a Mahlo cardinal, the conjunction of the following statements is consistent for some cardinal κ:

- $\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, 1, \mathcal{E}_\chi)$ fails for all $\chi < \kappa$;
- $\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, 1, \mathcal{E}_\kappa)$ holds.

Proof. Work in L, and let κ be a Mahlo cardinal that is not weakly compact (say, the first Mahlo). Since $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha\}$ is cofinal in κ, $\mathrm{P}^-(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, 1, \mathcal{E}_\chi)$ fails for all $\chi < \kappa$. On the other hand, by Theorem 3.12, $\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, \kappa, \mathcal{E}_\kappa)$ holds. $\square$

1.4. Organization of this paper

In Section 2, we fix some terminology and notation, demonstrate the microscopic approach by constructing various κ-Souslin trees from instances of the proxy principle, and highlight the differences between the classical approach and the new one.

Sections 3 to 6 are the heart of the matter, where we build the bridge between the old foundations and the new one. That is, we establish instances of the proxy principle from various hypotheses that were previously known to entail the existence of κ-Souslin trees. The division between these sections is based on the third parameter of the proxy principle. Specifically, Section 3 deals with $\sqsubseteq$, Section 4 with $\sqsubseteq_\chi$, Section 5 with ${}_\lambda\sqsubseteq$, and Section 6 deals with ${}_\chi\sqsubseteq^*$.

Finally, the paper concludes with an Appendix that briefly provides the necessary background regarding the combinatorial principles used in this context.

2. Constructing Souslin trees from the proxy principle

In this section we demonstrate the microscopic approach by constructing various κ-Souslin trees from instances of the proxy principle. We begin by recalling the relevant terminology and fixing some notation.

A *tree* is a partially ordered set $(T, <_T)$ with the property that for every $x \in T$, the downward cone $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. The *height* of $x \in T$, denoted $\mathrm{ht}(x)$, is the order-type of $(x_\downarrow, <_T)$. Then, the α^{th} level of $(T, <_T)$ is the set $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$. We also write $T \restriction X := \{t \in T \mid \mathrm{ht}(t) \in X\}$. A tree $(T, <_T)$ is said to be χ-*complete* if any $<_T$-increasing sequence of elements from T, and of length $< \chi$, has an upper bound in T. On the other extreme, the tree $(T,<_T)$ is said to be *slim* if $|T_\alpha| \leq \max\{|\alpha|, \aleph_0\}$ for every ordinal α. A tree $(T,<_T)$ is said to be *normal* if for all ordinals $\alpha<\beta$ and every $x\in T_\alpha$, if $T_\beta\neq\emptyset$ then there exists some $y\in T_\beta$ such that $x<_T y$. A tree $(T,<_T)$ is said to be *splitting* if every node in T admits at least two immediate successors.

Throughout, let κ denote a regular uncountable cardinal. A tree $(T,<_T)$ is a *κ-tree* whenever $\{\alpha \mid T_\alpha\neq\emptyset\}=\kappa$, and $|T_\alpha|<\kappa$ for all $\alpha<\kappa$. A subset $B\subseteq T$ is a *cofinal branch* if $(B,<_T)$ is linearly ordered and $\{\mathrm{ht}(t)\mid t\in B\}=\{\mathrm{ht}(t)\mid t\in T\}$. A *$\kappa$-Aronszajn tree* is a κ-tree with no cofinal branches. A *κ-Souslin tree* is a κ-Aronszajn tree that has no antichains of size κ.

Let $(T,<_T)$ denote a κ-tree. A function $\rho: T\to T$ is said to be *regressive* if $\rho(t)<_T t$ for every nonminimal node $t\in T$. Two nonminimal nodes $s,t\in T$ are said to be *ρ-compatible* if $\rho(s)<_T t$ and $\rho(t)<_T s$. The tree $(T,<_T)$ is said to be *regressive* if there exists a regressive function $\rho: T\to T$ such that for all $\alpha\in\mathrm{acc}(\kappa)$: $s,t\in T_\alpha$ are ρ-compatible iff $s=t$. It is *stationarily-regressive* (respectively, *club-regressive*), if, in addition, for every $\alpha\in E^\kappa_{>\omega}$ there exists a stationary (respectively, club) subset $e_\alpha\subseteq\alpha$ such that: $s,t\in T\restriction(e_\alpha\cup\{\alpha\})$ are ρ-compatible iff s and t are compatible. By an argument from [Tod81, p. 242], a regressive κ-tree contains no ν-Cantor subtrees for any infinite cardinal ν, and a stationarily-regressive κ-tree contains no ν-Aronszajn subtrees for all infinite regular $\nu<\kappa$.

Of special interest is the case where $<_T$ is simply $\subset$, and T is a downward-closed subset of ${}^{<\kappa}\kappa$. All of the trees that we construct here will be of this form. In such a setup, each node t of the tree T is a function $t:\alpha\to\kappa$ for some ordinal $\alpha<\kappa$, and we require that if $t:\alpha\to\kappa$ is in T, then $t\restriction\beta\in T$ for every $\beta<\alpha$. For any node $t\in T$, the height of t in T is just its domain, that is, $\mathrm{ht}(t)=\mathrm{dom}(t)$, and its set of predecessors, $t_\downarrow$, is simply $\{t\restriction\beta \mid \beta<\mathrm{dom}(t)\}$. Note that $T_\alpha=T\cap{}^\alpha\kappa$ for every $\alpha<\kappa$. Finally, any function $f:\kappa\to\kappa$ determines a cofinal branch through ${}^{<\kappa}\kappa$, namely $\{f\restriction\alpha\mid\alpha<\kappa\}$, which ought not to end up being a subset of T if $(T,\subset)$ is to form a κ-Aronszajn tree.

The main advantage of this approach is the ease of completing a branch at a limit level. Suppose that, during the process of constructing T, we have already inserted into T a $\subset$-increasing sequence of nodes $\eta:=\langle t_\alpha\mid\alpha<\beta\rangle$ for some limit ordinal $\beta<\kappa$. The (unique) limit of this sequence, which may or may not become a member of T, is nothing but $\bigcup\mathrm{Im}(\eta)$, that is, $\bigcup_{\alpha<\beta}t_\alpha$. Furthermore, compatibility of nodes in the tree is easily expressed: For $x,y\in T$, x and y are compatible iff $x\cup y\in T$.

A κ-tree is said to be *binary* if it is a downward-closed subset of the complete binary tree ${}^{<\kappa}2$. On the opposite extreme, a subtree $T\subseteq{}^{<\kappa}\kappa$ is *prolific* if it is downward-closed and for every $\alpha<\kappa$ and every $t\in T\cap{}^\alpha\kappa$, we have $\{t^\frown\langle i\rangle\mid i<\max\{\omega,\alpha\}\}\subseteq T$. Of course, a prolific tree is splitting.

A subtree $T\subseteq{}^{<\kappa}\kappa$ is *coherent* if it is downward-closed and for every $\delta<\kappa$ and $s,t\in T\cap{}^\delta\kappa$, the set $\{\gamma<\delta\mid s(\gamma)\neq t(\gamma)\}$ is finite. It is *uniformly coherent*, if, in addition, for all $\alpha<\beta<\kappa$, $s\in T_\alpha$ and $t\in T_\beta$: $s*t:=s\cup(t\restriction(\beta\setminus\alpha))$ is in T. Notice that if T is prolific and uniformly coherent, then for every two regressive functions $s,t\in{}^\delta\delta$ with a finite $\{\gamma<\delta\mid s(\gamma)\neq t(\gamma)\}$, $s\in T$ iff $t\in T$.

Finally, note that a coherent tree is regressive, and a regressive binary tree is slim.

While classical constructions of κ-Souslin trees typically involve a recursive process of determining a partial order $<_T$ over κ by consulting a $\diamondsuit(\kappa)$-sequence, here, the order is already known (being $\subset$), and the recursive process involves the determination of a subset of ${}^{<\kappa}\kappa$. For this reason, it is more convenient to work with the following richer variation of $\diamondsuit(\kappa)$:

Definition 2.1. $\diamondsuit(H_\kappa)$ asserts the existence of a partition $\langle R_i \mid i<\kappa\rangle$ of κ and a sequence $\langle \Omega_\beta \mid \beta<\kappa\rangle$ of elements of H_κ such that for every $p\in H_{\kappa^+}$, $i<\kappa$, and $\Omega\subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M}\prec H_{\kappa^+}$ such that:

- $p\in\mathcal{M}$;
- $\mathcal{M}\cap\kappa\in R_i$;
- $\mathcal{M}\cap\Omega=\Omega_{\mathcal{M}\cap\kappa}$.

Here, H_λ denotes the collection of all sets of hereditary cardinality less than λ (cf. [Kun80, IV, §6]).

Lemma 2.2. *$\diamondsuit(\kappa)$ is equivalent to $\diamondsuit(H_\kappa)$ for any regular uncountable cardinal κ.*

Proof. ($\Longleftarrow$): Given $\langle R_i \mid i<\kappa\rangle$ and $\langle \Omega_\beta \mid \beta<\kappa\rangle$ as in the definition of $\diamondsuit(H_\kappa)$, let for all $\beta<\kappa$:

$$Z_\beta := \begin{cases}\Omega_\beta, & \text{if } \Omega_\beta\subseteq\beta;\\ \emptyset, & \text{otherwise.}\end{cases}$$

To show that $\langle Z_\beta \mid \beta<\kappa\rangle$ is a $\diamondsuit(\kappa)$-sequence, consider any $Z\subseteq\kappa$, and we must show that $\{\beta<\kappa \mid Z\cap\beta=Z_\beta\}$ is stationary. Thus, let $D\subseteq\kappa$ be a club, and we must find some $\beta\in D$ such that $Z\cap\beta=Z_\beta$. Put $p:=D$. As $p\in H_{\kappa^+}$, and $Z\subseteq\kappa\subseteq H_\kappa$, we may pick $\mathcal{M}\prec H_{\kappa^+}$ with $D\in\mathcal{M}$ such that $\mathcal{M}\cap\kappa\in R_0$ and $\mathcal{M}\cap Z=\Omega_{\mathcal{M}\cap\kappa}$. Let $\beta:=\mathcal{M}\cap\kappa$. Then $\Omega_\beta=\mathcal{M}\cap Z=\mathcal{M}\cap\kappa\cap Z=\beta\cap Z$ and hence $Z_\beta=\Omega_\beta=Z\cap\beta$. As D is club in κ and $D\in\mathcal{M}$, we have (by elementarity of $\mathcal{M}$) $\beta=\sup(\mathcal{M}\cap\kappa)\in D$.

($\Longrightarrow$): Fix a bijection $\pi:\kappa\times\kappa\leftrightarrow\kappa$. By $\diamondsuit(\kappa)$, fix $\langle Z_\beta \mid \beta<\kappa\rangle$ such that $\{\beta<\kappa \mid Z\cap\beta=Z_\beta\}$ is stationary for all $Z\subseteq\kappa$. By $\diamondsuit(\kappa)$, we also have $|H_\kappa|=2^{<\kappa}=\kappa$, so let us fix a bijection $\phi:\kappa\setminus\{0\}\leftrightarrow H_\kappa$. Now, let:

- $\Omega_\beta:=\{\phi(\alpha) \mid \pi(\alpha,1)\in Z_\beta, \alpha\neq 0\}$;
- $R_0:=\kappa\setminus\bigcup_{0<i<\kappa}R_i$, where for nonzero $i<\kappa$:
- $R_i:=\{\beta\in\operatorname{acc}(\kappa) \mid \{j<\kappa \mid \pi(0,j)\in Z_\beta\}=\{i\}\}$.

To see that $\langle \Omega_\beta \mid \beta<\kappa\rangle$ and $\langle R_i \mid i<\kappa\rangle$ are as requested, let $p\in H_{\kappa^+}$, $i<\kappa$, and $\Omega\subseteq H_\kappa$ be arbitrary. Define $f:\kappa\to\kappa$ by letting for all $\alpha<\kappa$:

$$f(\alpha):=\begin{cases} i, & \text{if } \alpha=0;\\ 1, & \text{if } \alpha>0 \ \&\ \phi(\alpha)\in\Omega;\\ 0, & \text{otherwise.}\end{cases}$$

Let $p' := \{p, \phi, \pi, f\}$. Let $\langle M_\beta \mid \beta < \kappa\rangle$ be a continuous $\in$-chain of elementary submodels of H_{κ^+}, each of size $< \kappa$ and containing p'. Evidently, $E := \{\beta < \kappa \mid M_\beta \cap \kappa = \beta\}$ is a club in κ. Recalling that $f \subseteq \kappa \times \kappa$, so that $\pi[f] \subseteq \kappa$, consider the stationary set of guesses $G := \{\beta < \kappa \mid \pi[f] \cap \beta = Z_\beta\}$, and pick a nonzero $\beta \in E \cap G$. We shall show that $\mathcal{M} = M_\beta$ satisfies the required properties.

By $\pi, f \in M_\beta$ and $M_\beta \cap \kappa = \beta$, we have $f[\beta] \subseteq \beta$ and $\pi[\beta \times \beta] = \beta$. Thus, $\pi[f \restriction \beta] = \pi[f] \cap \beta = Z_\beta$.

Claim 2.2.1. *$\beta \in R_i$.*

Proof. Clearly $\beta \in \mathrm{acc}(\kappa)$ since $\beta = M_\beta \cap \kappa$. Since $Z_\beta \subseteq \beta$, also $\pi^{-1}[Z_\beta] \subseteq \beta \times \beta$, so that $\{j < \kappa \mid \pi(0, j) \in Z_\beta\} \subseteq \beta$. Also $i = f(0) < \beta$. Furthermore, for all $j < \beta$, we have

$$\begin{aligned} \pi(0, j) \in Z_\beta &\iff f(0) = j && \text{since } \pi[f \restriction \beta] = Z_\beta \\ &\iff j = i && \text{by the definition of } f. \end{aligned}$$

By the definition of R_i, it follows that $\beta \in R_i$. □

Claim 2.2.2. *$\phi[\beta \setminus \{0\}] = M_\beta \cap H_\kappa$.*

Proof. For all $x \in H_\kappa$, by $\phi \in M_\beta$, we have $x \in M_\beta$ iff $\phi^{-1}(x) \in M_\beta \cap \mathrm{dom}(\phi)$ iff $\phi^{-1}(x) \in \beta \setminus \{0\}$. □

Claim 2.2.3. *$M_\beta \cap \Omega = \Omega_\beta$*

Proof. We have

$$\begin{aligned} M_\beta \cap \Omega &= M_\beta \cap H_\kappa \cap \Omega && \text{since } \Omega \subseteq H_\kappa \\ &= \phi[\beta \setminus \{0\}] \cap \Omega && \text{by the previous claim} \\ &= \{\phi(\alpha) \mid 0 < \alpha < \beta, f(\alpha) = 1\} && \text{by the definition of } f \\ &= \{\phi(\alpha) \mid 0 \neq \alpha, \pi(\alpha, 1) \in Z_\beta\} && \text{since } \pi[f \restriction \beta] = Z_\beta \\ &= \Omega_\beta && \text{by the definition of } \Omega_\beta. \end{aligned}$$
□

So $\mathcal{M} = M_\beta$ is as required. □

We commence with a simple construction using $\boxtimes^-(\kappa)$.

Proposition 2.3. *If κ is a regular uncountable cardinal and $\boxtimes^-(\kappa) + \diamondsuit(\kappa)$ holds, then there exists a normal, binary, splitting, club-regressive κ-Souslin tree.*

Proof. Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a witness to $\boxtimes^-(\kappa)$. Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$. Let $\lhd$ be some well-ordering of H_κ. We shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of levels whose union will ultimately be the desired tree T.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let $T_{\alpha+1} := \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle \mid t \in T_\alpha\}$.

Next, suppose that $\alpha < \kappa$ is a nonzero limit ordinal, and that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been defined. Constructing the level T_α involves deciding which branches through $(T \restriction \alpha, \subset)$

will have their limits placed into the tree. We need T_α to contain enough nodes to ensure that the tree is normal, so the idea is to attach to each node $x \in T \restriction C_\alpha$ some node $\mathbf{b}^\alpha_x : \alpha \to 2$ above it, and then let

$$T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction C_\alpha\}.$$

Let $x \in T \restriction C_\alpha$ be arbitrary. As $\mathbf{b}^\alpha_x$ will be the limit of some branch through $(T \restriction \alpha, \subset)$ and above x, it makes sense to describe $\mathbf{b}^\alpha_x$ as the limit $\bigcup \mathrm{Im}(b^\alpha_x)$ of a sequence $b^\alpha_x \in \prod_{\beta \in C_\alpha \setminus \mathrm{dom}(x)} T_\beta$ such that:

- $b^\alpha_x(\mathrm{dom}(x)) = x$;
- $b^\alpha_x(\beta') \subset b^\alpha_x(\beta)$ for all $\beta' < \beta$ in $(C_\alpha \setminus \mathrm{dom}(x))$;
- $b^\alpha_x(\beta) = \bigcup \mathrm{Im}(b^\alpha_x \restriction \beta)$ for all $\beta \in \mathrm{acc}(C_\alpha \setminus \mathrm{dom}(x))$.

Of course, we have to define b^α_x carefully, so that the resulting tree doesn't include large antichains. We do this by recursion:

Let $b^\alpha_x(\mathrm{dom}(x)) := x$. Next, suppose $\beta^- < \beta$ are successive points of $(C_\alpha \setminus \mathrm{dom}(x))$, and $b^\alpha_x(\beta^-)$ has already been defined. In order to decide $b^\alpha_x(\beta)$, we consult the following set:

$$Q^{\alpha,\beta}_x := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup b^\alpha_x(\beta^-)) \subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q^{\alpha,\beta}_x \neq \emptyset$, then let $b^\alpha_x(\beta)$ be its $\lhd$-least element.
- Otherwise, let $b^\alpha_x(\beta)$ be the $\lhd$-least element of T_β that extends $b^\alpha_x(\beta^-)$. Such an element must exist, as the level T_β was constructed so as to preserve normality.

Finally, suppose $\beta \in \mathrm{acc}(C_\alpha \setminus \mathrm{dom}(x))$ and $b^\alpha_x \restriction \beta$ has already been defined. As promised, we let $b^\alpha_x(\beta) := \bigcup \mathrm{Im}(b^\alpha_x \restriction \beta)$. It is clear that $b^\alpha_x(\beta) \in {}^\beta 2$, but we need more than that:

Claim 2.3.1. *$b^\alpha_x(\beta) \in T_\beta$.*

Proof. It suffices to prove that $b^\alpha_x \restriction \beta = b^\beta_x$, as this will imply that $b^\alpha_x(\beta) = \bigcup \mathrm{Im}(b^\beta_x) = \mathbf{b}^\beta_x \in T_\beta$.

First, note that since $\beta \in \mathrm{acc}(C_\alpha)$ and $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\boxtimes^-(\kappa)$-sequence, we have $\mathrm{dom}(b^\beta_x) = C_\beta \setminus \mathrm{dom}(x) = C_\alpha \cap \beta \setminus \mathrm{dom}(x) = \mathrm{dom}(b^\alpha_x) \cap \beta$. Call the latter by d. Now, we prove by induction that for every $\gamma \in d$, the value of $b^\beta_x(\gamma)$ was determined in exactly the same way as $b^\alpha_x(\gamma)$:

- Clearly, $b^\beta_x(\min(d)) = x = b^\alpha_x(\min(d))$.
- Suppose $\gamma^- < \gamma$ are successive points of d. Notice that the definition of $Q^{\alpha,\gamma}_x$ depends only on $b^\alpha_x(\gamma^-)$, Ω_γ, and T_γ, and so if $b^\alpha_x(\gamma^-) = b^\beta_x(\gamma^-)$, then $Q^{\alpha,\gamma}_x = Q^{\beta,\gamma}_x$, and hence $b^\alpha_x(\gamma) = b^\beta_x(\gamma)$.
- For $\gamma \in \mathrm{acc}(d)$: If the sequences are identical up to γ, then their limits must be identical. $\square$

This completes the definition of b_x^α for each $x \in T \restriction C_\alpha$, and hence of the level T_α.

Having constructed all levels of the tree, we then let

$$T := \bigcup_{\alpha<\kappa} T_\alpha.$$

Notice that for every $\alpha < \kappa$, T_α is a subset of ${}^\alpha 2$ of size $\leq \max\{\aleph_0, |\alpha|\} < \kappa$. Altogether, $(T, \subset)$ is a normal, binary, slim, splitting κ-tree.

Our next task is proving that $(T, \subset)$ is κ-Souslin. As any splitting κ-tree with no antichains of size κ also has no chains of size κ, it suffices to prove Claim 2.3.3 below. For this, we shall need the following.

Claim 2.3.2. *Suppose that $A \subseteq T$ is a maximal antichain. Then the set*

$$B := \{\beta \in R_0 \mid A \cap (T \restriction \beta) = \Omega_\beta \textit{ is a maximal antichain in } T \restriction \beta\}.$$

is a stationary subset of κ.

Proof. Let $D \subseteq \kappa$ be an arbitrary club. We must show that $D \cap B \neq \emptyset$. Put $p := \{A, T, D\}$. Using the fact that the sequences $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ witness $\diamondsuit(H_\kappa)$, pick $\mathcal{M} \prec H_{\kappa^+}$ with $p \in \mathcal{M}$ such that $\beta := \mathcal{M} \cap \kappa$ is in R_0 and $\Omega_\beta = \mathcal{M} \cap A$. Since $D \in \mathcal{M}$ and D is club in κ, we have $\beta \in D$. We claim that $\beta \in B$.

For all $\alpha < \beta$, by $\alpha, T \in \mathcal{M}$, we have $T_\alpha \in \mathcal{M}$, and by $\mathcal{M} \models |T_\alpha| < \kappa$, we have $T_\alpha \subseteq \mathcal{M}$. So $T \restriction \beta \subseteq \mathcal{M}$. As $\mathrm{dom}(z) \in \mathcal{M}$ for all $z \in T \cap \mathcal{M}$, we conclude that $T \cap \mathcal{M} = T \restriction \beta$. Thus, $\Omega_\beta = A \cap (T \restriction \beta)$. As $H_{\kappa^+} \models A$ is a maximal antichain in T, it follows by elementarity that $\mathcal{M} \models A$ is a maximal antichain in T. Since $T \cap \mathcal{M} = T \restriction \beta$, we get that $A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$. □

Claim 2.3.3. *Suppose that $A \subseteq T$ is a maximal antichain. Then $|A| < \kappa$.*

Proof. By Claim 2.3.2, $B := \{\beta \in R_0 \mid A \cap (T \restriction \beta) = \Omega_\beta$ is a maximal antichain in $T \restriction \beta\}$ is a cofinal subset of κ. Thus, by the fact that $\langle C_\alpha \mid \alpha < \kappa\rangle$ witnesses $\boxtimes^-(\kappa)$, we obtain a limit ordinal $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

We shall prove that $A \subseteq T \restriction \alpha$, from which it follows that $|A| \leq |\alpha| < \kappa$.

To see that $A \subseteq T \restriction \alpha$, consider any $z \in T \restriction (\kappa \setminus \alpha)$, and we will show that $z \notin A$ by finding some element of $A \cap (T \restriction \alpha)$ compatible with z.

Since $\mathrm{dom}(z) \geq \alpha$, we can let $y := z \restriction \alpha$. Then $y \in T_\alpha$ and $y \subseteq z$. By construction, $y = \mathbf{b}_x^\alpha = \bigcup_{\beta \in C_\alpha \setminus \mathrm{dom}(x)} b_x^\alpha(\beta)$ for some $x \in T \restriction C_\alpha$. Fix $\beta \in \mathrm{nacc}(C_\alpha) \cap B$ with $\mathrm{dom}(x) < \beta < \alpha$. Denote $\beta^- := \sup(C_\alpha \cap \beta)$. Then $\beta^- < \beta$ are consecutive points of $C_\alpha \setminus \mathrm{dom}(x)$. Since $\beta \in B$, we know that $\Omega_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$, and hence there is some $s \in \Omega_\beta$ compatible with $b_x^\alpha(\beta^-)$, so that by the normality of the tree, $Q_x^{\alpha,\beta} \neq \emptyset$. It follows that we chose $b_x^\alpha(\beta)$ to extend some $s \in \Omega_\beta$. Altogether, $s \subseteq b_x^\alpha(\beta) \subset \mathbf{b}_x^\alpha = y \subseteq z$. Since s is an element of the antichain A, the fact that z extends s implies that $z \notin A$. □

Claim 2.3.4. *$(T, \subset)$ is club-regressive.*

Proof. Define a regressive map $\rho : T \to T$, as follows. Put $\rho(t) := \emptyset$ for all $t \in T \restriction \mathrm{nacc}(\kappa)$. Next, for all $\alpha \in \mathrm{acc}(\kappa)$ and $t \in T_\alpha$, let $\beta_t := \min\{\beta \in C_\alpha \mid t = \mathbf{b}^\alpha_{t\restriction\beta}\}$ and put $\rho(t) := t \restriction \beta_t$. It is not hard to see that $t = \mathbf{b}^\alpha_{t\restriction\beta}$ for all $\beta \in C_\alpha \setminus \beta_t$. We claim that ρ witnesses that $(T, \subset)$ is regressive. To see this, suppose that we are given $\alpha \in \mathrm{acc}(\kappa)$ and $s, t \in T_\alpha$ that are ρ-compatible. Without loss of generality, $\beta_t \leq \beta_s$. By $\rho(s) \subseteq t$, we get that $t = \mathbf{b}^\alpha_{t\restriction\beta_t} = \mathbf{b}^\alpha_{t\restriction\beta_s} = \mathbf{b}^\alpha_{s\restriction\beta_s} = s$.

Next, let us show that $(T, \subset)$ is moreover club-regressive. Let $\alpha \in E^\kappa_{>\omega}$ be arbitrary. Put $e_\alpha := \mathrm{acc}(C_\alpha)$. Suppose that $s, t \in T \restriction (e_\alpha \cup \{\alpha\})$ are ρ-compatible. Put $\gamma := \mathrm{ht}(t)$ and $\delta := \mathrm{ht}(s)$. Without loss of generality, $\gamma \leq \delta$. Since s, t are ρ-compatible, we have $s \restriction \beta_s = \rho(s) \subseteq t$, and hence $\beta_s \leq \gamma$.

► If $\beta_s = \gamma$, then $t = \rho(s) \subseteq s$, and we are done.

► If $\beta_s < \gamma$, then by $C_\gamma \sqsubseteq C_\delta$ and $s \restriction \gamma \subseteq s$, we have $\beta_{s\restriction\gamma} \leq \beta_s$. In particular, $s \restriction \gamma, t \in T_\gamma$ are ρ-compatible, and hence $t = s \restriction \gamma$. That is, $t \subseteq s$. □

This completes the proof. □

Let us briefly compare the above construction with Jensen's classical one [Jen72], as rendered in [Dev84, Theorem IV.2.4]. Both constructions consist of a recursive process of determining the levels T_α of the ultimate tree T, and both constructions face the same two challenges:

1. Maintaining the ability to climb up through the levels while keeping their width small;
2. Sealing antichains so that if $A \subseteq T$ is a maximal antichain, then there would be some level α, where every node placed into T_α is compatible with some element of $A \cap (T \restriction \alpha)$.

In both constructions, challenge (1) is addressed at limit stage $\alpha < \kappa$ by attaching, for cofinally many nodes $x \in T \restriction \alpha$, a canonical branch b^α_x that goes through x and climbs all the way up to level α. In both constructions, the coherence of the sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ entails that $b^{\bar\alpha}_x = b^\alpha_x \restriction \bar\alpha$ whenever $\bar\alpha \in \mathrm{acc}(C_\alpha)$, thereby ensuring that the recursive process of constructing b^α_x never gets stuck. So where is the difference?

The difference is in the way that we seal antichains. In the above construction, we let $T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction C_\alpha\}$ uniformly for all limit levels $\alpha < \kappa$, whereas in the classical one, there is some stationary set E of levels $\alpha < \kappa$, where maximal antichains are sealed by letting T_α be some carefully chosen subset of $\{\mathbf{b}^\alpha_x \mid x \in T \restriction \alpha\}$. But, of course, by omitting some nodes of the form $\mathbf{b}^\beta_x$ from the β^{th} level, one puts the analogue of Claim 2.3.1 in danger! To overcome this, Jensen ensured that the ordinals in E never occur as accumulation points of C_α for any $\alpha < \kappa$. While being a successful remedy, it means that the classical approach is based on *sealing antichains at the levels of some nonreflecting stationary set.*[13]

[13] As a matter of fact, we are unaware of any previous $\diamondsuit$-based construction whose sealing process does not center on a nonreflecting stationary set. The only candidate we could find in the literature that may involve sealing at a reflecting set is Theorem 4 from [BDS86], but that theorem must be false in light of Theorem 3.1 of [MS96].

In the microscopic approach, sealing of antichains can be done at the levels coming from any stationary set. For example, in [BR17c], we constructed a free $\aleph_{\omega+1}$-Souslin tree in a model of Martin's Maximum (MM),[14] where the sealing of antichains was done at levels $\alpha < \aleph_{\omega+1}$ of countable cofinality, even though MM implies that every stationary subset of $E^{\aleph_{\omega+1}}_{\omega}$ reflects. Furthermore, in response to our question, Lambie-Hanson recently proved [LH17] the consistency (modulo large cardinals) of the conjunction of the following:

- GCH;
- every stationary subset of $\aleph_{\omega+1}$ reflects;
- $\mathrm{P}(\aleph_{\omega+1}, 2, \sqsubseteq, \aleph_{\omega+1}, \{S \subseteq \aleph_{\omega+1} \mid S \text{ is stationary}\})$.

This establishes that the construction of Proposition 2.3 is genuinely different from the classical one, and provides κ-Souslin trees in new scenarios that are not suited to Jensen's method.

By strengthening the parameter of the principle $\boxtimes^-(\cdot)$ from κ to $E^\kappa_{\geq\chi}$ for some cardinal $\chi < \kappa$ (or, equivalently, strengthening the fifth parameter of the proxy principle $P(\cdots)$ from $\{\kappa\}$ to $\{E^\kappa_{\geq\chi}\}$), we can ensure that the ordinal α on which the "hitting" action takes place in the course of proving that the tree is Souslin has large cofinality. Thus, the careful limitation determining which nodes of limit height α are placed into the tree (as given in the recursive construction in Proposition 2.3) needs to be observed only for ordinals $\alpha \in E^\kappa_{\geq\chi}$. This gives us the flexibility to add as many nodes as we like at any height α of small cofinality, subject only to the constraint that the tree remain a κ-tree, that is, $|T_\alpha| < \kappa$. In particular, if for every cardinal $\lambda < \kappa$ we have $\lambda^{\mathrm{cf}(\alpha)} < \kappa$, then we can add a limit of every branch at level α into T, and if we can do this for every limit ordinal $\alpha \in E^\kappa_{<\chi}$ then we can ensure that our tree is χ-complete. Of course we must forgo the slimness of the tree (and therefore also the regressiveness) obtained in Proposition 2.3, but this is obvious, as these are contradictory concepts.

The next proof demonstrates that there is a transparent way of transforming any proxy-based construction of a slim tree into a construction of a complete tree.

Proposition 2.4. *If κ is a regular uncountable cardinal and $\chi < \kappa$ is a cardinal satisfying $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$, then $\boxtimes^-(E^\kappa_{\geq\chi}) + \diamondsuit(\kappa)$ entails the existence of a normal, binary, splitting, χ-complete κ-Souslin tree.*[15]

Proof. Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a witness to $\boxtimes^-(E^\kappa_{\geq\chi})$. Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$. Let $\lhd$ be some well-ordering of H_κ. As before, we shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of levels whose union will ultimately be the desired tree T.

[14] Indeed, in the model of Corollary 1.20 above.

[15] Recall that the model of Example 1.30 demonstrates the consistency of $\boxtimes(E^\kappa_{\geq\chi}) + \diamondsuit(\kappa)$ together with $\neg\diamondsuit(E^\kappa_{\geq\chi})$.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let $T_{\alpha+1} := \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle \mid t \in T_\alpha\}$.

Next, suppose that $\alpha < \kappa$ is a nonzero limit ordinal, and that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been defined. For each $x \in T\restriction C_\alpha$, define the sequence $b^\alpha_x \in \prod_{\beta\in C_\alpha\setminus\mathrm{dom}(x)} T_\beta$ exactly as in the proof of Proposition 2.3, and denote its limit by $\mathbf{b}^\alpha_x$. Now, there are two cases to consider:

▶ If $\mathrm{cf}(\alpha) < \chi$, let T_α consist of the limits of all branches through $(T\restriction\alpha, \subset)$.

▶ If $\mathrm{cf}(\alpha) \geq \chi$, let $T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T\restriction C_\alpha\}$.

Notice that in both cases, $\{\mathbf{b}^\alpha_x \mid x \in T\restriction C_\alpha\} \subseteq T_\alpha$, and $|T_\alpha| < \kappa$.[16]

Having constructed all levels of the tree, we then let

$$T := \bigcup_{\alpha<\kappa} T_\alpha.$$

Claim 2.4.1. *If $A \subseteq T$ is a maximal antichain, then $|A| < \kappa$.*

Proof. Let $A \subseteq T$ be a maximal antichain. By the proof of Claim 2.3.2,

$$B := \{\beta \in R_0 \mid A \cap (T\restriction\beta) = \Omega_\beta \text{ is a maximal antichain in } T\restriction\beta\}$$

is a cofinal subset of κ. Thus, by the fact that $\langle C_\alpha \mid \alpha < \kappa\rangle$ witnesses $\boxtimes^-(E^\kappa_{\geq\chi})$, we obtain an ordinal $\alpha \in E^\kappa_{\geq\chi}$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

To see that $A \subseteq T\restriction\alpha$, consider any $z \in T\restriction(\kappa\setminus\alpha)$. Let $y := z\restriction\alpha \in T_\alpha$. By $\mathrm{cf}(\alpha) \geq \chi$, we have $y = \mathbf{b}^\alpha_x$ for some $x \in T\restriction C_\alpha$. Then the same analysis as in the proof of Claim 2.3.3 entails the existence of $s \in A$ such that $s \subset \mathbf{b}^\alpha_x = y \subseteq z$. In particular, $z \notin A$. □

So $(T, \subset)$ is a normal, binary, splitting, χ-complete κ-Souslin tree. □

Let us emphasize that the hypothesis of Proposition 2.4 involves the principle $\diamondsuit(\kappa)$ rather than $\diamondsuit(E^\kappa_{\geq\chi})$, and indeed, we did not consult Ω_β when constructing the level T_β. Rather, we consulted Ω_β at levels $\alpha > \beta$ for which $\beta \in \mathrm{nacc}(C_\alpha)$. Specifically, if $\beta \in \mathrm{nacc}(C_\alpha)$, Ω_β is a maximal antichain in $T\restriction\beta$, and $\mathrm{dom}(x) < \beta < \alpha$, then the node $\mathbf{b}^\alpha_x$ placed into T_α must be compatible with some element of Ω_β. The proxy principle $\boxtimes^-(E^\kappa_{\geq\chi})$ then ensures that, for any maximal antichain $A \subseteq T$, we can find some ordinal $\alpha \in E^\kappa_{\geq\chi}$ such that A was gradually sealed when building the level T_α, and therefore $A \subseteq T\restriction\alpha$.

A construction of a uniformly coherent κ-Souslin tree for a successor cardinal κ may be found in [DJ74],[Lar99],[Vel86]. Here, we give a construction that applies also for inaccessible κ.

Proposition 2.5. *If κ is a regular uncountable cardinal and $\boxtimes(\kappa) + \diamondsuit(\kappa)$ holds, then there exists a normal, slim, prolific, club-regressive, uniformly coherent κ-Souslin tree.*

[16]Indeed, the case $\mathrm{cf}(\alpha) < \chi$ is where the arithmetic hypothesis comes into play.

Proof. Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a witness to $\boxtimes(\kappa)$. Without loss of generality, we may assume that $0 \in C_\alpha$ for all $\alpha < \kappa$. Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$. Let $\pi : \kappa \to \kappa$ be such that $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \kappa$. By $\diamondsuit(\kappa)$, we have $|H_\kappa| = \kappa$, thus let $\lhd$ be some well-ordering of H_κ of order-type κ, and let $\phi : \kappa \leftrightarrow H_\kappa$ witness the isomorphism $(\kappa, \in) \cong (H_\kappa, \lhd)$. Put $\psi := \phi \circ \pi$.

For two elements of η, τ of H_κ, we define $\eta * \tau$ to be the emptyset, unless $\eta, \tau \in {}^{<\kappa}\kappa$ with $\mathrm{dom}(\eta) < \mathrm{dom}(\tau)$, in which case $\eta * \tau : \mathrm{dom}(\tau) \to \kappa$ is defined by stipulating:

$$(\eta * \tau)(\beta) := \begin{cases} \eta(\beta), & \text{if } \beta \in \mathrm{dom}(\eta); \\ \tau(\beta), & \text{otherwise.} \end{cases}$$

We shall now recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of levels whose union will ultimately be the desired tree T.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let $T_{\alpha+1} := \{t^\frown\langle i\rangle \mid t \in T_\alpha, i < \max\{\omega, \alpha\}\}$.

Next, suppose that $\alpha < \kappa$ is a nonzero limit ordinal, and that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been defined. Similar to the proof of Proposition 2.3, to each node $x \in T \restriction \alpha$ we shall associate some node $\mathbf{b}^\alpha_x : \alpha \to \kappa$ above x, and then let $T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction \alpha\}$.

Unlike the proof of Proposition 2.3, we first define $\mathbf{b}^\alpha_x$ for $x = \emptyset$.

Define $b^\alpha_\emptyset \in \prod_{\beta \in C_\alpha} T_\beta$ by recursion. Let $b^\alpha_\emptyset(0) := \emptyset$. Next, suppose $\beta^- < \beta$ are successive points of C_α, and $b^\alpha_\emptyset(\beta^-)$ has already been defined. In order to decide $b^\alpha_\emptyset(\beta)$, we consult the following set:

$$Q^{\alpha,\beta} := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (\psi(\beta) * b^\alpha_\emptyset(\beta^-))) \subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q^{\alpha,\beta} \neq \emptyset$, then let t denote its $\lhd$-least element, and put $b^\alpha_\emptyset(\beta) := b^\alpha_\emptyset(\beta^-) * t$;
- Otherwise, let $b^\alpha_\emptyset(\beta)$ be the $\lhd$-least element of T_β that extends $b^\alpha_\emptyset(\beta^-)$.

Note that $Q^{\alpha,\beta}$ depends only on $T_\beta, \Omega_\beta, \psi(\beta)$ and $b^\alpha_\emptyset(\beta^-)$, and hence for every ordinal $\gamma < \kappa$, if $C_\alpha \cap (\beta + 1) = C_\gamma \cap (\beta + 1)$, then $b^\alpha_\emptyset \restriction (\beta + 1) = b^\gamma_\emptyset \restriction (\beta + 1)$. It follows that for all $\beta \in \mathrm{acc}(C_\alpha)$ such that $b^\alpha_\emptyset \restriction \beta$ has already been defined, we may let $b^\alpha_\emptyset(\beta) := \bigcup \mathrm{Im}(b^\alpha_\emptyset \restriction \beta)$ and infer that $b^\alpha_\emptyset(\beta) = \mathbf{b}^\beta_\emptyset \in T_\beta$. This completes the definition of $b^\alpha_\emptyset$ and its limit $\mathbf{b}^\alpha_\emptyset = \bigcup \mathrm{Im}(b^\alpha_\emptyset)$.

Next, for each $x \in T \restriction \alpha$, let $\mathbf{b}^\alpha_x := x * \mathbf{b}^\alpha_\emptyset$. This completes the definition of the level T_α.

Having constructed all levels of the tree, we then let

$$T := \bigcup_{\alpha<\kappa} T_\alpha.$$

Claim 2.5.1. *For every $\alpha < \kappa$, every two nodes of T_α differ on a finite set.*

Proof. Suppose not, and let α be the least counterexample. Clearly, α must be a limit nonzero ordinal. Pick $x, y \in T \restriction \alpha$ such that $\mathbf{b}^\alpha_x$ differs from $\mathbf{b}^\alpha_y$ on an infinite set. As $\mathbf{b}^\alpha_x = x * \mathbf{b}^\alpha_\emptyset$ and $\mathbf{b}^\alpha_y = y * \mathbf{b}^\alpha_\emptyset$, it follows that x and y differ on an infinite set, contradicting the minimality of α. □

In particular, for every $\alpha < \kappa$, T_α is a subset of ${}^\alpha\kappa$ of size $\leq \max\{\aleph_0, |\alpha|\} < \kappa$.

Claim 2.5.2. *$(T, \subset)$ is club-regressive.*

Proof. Define a regressive map $\rho : T \to T$, as follows. Put $\rho(t) := \emptyset$ for all $t \in T \restriction \mathrm{nacc}(\kappa)$. Next, for all $\alpha \in \mathrm{acc}(\kappa)$ and $t \in T_\alpha$, let $\beta_t := \min\{\beta < \alpha \mid \{\varepsilon < \alpha \mid t(\varepsilon) \neq \mathbf{b}^\alpha_\emptyset(\varepsilon)\} \subseteq \beta\}$ and put $\rho(t) := t \restriction \beta_t$. It is easy to see that ρ and $\langle \mathrm{acc}(C_\alpha) \mid \alpha \in E^\kappa_{>\omega}\rangle$ witness together that $(T, \subset)$ is club-regressive. □

Finally, to establish that $(T, \subset)$ is κ-Souslin, we prove the following.

Claim 2.5.3. *If $A \subseteq T$ is a maximal antichain, then $|A| < \kappa$.*

Proof. Let $A \subseteq T$ be a maximal antichain. By the proof of Claim 2.3.2, for every $i < \kappa$, the set

$$B_i := \{\beta \in R_i \mid A \cap (T \restriction \beta) = \Omega_\beta \text{ is a maximal antichain in } T \restriction \beta\}$$

is stationary. Thus, by the fact that $\langle C_\alpha \mid \alpha < \kappa\rangle$ witnesses $\boxtimes(\kappa)$, we apply the hitting property of $\boxtimes(\kappa)$ to the sequence $\langle B_i \mid i < \kappa\rangle$ and the club $D := \{\alpha < \kappa \mid T \restriction \alpha \subseteq \phi[\alpha]\}$ to obtain an ordinal $\alpha \in D$ such that for all $i < \alpha$:

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha. \tag{1}$$

To see that $A \subseteq T \restriction \alpha$, consider any $z \in T \restriction (\kappa \setminus \alpha)$. Let $y := z \restriction \alpha \in T_\alpha$. By construction, $y = \mathbf{b}^\alpha_x = x * \mathbf{b}^\alpha_\emptyset$ for some $x \in T \restriction \alpha$. As $\alpha \in D$ and $x \in T \restriction \alpha$, we can fix $i < \alpha$ such that $\phi(i) = x$.

Fix $\beta \in \mathrm{nacc}(C_\alpha) \cap B_i$ with $\mathrm{dom}(x) < \beta < \alpha$. Clearly, $\psi(\beta) = \phi(\pi(\beta)) = \phi(i) = x$. Since $\beta \in B_i$, we know that $\Omega_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$, and hence $Q^{\alpha,\beta} \neq \emptyset$. Let $t := \min(Q^{\alpha,\beta}, \lhd)$ and $\beta^- := \sup(C_\alpha \cap \beta)$. Then $b^\alpha_\emptyset(\beta) = b^\alpha_\emptyset(\beta^-) * t$, and there exists some $s \in \Omega_\beta$ such that $(s \cup (x * b^\alpha_\emptyset(\beta^-))) \subseteq t$. In particular, $x * b^\alpha_\emptyset(\beta)$ extends an element of Ω_β. Altogether, there exists some $s \in A \cap (T \restriction \beta)$ such that $s \subseteq x * b^\alpha_\emptyset(\beta) \subseteq x * \mathbf{b}^\alpha_\emptyset = \mathbf{b}^\alpha_x = y \subseteq z$, and hence $z \notin A$. □

This completes the proof. □

Remark 2.6. As Equation (1) above makes explicit, the preceding proof did not utilize the full force of the axiom $\boxtimes(\kappa)$, and in fact $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa)$ suffices. For an application of the full force of $\boxtimes(\kappa)$, we refer the reader to [BR17c, §6].

Proposition 2.5 partially demonstrates the *microscopic approach*: there is a stationary set $S \subseteq \kappa$, such that for every $\alpha \in S$, every node of T_α is determined by an element $x \in T \restriction \alpha$, a club $C \in \mathcal{C}_\alpha$ and some increasing and continuous sequence $b^C_y \in \prod_{\beta \in C \setminus \mathrm{dom}(y)} T_\beta$ for some $y \in T \restriction C$. The move from $b^C_y(\beta^-)$ to its successor $b^C_y(\beta)$ depends only on $b^C_y(\beta^-)$, and the restriction of some structure

$$\mathcal{M} = (H(\kappa), \in, \lhd, \psi, T, \langle R_i \mid i < \kappa\rangle, \langle \Omega_\eta \mid \eta < \kappa\rangle, \ldots)$$

to level $\beta+1$. As we are only allowed to "look down", the coherence (à la Claim 2.3.1) is guaranteed.

Underlying that, we have a predefined library of *actions*, each labeled by a member of H_κ, and each, given a restricted structure $\mathcal{M}\restriction(\gamma+1)$ and an element $z\in T\restriction\gamma$, will determine an extension of z that belongs to the top level of the normal tree $T\restriction(\gamma+1)$. Fix at the outset a subset $\mathtt{h}$ of H_κ. Then, the microscopic steps from $b^C_y(\beta^-)$ to $b^C_y(\beta)$ are the outcome of feeding $\mathcal{M}\restriction(\beta+1)$ and $b^C_y(\beta^-)$ to the action $\psi(\beta)$ provided that $\psi(\beta)\in\mathtt{h}$, or feeding them to the default action (labeled by $\emptyset\in H_\kappa$) otherwise. Needless to say that these microscopic steps do not know where they are heading, and are certainly not aware of α.

Nota bene that for different choices of $\mathtt{h}$ and $\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$, the above machine will produce different κ-trees. Of interest is the analysis of $\mathtt{h}$ files that include two actions of contradictory purpose (e.g., one for making the tree rigid, and the other for making the tree homogeneous).

3. The coherence relation $\sqsubseteq$

The relation $\sqsubseteq$ is the strongest coherence relation one can expect in this context. Note that in Section 2, a uniformly coherent κ-Souslin tree was derived from $\mathrm{P}(\kappa,2,\sqsubseteq,\kappa)$.

Lemma 3.1. *For any infinite cardinal λ, $\mathrm{P}^-(\lambda^+,2,\sqsubseteq,1,\{\lambda^+\},2,0,\mathcal{E}_\lambda)$ is equivalent to $\square_\lambda$.*

Proof. Straight-forward, but see also Lemma 4.1 below. □

Lemma 3.2. $\mathrm{P}^-(\kappa,2,\sqsubseteq)$ *entails $\square(\kappa)$ for every regular uncountable cardinal κ.*

Proof. Let $\langle C_\alpha\mid\alpha<\kappa\rangle$ witness $\mathrm{P}^-(\kappa,2,\sqsubseteq,1,\{\kappa\},2,1)$. Towards a contradiction suppose that $\square(\kappa)$ fails. Then, there exists a club $C\subseteq\kappa$ such that $C\cap\alpha=C_\alpha$ for all $\alpha\in\operatorname{acc}(C)$. Since κ is regular uncountable and C is club in κ, it follows that also $\operatorname{acc}(C)$ is club. Let $A_0:=\operatorname{acc}(C)$. Pick $\alpha\in\operatorname{acc}(C)$ such that $\sup(\operatorname{nacc}(C_\alpha)\cap A_0)=\alpha$. This is a contradiction to the fact that $\operatorname{nacc}(C_\alpha)\subseteq\operatorname{nacc}(C)$, $A_0=\operatorname{acc}(C)$, and $\operatorname{nacc}(C)\cap\operatorname{acc}(C)=\emptyset$. □

In the inverse direction, the second author recently proved [Rin17a] that $\mathsf{GCH}+\square(\lambda^+)$ entails $\mathrm{P}(\lambda^+,2,\sqsubseteq)$ for every uncountable cardinal λ. This was further improved to show that $\mathsf{GCH}+\square(\lambda^+)$ entails $\mathrm{P}(\lambda^+,2,\sqsubseteq,\lambda^+)$ assuming either that λ is singular [BR17a] or that there exists a stationary $S\subseteq\lambda^+$ such that every stationary subset of S reflects [LHR16].

Lemma 3.3. *Let $S\subseteq\omega_1$ be stationary. Then:*

1. *The following are equivalent:*
   (a) $\clubsuit_w(S)$*;*
   (b) $\mathrm{P}^-(\aleph_1,2,\sqsubseteq,1,\{S\},2,\omega_1,\mathcal{E}_\omega)$*;*
   (c) $\mathrm{P}^-(\aleph_1,2,\mathcal{R},1,\{S\},2,\omega_1)$ *for some $\mathcal{R}$.*
2. *The following are equivalent:*
   (a) $\diamondsuit(S)$*;*
   (b) $\mathrm{P}(\aleph_1,2,\sqsubseteq,1,\{S\},2,\omega_1,\mathcal{E}_\omega)$*;*
   (c) $\mathrm{P}(\aleph_1,2,\mathcal{R},1,\{S\},2,\omega_1)$ *for some $\mathcal{R}$.*

Proof. Recalling that ${}_{\aleph_0}\sqsubseteq$ is the same as $\sqsubseteq$, this is the case $\lambda = \aleph_0$ of Theorem 5.1 below. □

Lemma 3.4. *If λ is an uncountable cardinal and $\boxdiamond_\lambda$ holds, then there exists a sequence $\langle (C_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ such that:*

1. *for every limit $\alpha < \lambda^+$, C_α is a club in α of order-type $\le \lambda$, and $X_\alpha \subseteq \alpha$;*
2. *if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then $C_\alpha \cap \bar\alpha = C_{\bar\alpha}$;*
3. *for every subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists a limit $\alpha < \lambda^+$ with $\operatorname{otp}(C_\alpha) = \lambda$ such that $C_\alpha \subseteq \{\gamma \in E \mid X \cap \gamma = X_\gamma\}$.*

Proof. Pick a sequence $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ as in Definition 8.17. For every $\alpha < \lambda^+$ such that $\sup(\operatorname{acc}(D_\alpha)) < \alpha$, we have $\operatorname{cf}(\alpha) = \omega$, so let us fix some cofinal subset c_α of α of order-type ω. Then, for every limit $\alpha < \lambda^+$, let:

$$C_\alpha := \begin{cases} \operatorname{acc}(D_\alpha), & \text{if } \sup(\operatorname{acc}(D_\alpha)) = \alpha; \\ c_\alpha, & \text{otherwise.} \end{cases}$$

Let $C_{\alpha+1} := \emptyset$ for all $\alpha < \omega_1$.

To see that $\langle (C_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ is as sought:

1. Immediate.
2. If $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then in particular $\operatorname{acc}(C_\alpha) \neq \emptyset$, so that $\operatorname{otp}(C_\alpha) > \omega$, meaning that $C_\alpha = \operatorname{acc}(D_\alpha)$, and $\bar\alpha \in \operatorname{acc}(\operatorname{acc}(D_\alpha)) \subseteq \operatorname{acc}(D_\alpha)$, so that $D_\alpha \cap \bar\alpha = D_{\bar\alpha}$, thus $\operatorname{acc}(D_\alpha) \cap \bar\alpha = \operatorname{acc}(D_{\bar\alpha})$. Then $\bar\alpha = \sup(\operatorname{acc}(D_\alpha) \cap \bar\alpha) = \sup(\operatorname{acc}(D_{\bar\alpha}))$, so that $C_{\bar\alpha} = \operatorname{acc}(D_{\bar\alpha}) = \operatorname{acc}(D_\alpha) \cap \bar\alpha = C_\alpha \cap \bar\alpha$.
3. Given a subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, take $\alpha < \lambda^+$ as in Clause (3) of Definition 8.17. We will show that this α is as required: Since $\operatorname{otp}(\operatorname{acc}(D_\alpha)) = \lambda$ while (by Clause (1)) $\operatorname{otp}(D_\alpha) = \lambda = \omega \cdot \lambda$, it follows that $\sup(\operatorname{acc}(D_\alpha)) = \alpha$, so that $C_\alpha = \operatorname{acc}(D_\alpha)$, and $\operatorname{otp}(C_\alpha) = \operatorname{otp}(\operatorname{acc}(D_\alpha)) = \lambda$. Furthermore, $C_\alpha = \operatorname{acc}(D_\alpha) \subseteq E$. Finally, for any $\gamma \in C_\alpha = \operatorname{acc}(D_\alpha)$, the combination of Clauses (2) and (3) of Definition 8.17 gives $X_\gamma = X_\alpha \cap \gamma = (X \cap \alpha) \cap \gamma = X \cap \gamma$, as required. □

Lemma 3.5. *Suppose that κ is a regular uncountable cardinal, $S \subseteq E^\kappa_\omega$ is stationary, and $\diamondsuit(S)$ holds.*

Then there exists a sequence $\langle (C_\alpha, X_\alpha) \mid \alpha \in E^\kappa_\omega\rangle$ such that:

- *for every $\alpha \in E^\kappa_\omega$, C_α is a countable club in α, and $X_\alpha \subseteq \alpha$;*
- *if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then $\bar\alpha \in S$ and $C_\alpha \cap \bar\alpha = C_{\bar\alpha}$;*
- *for every subset $X \subseteq \kappa$, club $E \subseteq \kappa$ and nonzero $\varepsilon < \omega_1$, there exists $\alpha \in S$ with $\operatorname{otp}(C_\alpha) = \omega \cdot \varepsilon$ such that $C_\alpha \subseteq \{\gamma \in E \cap S \mid X \cap \gamma = X_\gamma\}$.*

Proof. By $\diamondsuit(S)$ and the implication (1) $\Rightarrow$ (3) of Fact 8.5, let us fix a sequence $\langle (X_\delta, Y_\delta) \mid \delta < \kappa\rangle$, and functions $\phi_0 : \kappa \to \omega_1$, $\phi_1 : \kappa \to \kappa$ such that for every $X, Y \in \mathcal{P}(\kappa)$, $\varepsilon < \omega_1$ and $\alpha < \kappa$, the following set is stationary:

$$\{\delta \in S \mid X_\delta = X \cap \delta, Y_\delta = Y \cap \delta, \phi_0(\delta) = \varepsilon, \phi_1(\delta) = \alpha\}.$$

In particular, we may assume that X_δ and Y_δ are subsets of δ for all $\delta < \kappa$.

We now tailor the arguments of [Rin15b]. For every $\delta \in E^\kappa_\omega$, let D_δ be some cofinal subset of δ of order-type ω. If $\phi_1(\delta) < \delta$, then we require in addition that $\min(D_\delta) > \phi_1(\delta)$. If $\{\gamma \in Y_\delta \mid X_\gamma = X_\delta \cap \gamma \ \&\ Y_\gamma = Y_\delta \cap \gamma\}$ is cofinal in δ, then we require in addition that D_δ be a subset of the latter.

We shall define a sequence $\langle C_\delta \mid \delta \in E^\kappa_\omega \cup \{0\}\rangle$ by recursion over δ, as follows.

Let $C_0 := \emptyset$. Suppose that $\delta \in E^\kappa_\omega$, and $\langle C_\alpha \mid \alpha \in E^\delta_\omega \cup \{0\}\rangle$ has already been defined. The definition of C_δ now splits into three cases:

- ▶ If $\phi_0(\delta)$ is a successor ordinal > 1, $\phi_1(\delta) \in S \cap \delta$ and $\mathrm{otp}(C_{\phi_1(\delta)}) = \omega \cdot (\phi_0(\delta) - 1)$, then let
$$C_\delta := C_{\phi_1(\delta)} \cup \{\phi_1(\delta)\} \cup D_\delta.$$
- ▶ If $\phi_0(\delta)$ is a limit ordinal > 0, and there exists an increasing sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ that converges to δ, such that $\alpha_0 = \phi_1(\delta)$, $\{\alpha_n \mid 0 < n < \omega\} \subseteq S \cap \delta$, $\langle C_{\alpha_n} \mid n < \omega\rangle$ is $\sqsubseteq$-increasing, $X_\beta = X_\delta \cap \beta$ and $Y_\beta = Y_\delta \cap \beta$ for all $\beta \in (\bigcup_{n<\omega} C_{\alpha_n})$, and $\mathrm{otp}(\bigcup_{n<\omega} C_{\alpha_n}) = \omega \cdot \phi_0(\delta)$, then fix such a sequence, and let $C_\delta = \bigcup_{n<\omega} C_{\alpha_n}$.
- ▶ Otherwise, let $C_\delta := D_\delta$.

Clearly, for any $\delta \in E^\kappa_\omega$, C_δ is a club subset of δ, and either $\mathrm{otp}(C_\delta) = \omega \cdot \phi_0(\delta)$ or $\mathrm{otp}(C_\delta) = \omega$. In particular, for every $\delta \in E^\kappa_\omega \cup \{0\}$, there exists some $\varepsilon < \omega_1$ for which $\mathrm{otp}(C_\delta) = \omega \cdot \varepsilon$.

Claim 3.5.1. *If $\delta \in E^\kappa_\omega$ and $\bar\delta \in \mathrm{acc}(C_\delta)$, then $\bar\delta \in S$ and $C_\delta \cap \bar\delta = C_{\bar\delta}$.*

Proof. Suppose not, and let δ be the least counterexample. Clearly, C_δ was defined according to the first case. That is, $C_\delta = C_{\phi_1(\delta)} \cup \{\phi_1(\delta)\} \cup D_\delta$. In particular, $\phi_1(\delta) \in S \cap \delta$ and $C_\delta \cap \phi_1(\delta) = C_{\phi_1(\delta)}$.

Let $\bar\delta \in \mathrm{acc}(C_\delta)$ be such that $C_\delta \cap \bar\delta \neq C_{\bar\delta}$.

▶ If $\bar\delta = \phi_1(\delta)$, then we get a contradiction to the fact that $C_\delta \cap \phi_1(\delta) = C_{\phi_1(\delta)}$.

▶ If $\bar\delta < \phi_1(\delta)$, then already $\bar\delta \in \mathrm{acc}(C_{\phi_1(\delta)})$ with $C_{\phi_1(\delta)} \cap \bar\delta \neq C_{\bar\delta}$, contradicting the minimality of δ. ☐

For subsets E, X of κ, denote:

$$G(E,X) := \{\gamma \in E \cap S \mid X \cap \gamma = X_\gamma \ \&\ E \cap S \cap \gamma = Y_\gamma\};$$
$$F(E,X) := \{\alpha \in G(E,X) \cup \{0\} \mid C_\alpha \subseteq G(E,X)\}.$$

Claim 3.5.2. *For all subsets $E, X \subseteq \kappa$, if $\delta \in \mathrm{acc}(G(E,X))$ and $\phi_1(\delta) < \delta$, then $D_\delta \subseteq G(E,X)$.*

Proof. Fix $E, X \subseteq \kappa$ and $\delta \in \mathrm{acc}(G(E,X))$. By $\delta \in G(E,X)$, we have $E \cap S \cap \delta = Y_\delta$ and $X \cap \delta = X_\delta$. Then,

$$\begin{aligned}\{\gamma \in Y_\delta \mid X_\gamma = X_\delta \cap \gamma \ \&\ Y_\gamma = Y_\delta \cap \gamma\} &= \\ = \{\gamma \in E \cap S \cap \delta \mid X_\gamma = (X \cap \delta) \cap \gamma \ \&\ Y_\gamma = (E \cap S \cap \delta) \cap \gamma\}& \\ = \{\gamma \in E \cap S \cap \delta \mid X_\gamma = X \cap \gamma \ \&\ Y_\gamma = E \cap S \cap \gamma\} = G(E,X) \cap \delta,&\end{aligned}$$

which is cofinal in δ, so that D_δ was chosen to be a subset of it. $\square$

Claim 3.5.3. *Suppose that $X \subseteq \kappa$ is some set and $E \subseteq \kappa$ is some club.*

For every $\varepsilon < \omega_1$ and every $\alpha \in F(E,X)$ satisfying $\mathrm{otp}(C_\alpha) < \omega \cdot \varepsilon$, the set

$$S^{E,X}_{\alpha,\varepsilon} := \{\delta \in F(E,X) \mid C_\alpha \sqsubseteq C_\delta \text{ and } \mathrm{otp}(C_\delta) = \omega \cdot \varepsilon\}$$

is stationary.

Proof. Note that by our choice of the diamond sequence, the set $G(E,X)$ is a stationary subset of ω_1, being the intersection of the club set E with a stationary set. Thus, in particular, $\mathrm{acc}^+(G(E,X))$ is club in κ.

We now prove the claim by induction over ε. First, notice that when $\varepsilon = 0$ there is nothing to show, as there is no α satisfying $\mathrm{otp}(C_\alpha) < 0$. Thus the induction begins with $\varepsilon = 1$.

Base case, $\varepsilon = 1$ We consider only α satisfying $\mathrm{otp}(C_\alpha) = 0$.

By our choice of the diamond sequence, the following set is stationary:

$$Z := \{\delta \in G(E,X) \mid \phi_0(\delta) = 1, \phi_1(\delta) = \alpha\} \setminus (\alpha + 1).$$

Since $\mathrm{acc}^+(G(E,X))$ is a club subset of κ, it follows that $Z \cap \mathrm{acc}^+(G(E,X))$ is stationary. We shall show that $Z \cap \mathrm{acc}^+(G(E,X)) \subseteq S^{E,X}_{\alpha,1}$.

Let $\delta \in Z \cap \mathrm{acc}^+(G(E,X))$ be arbitrary. We have $\delta \in Z \subseteq G(E,X)$ and $\delta \in \mathrm{acc}^+(G(E,X))$, so that $\delta \in \mathrm{acc}(G(E,X))$. Thus Claim 3.5.2 gives $D_\delta \subseteq G(E,X)$. By $\delta \in Z$, we have $\phi_0(\delta) = 1$, and so by definition of C_δ in this case it follows that $C_\delta = D_\delta \subseteq G(E,X)$, so that $\delta \in F(E,X)$. Clearly $C_\alpha = \emptyset \sqsubseteq C_\delta$ and $\mathrm{otp}(C_\delta) = \omega$, so that $\delta \in S^{E,X}_{\alpha,1}$.

Successor case, $\varepsilon = \epsilon' + 1$ for some nonzero $\epsilon' < \omega_1$ We assume the claim holds for ϵ'. Fix $\alpha \in F(E,X)$ satisfying $\mathrm{otp}(C_\alpha) < \omega \cdot \varepsilon$, and we must show that the set $S^{E,X}_{\alpha,\varepsilon}$ is stationary. We find $\alpha' \in F(E,X)$ satisfying $C_\alpha \sqsubseteq C_{\alpha'}$ and $\mathrm{otp}(C_{\alpha'}) = \omega \cdot \epsilon'$ by considering two cases:

- If $\mathrm{otp}(C_\alpha) = \omega \cdot \epsilon'$, then let $\alpha' := \alpha$.
- If $\mathrm{otp}(C_\alpha) < \omega \cdot \epsilon'$, then apply the induction hypothesis to obtain $\alpha' \in S^{E,X}_{\alpha,\epsilon'}$.

By our choice of the diamond sequence, the following set is stationary:

$$Z := \{\delta \in G(E,X) \mid \phi_0(\delta) = \varepsilon, \phi_1(\delta) = \alpha'\} \setminus (\alpha' + 1).$$

Thus, it suffices to prove that $S^{E,X}_{\alpha,\varepsilon}$ covers the stationary set $Z \cap \mathrm{acc}^+(G(E,X))$.

Let $\delta \in Z \cap \mathrm{acc}^+(G(E,X))$ be arbitrary. Again, Claim 3.5.2 gives $D_\delta \subseteq G(E,X)$. Since $\delta \in Z$, we have $\phi_0(\delta) = \varepsilon$ is a successor ordinal > 1, $\phi_1(\delta) < \delta$, and $\mathrm{otp}(C_{\phi_1(\delta)}) = \mathrm{otp}(C_{\alpha'}) = \omega \cdot \epsilon' = \omega \cdot (\varepsilon - 1) = \omega \cdot (\phi_0(\delta) - 1)$, so that by definition of C_δ in this case, we have $C_\delta = C_{\alpha'} \cup \{\alpha'\} \cup D_\delta$, where $D_\delta \subseteq \delta \setminus (\alpha'+1)$. In particular, $C_\alpha \sqsubseteq C_{\alpha'} \sqsubseteq C_\delta$ and $\mathrm{otp}(C_\delta) = \mathrm{otp}(C_{\alpha'}) + \omega = \omega \cdot \epsilon' + \omega = \omega \cdot (\epsilon'+1) = \omega \cdot \varepsilon$. Since $\alpha' \in F(E,X)$, we have $C_{\alpha'} \subseteq G(E,X)$ and $\alpha' \in G(E,X)$.[17] Thus, altogether, $C_\delta = C_{\alpha'} \cup \{\alpha'\} \cup D_\delta \subseteq G(E,X)$, so that $\delta \in F(E,X)$, and hence $\delta \in S^{E,X}_{\alpha,\varepsilon}$.

Limit case Suppose that $\varepsilon < \omega_1$ is a nonzero limit ordinal, and for every $\epsilon' < \varepsilon$ and $\alpha \in F(E,X)$ satisfying $\mathrm{otp}(C_\alpha) < \omega \cdot \epsilon'$, the set $S^{E,X}_{\alpha,\epsilon'}$ is stationary. Now fix $\alpha \in F(E,X)$ satisfying $\mathrm{otp}(C_\alpha) < \omega \cdot \varepsilon$, and we must show that the set $S^{E,X}_{\alpha,\varepsilon}$ is stationary. Let $D \subseteq \omega_1$ be an arbitrary club. Since the set

$$Z := \{\delta \in G(E,X) \mid \phi_0(\delta) = \varepsilon, \phi_1(\delta) = \alpha\}$$

is stationary in ω_1, pick an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$, with $\mathcal{M} \cap \kappa \in Z \cap D$, such that $\alpha, \varepsilon \in \mathcal{M}$ and $\langle S^{E,X}_{\alpha',\epsilon'} \mid \epsilon' < \omega_1, \alpha' < \kappa\rangle \in \mathcal{M}$. Denote $\delta := \mathcal{M} \cap \kappa$. We shall show that $\delta \in S^{E,X}_{\alpha,\varepsilon}$.

Let $\langle \delta_n \mid n < \omega\rangle$ be a strictly increasing sequence of ordinals converging to δ. Let $\langle \varepsilon_n \mid n < \omega\rangle$ be a strictly increasing sequence of ordinals converging to ε, with $\mathrm{otp}(C_\alpha) = \omega \cdot \varepsilon_0$. Now, define a sequence $\langle \alpha_n \mid n < \omega\rangle$ by recursion as follows, where we will ensure, for each $n < \omega$, that $\alpha_n \in F(E,X) \cap \delta$ and $\mathrm{otp}(C_{\alpha_n}) = \omega \cdot \varepsilon_n$.

Let $\alpha_0 := \alpha$. Next, fix $n < \omega$, and suppose α_n has already been defined. Since $\varepsilon_n < \varepsilon_{n+1} < \varepsilon$, the induction hypothesis guarantees that $S^{E,X}_{\alpha_n,\varepsilon_{n+1}}$ is stationary, and since $\alpha_n, \varepsilon_{n+1} \in \mathcal{M}$, it follows that $S^{E,X}_{\alpha_n,\varepsilon_{n+1}} \in \mathcal{M}$. Since also $\delta_n < \delta$, it follows by elementarity of $\mathcal{M}$ that we can pick $\alpha_{n+1} \in S^{E,X}_{\alpha_n,\varepsilon_{n+1}} \cap \mathcal{M}$ with $\alpha_{n+1} > \delta_n$.

Since $\alpha_{n+1} \in S^{E,X}_{\alpha_n,\varepsilon_{n+1}}$ for all $n < \omega$, it follows that $\langle C_{\alpha_n} \mid n < \omega\rangle$ is $\sqsubseteq$-increasing, and $\mathrm{otp}(C_{\alpha_n}) = \omega \cdot \varepsilon_n$ for all $n < \omega$. Consequently, $\langle \alpha_n \mid n < \omega\rangle$ is increasing and converging to δ, $\{\alpha_n \mid 0 < n < \omega\} \subseteq S \cap \delta$, and

$$\mathrm{otp}(\bigcup_{n<\omega} C_{\alpha_n}) = \sup_{n<\omega}(\mathrm{otp}(C_{\alpha_n})) = \sup_{n<\omega}(\omega \cdot \varepsilon_n) = \omega \cdot \sup_{n<\omega} \varepsilon_n = \omega \cdot \varepsilon = \omega \cdot \phi_0(\delta)$$

Furthermore, since $\alpha_n \in F(E,X)$ for all $n < \omega$, it follows that $\bigcup_{n<\omega} C_{\alpha_n} \subseteq G(E,X)$. Together with $\delta \in Z \subseteq G(E,X)$, this implies that for every $\beta \in (\bigcup_{n<\omega} C_{\alpha_n})$ we have $X_\beta = X \cap \beta = (X \cap \delta) \cap \beta = X_\delta \cap \beta$ and $Y_\beta = E \cap S \cap \beta = (E \cap S \cap \delta) \cap \beta = Y_\delta \cap \beta$. Finally, $\delta \in Z$ gives $\alpha_0 = \alpha = \phi_1(\delta)$. Altogether, the conditions are satisfied for C_δ to be chosen according to second case of the definition, so that $\mathrm{otp}(C_\delta) = \omega \cdot \varepsilon$, $C_\delta \subseteq G(E,X)$, and $C_\alpha = C_{\phi_1(\delta)} \sqsubseteq C_\delta$. It follows that $\delta \in S^{E,X}_{\alpha,\varepsilon}$. But δ is an element of the arbitrary club set D. Thus $S^{E,X}_{\alpha,\varepsilon}$ is stationary, as required. □

Given a subset $X \subseteq \kappa$, club $E \subseteq \kappa$, and nonzero $\varepsilon < \omega_1$, apply the last claim with $\alpha = 0$ to obtain $\delta \in S^{E,X}_{0,\varepsilon} \subseteq F(E,X) \setminus \{0\} \subseteq G(E,X) \subseteq S$, so that $\mathrm{otp}(C_\delta) = \omega \cdot \varepsilon$, and $C_\delta \subseteq G(E,X) \subseteq \{\gamma \in E \cap S \mid X \cap \gamma = X_\gamma\}$, completing the proof of the lemma. □

[17]The case $\alpha' = 0$ is ruled out by the fact that $\mathrm{otp}(C_{\alpha'}) > 0$.

In addition to its importance in the present context, the next theorem also has applications to infinite graph theory [Rin17b].

Theorem 3.6. *Assume any of the following:*

- $\lambda = \aleph_0$, $S \subseteq \omega_1$ *and* $\diamondsuit(S)$ *holds; or*
- λ *is an uncountable cardinal,* $S = E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *and* $\boxtimes_\lambda$ *holds.*

Denote $\chi := \omega \cdot \lambda$ *(ordinal multiplication), and let* $\sigma < \chi$ *be any ordinal. Then* $\mathrm{P}(\lambda^+, 2, \sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\chi)$ *holds. Moreover, there exist a sequence* $\langle C_\alpha \mid \alpha < \lambda^+\rangle$*, and a function* $h : \lambda^+ \to \lambda^+$ *satisfying the following:*

- *if* $\alpha < \lambda^+$ *is a limit, then* C_α *is a club subset of* $\alpha \setminus \{0\}$ *of order-type* $\leq \chi$*;*
- *if* $\bar\alpha \in \mathrm{acc}(C_\alpha)$*, then* $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ *and* $h(\bar\alpha) = h(\alpha)$*;*
- *for every sequence* $\langle A_\delta \mid \delta < \lambda^+\rangle$ *of cofinal subsets of* λ^+*, every club* $D \subseteq \lambda^+$*, and every* $\varsigma < \lambda^+$*, there exists* $\alpha \in S$ *for which all of the following hold:*
  1. $h(\alpha) = \varsigma$*;*
  2. $\mathrm{nacc}(C_\alpha) \subseteq \bigcup_{\delta<\alpha} A_\delta$*;*
  3. $\mathrm{otp}(\{\beta \in \mathrm{acc}(C_\alpha) \mid \mathrm{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_\delta\}) = \lambda$ *for every* $\delta < \alpha$*;*
  4. *for every* $\beta < \gamma$ *in* C_α*, there exists* $\eta \in D$*, with* $\beta < \eta < \gamma$*.*

Proof. First, notice that the given hypotheses (either $\diamondsuit(S)$ in case $\lambda = \aleph_0$ or $\boxtimes_\lambda$ in case λ is uncountable) imply $\diamondsuit(\lambda^+)$, which implies CH_λ. Thus, fix a function $\pi : \lambda^+ \to {}^{\lambda^+}\lambda^+$ such that $\{\alpha < \lambda^+ \mid f \subseteq \pi(\alpha)\}$ is cofinal in λ^+ for all $f \in {}^{<\lambda^+}\lambda^+$.

Using Lemma 3.5 (in case $\lambda = \aleph_0$) or Lemma 3.4 (in case $\lambda > \aleph_0$), pick a sequence $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

- for every limit $\alpha < \lambda^+$, D_α is a club in α of order-type $\leq \chi$, and $X_\alpha \subseteq \alpha$;
- if $\bar\alpha \in \mathrm{acc}(D_\alpha)$, then $D_\alpha \cap \bar\alpha = D_{\bar\alpha}$;
- for every subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists a limit $\alpha \in S$ with $\mathrm{otp}(D_\alpha) = \chi$ such that $\mathrm{nacc}(D_\alpha) \subseteq \{\gamma \in E \mid X \cap \gamma = X_\gamma\}$.[18]

Of course, we may also assume that $0 \notin D_\alpha$ for all $\alpha \in \mathrm{acc}(\lambda^+)$.

Define $h : \lambda^+ \to \lambda^+$ by letting, for all $\alpha < \lambda^+$:

$$h(\alpha) := \begin{cases} \min(X_{\min(D_\alpha)}), & \text{if } \alpha \in \mathrm{acc}(\lambda^+) \text{ and } X_{\min(D_\alpha)} \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

[18] Lemmas 3.4 and 3.5 give $D_\alpha \subseteq \{\gamma \in E \mid X \cap \gamma = X_\gamma\}$, but $\mathrm{nacc}(D_\alpha) \subseteq \{\gamma \in E \mid \dots\}$ is all we need here. Also, Lemma 3.5 gives us a sequence of local clubs of arbitrarily large order-type, and we can simply replace any club of order-type $> \omega^2$ with a club of order-type ω (of the same sup).

Notice that if $\bar\alpha \in \mathrm{acc}(D_\alpha)$, then $\min(D_{\bar\alpha}) = \min(D_\alpha)$, and hence $h(\bar\alpha) = h(\alpha)$.

For every $\gamma < \lambda^+$, fix an injection $\psi_\gamma : \gamma + 1 \to \lambda$. Using $|\chi| = \lambda$, fix a function $\psi : \lambda \to \chi \times \lambda$ such that $\{k < \lambda \mid (i,j) = \psi(k)\}$ has order-type λ for all $(i,j) \in \chi \times \lambda$.

Set $C_0 := \emptyset$, and for every $\alpha < \lambda^+$, set $C_{\alpha+1} := \emptyset$. Now, fix any nonzero limit $\alpha < \lambda^+$, and we will show how to construct C_α.

Without loss of generality, assume σ is an infinite limit ordinal. By $\mathrm{otp}(D_\alpha) \le \chi = \omega \cdot \lambda = \sigma \cdot \lambda$, we can let $o_\alpha : D_\alpha \to \lambda$ be the unique function satisfying $\mathrm{otp}(D_\alpha \cap \beta) \in [\sigma \cdot o_\alpha(\beta), \sigma \cdot o_\alpha(\beta) + \sigma)$ for each $\beta \in D_\alpha$. Then, define $\varphi_\alpha : D_\alpha \to \alpha$ by letting for all $\beta \in D_\alpha$:

$$\varphi_\alpha(\beta) := \begin{cases} \delta, & \text{if } \delta < \beta \ \&\ \psi(o_\alpha(\beta)) = (\mathrm{otp}(D_\alpha \cap \delta), \psi_{\min(D_\alpha \setminus \delta)}(\delta)); \\ 0, & \text{otherwise.} \end{cases}$$

Note that φ_α is well-defined, since for every $(i,j) \in \chi \times \lambda$, the set $\{\delta < \alpha \mid \mathrm{otp}(D_\alpha \cap \delta) = i \ \&\ \psi_{\min(D_\alpha \setminus \delta)}(\delta) = j\}$ contains at most a single element.

Next, define $d_\alpha : D_\alpha \to \lambda^+$ by letting for all $\beta \in D_\alpha$:

$$d_\alpha(\beta) := \begin{cases} \pi(\min(X_{\min(D_\alpha \setminus (\beta+1))} \setminus (\beta+1)))(\varphi_\alpha(\beta)), & \text{if } X_{\min(D_\alpha \setminus (\beta+1))} \nsubseteq \beta + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then, define $c_\alpha : D_\alpha \to \lambda^+$ by letting for all $\beta \in D_\alpha$:

$$c_\alpha(\beta) := \begin{cases} d_\alpha(\beta), & \text{if } \beta < d_\alpha(\beta) < \min(D_\alpha \setminus (\beta+1)); \\ \min(D_\alpha \setminus (\beta+1)), & \text{otherwise.} \end{cases}$$

Finally, let

$$C_\alpha := \mathrm{acc}(D_\alpha) \cup \{c_\alpha(\beta) \mid \beta \in D_\alpha\}.$$

The very definition of c_α (regardless of the fact that it relies on d_α) makes it clear that:

Claim 3.6.1. *For all $\beta \in D_\alpha$, $\beta < c_\alpha(\beta) \le \min(D_\alpha \setminus (\beta+1))$, so that $\mathrm{otp}(C_\alpha) = \mathrm{otp}(D_\alpha) \le \chi$, $\mathrm{acc}^+(C_\alpha) = \mathrm{acc}(D_\alpha)$, $\mathrm{nacc}(C_\alpha) = \{c_\alpha(\beta) \mid \beta \in D_\alpha\}$, and C_α is a club in $\alpha \setminus \{0\}$.* □

Having constructed C_α for all $\alpha < \lambda^+$, we will show that the sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ and the function h satisfy the requirements of the theorem.

Fix $\bar\alpha \in \mathrm{acc}(C_\alpha)$. Then $\bar\alpha \in \mathrm{acc}(D_\alpha)$, and hence $D_{\bar\alpha} = D_\alpha \cap \bar\alpha$, $o_{\bar\alpha} = o_\alpha \restriction \bar\alpha$, $\varphi_{\bar\alpha} = \varphi_\alpha \restriction \bar\alpha$, $d_{\bar\alpha} = d_\alpha \restriction \bar\alpha$, $c_{\bar\alpha} = c_\alpha \restriction \bar\alpha$, and $h(\bar\alpha) = h(\alpha)$, and it follows that $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

Thus, to see that $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ meets our needs, let us fix a sequence $\langle A_\delta \mid \delta < \lambda^+ \rangle$ of cofinal subsets of λ^+, together with a club $D \subseteq \lambda^+$, and some ordinal $\varsigma < \lambda^+$.

Define an increasing function $f : \lambda^+ \to \lambda^+$ recursively by letting $f(0) := \varsigma$, and for all nonzero $\beta < \lambda^+$:

$$f(\beta) := \min\{\tau \in \lambda^+ \setminus (\sup(f[\beta]) + 1) \mid \bigwedge_{\delta < \beta} \pi(\tau)(\delta) = \min(A_\delta \setminus (\min(D \setminus (\beta+1)) + 1))\}.$$

Consider the set $X := f[\lambda^+]$, and the club

$$E := \{\beta \in \operatorname{acc}(D) \cap \mathop{\triangle}_{\delta<\lambda^+} \operatorname{acc}^+(A_\delta) \mid f[\beta] \subseteq \beta\} \setminus (\varsigma + 1).$$

Pick a limit $\alpha \in S$ with $\operatorname{otp}(D_\alpha) = \chi = \sigma \cdot \lambda$ such that $\operatorname{nacc}(D_\alpha) \subseteq \{\gamma \in E \mid X \cap \gamma = X_\gamma\}$. In particular, $D_\alpha \subseteq E$, so that $\gamma := \min(D_\alpha) > \varsigma$, $X_\gamma = X \cap \gamma \supseteq f[\gamma] \neq \emptyset$, and $h(\alpha) = \min(X_{\min(D_\alpha)}) = \min(X_\gamma) = \min(X) = f(0) = \varsigma$.

Claim 3.6.2. *For every $\beta \in D_\alpha$, there exists $\eta \in D$ such that $\beta \in \eta \in c_\alpha(\beta) \in A_{\varphi_\alpha(\beta)}$. In particular:*

- $\operatorname{nacc}(C_\alpha) \subseteq \bigcup_{\delta<\alpha} A_\delta$*;*
- *for every $\beta < \gamma$ in C_α, there exists $\eta \in D$ with $\beta < \eta < \gamma$.*[19]

Proof. Fix $\beta \in D_\alpha$. Denote $\beta^+ := \min(D_\alpha \setminus (\beta + 1))$, $\gamma := c_\alpha(\beta)$, $\delta := \varphi_\alpha(\beta)$, and $\eta := \min(D \setminus (\beta+1))$. Then $\delta < \beta < \gamma \leq \beta^+ < \alpha$ and $\gamma = c_\alpha(\beta) = \min(C_\alpha \setminus (\beta+1)) \in \operatorname{nacc}(C_\alpha)$. As $\beta, \beta^+ \in D_\alpha \subseteq E$, f is increasing, $X = f[\lambda^+]$, and $X_{\beta^+} = X \cap \beta^+$, we have $\sup(f[\beta]) = \beta$, $\sup(X_{\beta^+}) = \beta^+$, $\min(X \setminus (\beta + 1)) = f(\beta)$, and

$$d_\alpha(\beta) = \pi(f(\beta))(\delta) = \min(A_\delta \setminus (\min(D \setminus (\beta + 1)) + 1)) = \min(A_\delta \setminus (\eta + 1)).$$

Since $\beta < \beta^+$ and $\beta^+ \in E \subseteq \operatorname{acc}(D)$, we have $\eta < \beta^+$. Then, since $\beta^+ \in E \setminus (\delta + 1) \subseteq \operatorname{acc}^+(A_\delta)$, it follows that $\beta < \eta < d_\alpha(\beta) < \beta^+$, and hence $c_\alpha(\beta) = d_\alpha(\beta)$. Altogether, $\beta \in \eta \in c_\alpha(\beta) = \gamma$, where $\eta \in D$ and $\gamma \in A_\delta$. $\square$

Fix $\delta < \alpha$. Put $i := \operatorname{otp}(D_\alpha \cap \delta)$ and $j := \psi_{\min(D_\alpha \setminus \delta)}(\delta)$. By the choice of ψ, the set $\{k < \lambda \mid \psi(k) = (i, j)\}$ has order-type λ, and hence $B := \{\beta \in D_\alpha \setminus (\delta+1) \mid \psi(o_\alpha(\beta)) = (i, j)\}$ contains λ-many intervals (relativized to D_α) of length σ, with each interval beginning at a point from $\operatorname{acc}(D_\alpha)$. By definition of φ_α, we have $\varphi_\alpha(\beta) = \delta$ for all $\beta \in B$. Then the preceding claim shows that $\{c_\alpha(\beta) \mid \beta \in B\} \subseteq \operatorname{nacc}(C_\alpha) \cap A_\delta$. In particular, $\operatorname{otp}(\{\beta \in \operatorname{acc}(C_\alpha) \mid \operatorname{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_\delta\}) = \lambda$. $\square$

Of course, in the case of $\lambda = \aleph_0$ we cannot improve $\mathcal{E}_{\omega^2}$ to $\mathcal{E}_\omega$ while maintaining $\sigma = \omega$ in the preceding,[20] but note we can do the following.

Theorem 3.7. *Assume $\diamondsuit(S)$ holds for a given $S \subseteq \omega_1$.*
Then $\mathrm{P}(\omega_1, 2, \sqsubseteq, \omega_1, \{S\}, 2, n, \mathcal{E}_\omega)$ holds for every $n < \omega$.

---

[19] Recall Claim 3.6.1.

[20] A set of ordinals C of order-type ω cannot satisfy $\operatorname{succ}_\omega(C \setminus \gamma_0) \subseteq A_0$ together with $\operatorname{succ}_\omega(C \setminus \gamma_1) \subseteq A_1$ for disjoint sets A_0, A_1 and ordinals $\gamma_0, \gamma_1 < \sup(C)$.

Proof. Modify the construction of Theorem 3.6, as follows. Let $\chi := \omega$ so that in particular, $\mathrm{otp}(D_\alpha) \leq \omega$ for all $\alpha < \omega_1$. Then, given a positive integer n, define for all $\alpha < \omega_1$, $o_\alpha : D_\alpha \to \omega$ by stipulating:

$$o_\alpha(\beta) := \left\lfloor \frac{\mathrm{otp}(D_\alpha \cap \beta)}{n} \right\rfloor .$$

The rest of the construction remains intact. □

Lemma 3.8. *Suppose that $\lambda \leq \mathrm{cf}(\kappa) = \kappa$ are uncountable cardinals, and $\langle C_\alpha \mid \alpha < \kappa\rangle$ is a sequence satisfying the following:*

(i) For every limit ordinal $\alpha < \kappa$, C_α is a club subset of α;

(ii) For every limit ordinal $\Theta < \lambda$, and every sequence $\langle B_\iota \mid \iota < \Theta\rangle$ of cofinal subsets of κ, there exists some limit ordinal $\alpha < \kappa$ such that:

- (a) $\mathrm{otp}(C_\alpha) = \Theta$*; and*
- (b) $C_\alpha(\iota + 1) \in B_\iota$ *for co-boundedly many* $\iota < \Theta$.

Then

1. *For every infinite cardinal $\theta < \lambda$, every ordinal $\sigma < \lambda$, every sequence $\langle A_i \mid i < \theta\rangle$ of cofinal subsets of κ, and every infinite regular cardinal $\chi < \lambda$, there exist stationarily many $\alpha \in E^\kappa_\chi$ satisfying, for every $i < \theta$:*

$$\sup\{\beta \in C_\alpha \mid \mathrm{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_i\} = \alpha.$$

2. *For every cofinal subset $A \subseteq \kappa$, every infinite regular cardinal $\chi < \lambda$, and every limit ordinal $\theta < \lambda$, there exist stationarily many $\alpha \in E^\kappa_\chi$ satisfying* $\mathrm{otp}(C_\alpha) \geq \theta$ *for which there exist $\beta < \alpha$ such that*

$$\mathrm{succ}_\kappa(C_\alpha \setminus \beta) \subseteq A.$$

3. $\kappa^{<\lambda} = \kappa$.

Proof. 1. Suppose that we are given $\theta, \sigma < \lambda$, $\langle A_i \mid i < \theta\rangle$, some infinite regular cardinal $\chi < \lambda$, and some club $D \subseteq \kappa$. Fix a bijection $\psi : \kappa \leftrightarrow \theta \times \kappa$. Let $\psi_0 : \kappa \to \theta$ be the function such that for all $\alpha < \kappa$, if $\psi(\alpha) = (i, j)$, then $\psi_0(\alpha) = i$.
Define a function $f : \kappa \to \kappa$ by recursion:

- $f(0) := \min(A_{\psi_0(0)})$;
- $f(\alpha) := \min(A_{\psi_0(\alpha)} \setminus (\min(D \setminus (\sup(f[\alpha]) + 1)) + 1))$ for all nonzero $\alpha < \kappa$.

Let $I := \sigma \times \theta \times \chi$ be the Cartesian product, and let $\lhd$ denote the reverse-lexicographic ordering of I induced from $\in$, so that $(I, \lhd)$ is isomorphic to $(\Theta, \in)$, where $\Theta := \sigma \cdot \theta \cdot \chi$ (ordinal multiplication). Then $\mathrm{cf}(\Theta) = \chi$, and since σ, θ, χ are all smaller than the cardinal λ, we have $\Theta < \lambda$. Fix a bijection $\pi : \Theta \leftrightarrow I$ such that $\alpha \in \beta \in \Theta$ implies $\pi(\alpha) \lhd \pi(\beta)$. That is, for every $j < \sigma$, $i < \theta$, and $\eta < \chi$, $\pi(\sigma \cdot \theta \cdot \eta + \sigma \cdot i + j) = (j, i, \eta)$.
Define $\langle B_\iota \mid \iota < \Theta\rangle$ by letting

$$B_\iota = \{f(\alpha) \mid \psi_0(\alpha) = i\}$$

for the unique $i < \theta$ such that $\pi(\iota)$ is of the form $(\cdot, i, \cdot)$. Evidently, B_ι is a cofinal subset of the corresponding A_i. Now, fix a limit ordinal $\alpha < \lambda^+$ satisfying satisfying properties (ii)(a) and (ii)(b) of the hypothesis.
By (i) and (ii)(a), we have $\mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{otp}(C_\alpha)) = \mathrm{cf}(\Theta) = \chi$.
By (ii)(b) and the fact that $B_\iota \subseteq \mathrm{Im}(f)$ for all ι, we get that $\alpha \in \mathrm{acc}^+(\mathrm{Im}(f))$. But the definition of f ensures that for all $\alpha < \beta < \kappa$, there exists some $\eta \in D$ such that $f(\alpha) < \eta < f(\beta)$, and hence also $\alpha \in \mathrm{acc}^+(D)$. As D is a club, we altogether have $\alpha \in D \cap E^\kappa_\chi$.
By (ii)(b), fix $\iota' < \theta$ such that $C_\alpha(\iota + 1) \in B_\iota$ whenever $\iota' < \iota < \Theta$. Then by the definition of $\langle B_\iota \mid \iota < \Theta\rangle$, for every $j < \sigma$, $i < \theta$, and $\eta < \chi$, either $(\sigma\cdot\theta\cdot\eta+\sigma\cdot i+j) \le \iota'$, or
$$C_\alpha(\sigma \cdot \theta \cdot \eta + \sigma \cdot i + j + 1) = C_\alpha(\pi^{-1}(j, i, \eta) + 1) \in B_{\pi^{-1}(j,i,\eta)} \subseteq A_i,$$
so that for any large enough β of the form $C_\alpha(\sigma\cdot\theta\cdot\eta+\sigma\cdot i)$, we have $\mathrm{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_i$. For any fixed $i < \theta$ the set $\{\sigma\cdot\theta\cdot\eta+\sigma\cdot i \mid \eta < \chi\}$ is cofinal in Θ, so that $\{C_\alpha(\sigma\cdot\theta\cdot\eta+\sigma\cdot i) \mid \eta < \chi\}$ is cofinal in α, and the required result follows.

2. Suppose that we are given A, χ, θ as in the hypothesis. Let $D \subseteq \kappa$ be an arbitrary club. Let A' be a cofinal subset of A with the property that for all $\alpha < \beta$ in A', there exists some $\eta \in D$ with $\alpha' < \eta < \beta'$. Let $\Theta := \theta + \chi$, and let $B_\iota = A'$ for all $\iota < \Theta$. Now, fix a limit ordinal $\alpha < \kappa$ satisfying properties (ii)(a) and (ii)(b) of the hypothesis. In particular, $\mathrm{otp}(C_\alpha) \ge \theta$.
By (i) and (ii)(a), we have $\mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{otp}(C_\alpha)) = \mathrm{cf}(\Theta) = \mathrm{cf}(\chi) = \chi$. By (ii)(b), fix $\iota' < \theta$ such that $C_\alpha(\iota + 1) \in B_\iota = A'$ whenever $\iota' < \iota < \Theta$. Put $\beta := C_\alpha(\iota')$. Then $\mathrm{succ}_\kappa(C_\alpha \setminus \beta) = \{C_\alpha(\iota + 1) \mid \iota' \le \iota < \Theta\} \subseteq A' \subseteq A$, and $\alpha \in \mathrm{acc}^+(A') \subseteq D$. In particular, $\alpha \in D \cap E^\kappa_\chi$.
3. Let $\langle A_i \mid i < \kappa\rangle$ be some partition of κ into cofinal sets. For every $\beta < \alpha < \kappa$, let $X^\alpha_\beta := \{i < \kappa \mid \mathrm{nacc}(C_\alpha) \cap A_i \nsubseteq \beta\}$. To see that $[\kappa]^{<\lambda} \subseteq \{X^\alpha_\beta \mid \beta < \alpha < \kappa\}$, let $x \in [\kappa]^{<\lambda}$ be arbitrary. Put $\Theta = \max\{|x|, \aleph_0\}$, and fix an enumeration $\{\xi_\iota \mid \iota < \Theta\}$ of x such that each $\xi \in x$ is enumerated cofinally often. Put $B_\iota := A_{\xi_\iota}$ for all $\iota < \Theta$. Pick $\alpha < \kappa$ with $\mathrm{otp}(C_\alpha) = \Theta$ along with some $\iota^* < \Theta$ such that $C_\alpha(\iota + 1) \in B_\iota$ for all $\iota \in [\iota^*, \Theta)$. Put $\beta := C_\alpha(\iota^* + 1)$. Then $i \in X^\alpha_\beta$ iff $\mathrm{nacc}(C_\alpha) \cap A_i \nsubseteq \beta$ iff there exists some $\iota \in [\iota^*, \Theta)$ with $C_\alpha(\iota + 1) \in A_i$ iff $i \in x$. □

Corollary 3.9. *For any uncountable cardinal λ, the following are equivalent:*

1. $\square_\lambda + \mathrm{CH}_\lambda$*;*
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq, \theta, \{E^{\lambda^+}_\chi \mid \aleph_0 \le \mathrm{cf}(\chi) = \chi < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ *for every cardinal $\theta < \lambda$ and every ordinal $\sigma < \lambda$;*
3. $\mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_\chi \mid \aleph_0 \le \mathrm{cf}(\chi) = \chi < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$*;*
4. $\mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda)$*.*

Proof. (2) $\implies$ (4) and (3) $\implies$ (4) are immediate by monotonicity of the parameters.

(4) $\implies$ (1): By $\mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda)$, we have $\diamondsuit(\lambda^+)$, and hence CH_λ holds. $\square_\lambda$ follows using Lemma 3.1.

(1) $\implies$ (2) & (3): Since λ is uncountable, we get from Fact 8.2 that $\diamondsuit(\lambda^+)$ holds. Next, by Fact 8.15, we can fix a $\clubsuit_\lambda$-sequence, $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+\rangle$. In particular, $\vec{C}$ is a $\square_\lambda$-sequence that satisfies the hypotheses of Lemma 3.8.

Since $\vec{C}$ is a $\square_\lambda$-sequence, for all $\alpha < \lambda^+$, C_α has order-type $\leq \lambda$, and is a club in α if α is a limit ordinal, and if $\bar{\alpha} \in \operatorname{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$, and hence $C_{\bar{\alpha}} \sqsubseteq C_\alpha$.

Thus, the fact that $\vec{C}$ witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq, \theta, \{E^{\lambda^+}_\chi \mid \aleph_0 \leq \operatorname{cf}(\chi) = \chi < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ and $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_\chi \mid \aleph_0 \leq \operatorname{cf}(\chi) = \chi < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$ follows from the two respective parts of Lemma 3.8. $\square$

Corollary 3.10. *For every singular cardinal λ, the following are equivalent:*

1. $\square_\lambda + \mathrm{CH}_\lambda$;
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq, \lambda^+, \{E^{\lambda^+}_{\operatorname{cf}(\lambda)}\}, 2, \sigma, \mathcal{E}_\lambda)$ *for every* $\sigma < \lambda$.

Proof. The forward implication follows from Fact 8.18 and Theorem 3.6. The backward implication follows from Corollary 3.9 ((4) $\implies$ (1)). $\square$

Corollary 3.11. *For every uncountable cardinal λ, the following are equivalent:*

1. $\boxtimes_\lambda$;
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_{\operatorname{cf}(\lambda)}\}, 2, \lambda^+, \mathcal{E}_\lambda)$.

Proof. The forward implication is obtained by applying the last bullet of Theorem 3.6 to the constant sequence $\langle A_0 \mid \delta < \lambda^+\rangle$, yielding stationarily many $\alpha \in E^{\lambda^+}_{\operatorname{cf}(\lambda)}$ such that $\operatorname{succ}_{\lambda^+}(C_\alpha \setminus \beta) \subseteq \operatorname{nacc}(C_\alpha) \subseteq A_0$ whenever $\beta < \alpha$.

For the backward implication, we consider two cases. If λ is singular, then by Corollary 3.9, $\square_\lambda + \mathrm{CH}_\lambda$ holds, and then by Fact 8.18, so does $\boxtimes_\lambda$.[21]

Thus, from now on, suppose that λ is a regular cardinal, $\diamondsuit(\lambda^+)$ holds, and $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+\rangle$ is a witness to $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_\lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$. By $\diamondsuit(\lambda^+)$, fix a matrix $\langle Z^i_\beta \mid i < \lambda, \beta < \lambda^+\rangle$ satisfying that for every sequence $\langle Z^i \mid i < \lambda\rangle$ of subsets of λ^+, there exist stationarily many $\beta < \lambda^+$ such that $\bigwedge_{i<\lambda} Z^i_\beta = Z^i \cap \beta$.

Claim 3.11.1. *There exists some $i < \lambda$ satisfying that for every subset $Z \subseteq \lambda^+$ and every club $D \subseteq \lambda^+$, there exist some $\alpha \in E^{\lambda^+}_\lambda$ for which $C_\alpha(j+1) \in \{\beta \in D \mid Z^i_\beta = Z \cap \beta\}$ for all $j \in [i, \lambda)$.*

Proof. Suppose not. Then, for each $i < \lambda$, pick a counterexample (Z^i, D^i). Put $B := \{\beta \in \bigcap_{i<\lambda} D^i \mid \bigwedge_{i<\lambda} Z^i_\beta = Z^i \cap \beta\}$, so that B is stationary. As $\vec{C}$ witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_\lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$, let us fix some $\alpha \in E^{\lambda^+}_\lambda$ and $\gamma < \alpha$ such that $\operatorname{succ}_{\lambda^+}(C_\alpha \setminus \gamma) \subseteq B$. Fix a large enough $i < \lambda$ such that $C_\alpha(i) \geq \gamma$. Then $C_\alpha(j+1) \in B$ for all $j \in [i, \lambda)$. In particular, $C_\alpha(j+1) \in \{\beta \in D^i \mid Z^i_\beta = Z^i \cap \beta\}$ for all $j \in [i, \lambda)$, contradicting the choice of (Z^i, D^i). $\square$

[21] This is actually the harder case, since, for instance, the very definition of $\square_\lambda$ does not ensure the existence of a coherent sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ for which $\{\alpha \in E^{\lambda^+}_{\operatorname{cf}(\lambda)} \mid \operatorname{otp}(C_\alpha) = \lambda\}$ is stationary. See [Rin15b] for a detailed discussion.

Let $i<\lambda$ be given by the preceding. For each $\alpha<\lambda^+$, let:

$$D_\alpha := \begin{cases} C_\alpha \setminus C_\alpha(i+1), & \text{if } i\in \operatorname{otp}(C_\alpha);\\ C_\alpha, & \text{otherwise,}\end{cases}$$

and then let

$$X_\alpha := \bigcup\{Z^i_\beta \cap [\sup(D_\alpha\cap\beta),\beta) \mid \beta\in\operatorname{nacc}(D_\alpha)\}.$$

Clearly, $\langle (D_\alpha, X_\alpha) \mid \alpha<\lambda^+\rangle$ witnesses $\boxed{\Diamond}_\lambda$. $\square$

Theorem 3.12. *Suppose that $V=L$, and that κ is an inaccessible cardinal that is not weakly compact.*

Then $\mathrm{P}(\kappa,2,\sqsubseteq,1,\{E^\kappa_\chi \mid \aleph_0\leq \operatorname{cf}(\chi)=\chi<\kappa\},2,\kappa)$ holds. In particular, $\mathrm{P}(\kappa,2,\sqsubseteq,\theta,\{E^\kappa_{\geq\chi} \mid \chi<\kappa\},2,\sigma)$ holds for all $\theta,\sigma<\kappa$.

Proof sketch. Work in L. As hinted in [She90, Theorem 3.2], the proof of [ASS87, §2] essentially shows that for every inaccessible cardinal κ that is not weakly compact, there exists a sequence $\langle (D_\alpha,X_\alpha)\mid \alpha<\kappa\rangle$ such that for every limit $\alpha<\kappa$, D_α is a club in α, and if $\bar\alpha\in\operatorname{acc}(D_{\bar\alpha})$, then $D_{\bar\alpha}=D_\alpha\cap\bar\alpha$ and $X_{\bar\alpha}=X_\alpha\cap\bar\alpha$. Moreover, for every club $E\subseteq\kappa$, subset $X\subseteq\kappa$, and a limit ordinal $\Theta<\kappa$, there exists a singular limit ordinal $\alpha<\kappa$ with $\operatorname{otp}(D_\alpha)=\Theta$, satisfying $X\cap\alpha=X_\alpha$ and $D_\alpha\subseteq E$. Thus, fix a sequence $\langle (D_\alpha,X_\alpha)\mid\alpha<\kappa\rangle$ as above. For all $\alpha<\kappa$, define $f_\alpha: D_\alpha\rightarrow\alpha$ by stipulating:

$$f_\alpha(\beta) := \min((X_\beta\cup\{\beta\})\setminus \sup(D_\alpha\setminus\beta)).$$

Put $C_\alpha:=\operatorname{Im}(f_\alpha)$. It is not hard to verify that $\langle C_\alpha\mid\alpha<\kappa\rangle$ witnesses that $\mathrm{P}^-(\kappa,2,\sqsubseteq,1,\{E^\kappa_\chi \mid \aleph_0\leq\operatorname{cf}(\chi)=\chi<\kappa\},2,\kappa)$ holds. As $\langle X_\alpha\mid\alpha<\kappa\rangle$ witnesses that $\diamondsuit(\kappa)$ holds, we altogether infer that $\mathrm{P}(\kappa,2,\sqsubseteq,1,\{E^\kappa_\chi\mid\aleph_0\leq\operatorname{cf}(\chi)=\chi<\kappa\},2,\kappa)$ holds.

The fact that, modulo $\kappa^{<\kappa}=\kappa$, $\mathrm{P}^-(\kappa,2,\sqsubseteq,1,\{E^\kappa_\chi\mid\aleph_0\leq\operatorname{cf}(\chi)=\chi<\kappa\},2,\kappa)$ entails the existence of a simultaneous witness to $\mathrm{P}^-(\kappa,2,\sqsubseteq,\theta,\{E^\kappa_{\geq\chi}\mid\chi<\kappa\},2,\sigma)$ for all $\theta,\sigma<\kappa$, is proven using the coding+decoding techniques of the proof of Theorem 3.6 augmented by the ordinal arithmetic considerations of Lemma 3.8. $\square$

In a recent work, we have generalized the preceding to the following.

Fact 3.13 ([BR17a])**.** *Suppose that $V=L$, and that κ is an inaccessible cardinal that is not weakly compact. Then $\mathrm{P}(\kappa,2,\sqsubseteq,\kappa,\{E^\kappa_\chi\mid\operatorname{cf}(\chi)=\chi<\kappa\},2,\sigma)$ holds for all $\sigma<\kappa$.*

4. The coherence relation $\sqsubseteq_\chi$

Various constructions of Souslin-trees using the relation $\sqsubseteq_\chi$ may be found in [BR17c].

Lemma 4.1. *Suppose that λ is an uncountable cardinal, and $\chi,\eta\leq\lambda$ are infinite regular cardinals.*

The following are equivalent:

1. $\boxminus_{\lambda,\geq\chi}$ *holds.*
2. *For every stationary* $S\subseteq\lambda^+$*, there exist a stationary subset* $S'\subseteq S$ *and a sequence* $\langle C_\alpha\mid\alpha\in\Gamma\rangle$ *satisfying:*
    - $E^{\lambda^+}_{\geq\chi}\subseteq\Gamma\subseteq\mathrm{acc}(\lambda^+)$*;*
    - *if* $\alpha\in\Gamma$*, then* C_α *is a club subset of* α *of order-type* $\leq\lambda$*;*
    - *if* $\alpha\in\Gamma$ *and* $\bar\alpha\in\mathrm{acc}(C_\alpha)$*, then* $\bar\alpha\in\Gamma\setminus S'$ *and* $C_{\bar\alpha}=C_\alpha\cap\bar\alpha$*;*
    - *for every club* $D\subseteq\lambda^+$*, there exist stationarily many* $\alpha\in\Gamma\cap E^{\lambda^+}_\eta$ *such that* $\min(C_\alpha)\in D$*.*
3. $\mathrm{P}^-(\lambda^+,2,\sqsubseteq_\chi,1,\{\lambda^+\},2,0,\mathcal{E}_\lambda)$.

In particular, $\boxminus_{\lambda,\geq\aleph_0}$, $\square_\lambda$, and $\mathrm{P}^-(\lambda^+,2,\sqsubseteq,1,\{\lambda^+\},2,0,\mathcal{E}_\lambda)$ are all equivalent.

Proof. (1) $\implies$ (2): Let $\langle C_\alpha\mid\alpha\in E^{\lambda^+}_{\geq\chi}\rangle$ be a $\boxminus_{\lambda,\geq\chi}$-sequence. First, we make the following adjustment. If $\bar\alpha<\alpha$ are two elements of $E^{\lambda^+}_{\geq\chi}$ such that $\bar\alpha\in\mathrm{acc}(C_\alpha)$, then replace $C_{\bar\alpha}$ with $C_\alpha\cap\bar\alpha$. Notice that this adjustment is well-defined as a result of the second clause of Definition 8.12. Then, let $\Gamma:=\bigcup\{\mathrm{acc}(C_\alpha)\cup\{\alpha\}\mid\alpha\in E^{\lambda^+}_{\geq\chi}\}$, and define for every $\bar\alpha\in\Gamma\cap E^{\lambda^+}_{<\chi}$, $C_{\bar\alpha}=C_\alpha\cap\bar\alpha$ for some (and hence, for every) $\alpha\in E^{\lambda^+}_{\geq\chi}$ satisfying $\bar\alpha\in\mathrm{acc}(C_\alpha)$. Again, this is well-defined. The following is clear:

- $E^{\lambda^+}_{\geq\chi}\subseteq\Gamma\subseteq\mathrm{acc}(\lambda^+)$;
- if $\alpha\in\Gamma$, then C_α is a club subset of α of order-type $\leq\lambda$;
- if $\alpha\in\Gamma$ and $\bar\alpha\in\mathrm{acc}(C_\alpha)$, then $\bar\alpha\in\Gamma$ and $C_{\bar\alpha}=C_\alpha\cap\bar\alpha$.

If $S_0:=S\setminus\Gamma$ is stationary, let $\epsilon:=0$. Otherwise, $S\cap\Gamma$ is stationary in λ^+, and since $\{\mathrm{otp}(C_\alpha)\mid\alpha\in S\cap\Gamma\}$ is a subset of $\mathrm{acc}(\lambda+1)$, there must exist some nonzero limit ordinal $\epsilon\leq\lambda$ such that $S_\epsilon:=\{\alpha\in S\cap\Gamma\mid\mathrm{otp}(C_\alpha)=\epsilon\}$ is stationary, so let ϵ denote the least such ordinal.

For all $\alpha\in\Gamma$, set:

$$c_\alpha:=\begin{cases}C_\alpha, & \text{if }\mathrm{otp}(C_\alpha)\leq\epsilon;\\ C_\alpha\setminus C_\alpha(\epsilon), & \text{otherwise.}\end{cases}$$

Evidently:

- $S':=S_\epsilon$ is a stationary subset of S;
- if $\alpha\in\Gamma$, then c_α is a club subset of α of order-type $\leq\lambda$;
- if $\alpha\in\Gamma$ and $\bar\alpha\in\mathrm{acc}(c_\alpha)$, then $\bar\alpha\in\Gamma\setminus S'$ and $c_{\bar\alpha}=c_\alpha\cap\bar\alpha$.

Now, for all $i<\lambda$ and $\alpha\in\Gamma$, define:

$$c^i_\alpha := \begin{cases} c_\alpha, & \text{if } \operatorname{otp}(c_\alpha)\le i;\\ c_\alpha\setminus c_\alpha(i), & \text{otherwise.}\end{cases}$$

We claim that there exists a limit ordinal $i<\lambda$, such that for every club $D\subseteq\lambda^+$, there exist stationarily many $\alpha\in\Gamma\cap E^{\lambda^+}_\eta$ with $\min(c^i_\alpha)\in D$. Of course, we then could simply fix such an i, and conclude that S' and $\langle c^i_\alpha \mid \alpha\in\Gamma\rangle$ are as sought.

Thus, suppose there is no such i. Then, we may find a sequence $\langle (D_i,E_i)\mid i<\lambda\rangle$ of pairs of club subsets of λ^+, such that for every limit $i<\lambda$ and every $\alpha\in\Gamma\cap E^{\lambda^+}_\eta\cap E_i$, we have $\min(c^i_\alpha)\notin D_i$. Consider the club $D:=\bigcap_{i\in\operatorname{acc}(\lambda)}(D_i\cap E_i)$. Pick $\alpha\in E^{\lambda^+}_{\max\{\aleph_1,\chi,\eta\}}\cap\operatorname{acc}(D)$. Then $\alpha\in\Gamma$ and $c_\alpha\cap D$ is a club in α. Put $\beta:=\min(c_\alpha\cap D)$, and $i:=\operatorname{otp}(c_\alpha\cap\beta)$. Pick $\bar\alpha\in(\operatorname{acc}(c_\alpha\cap D)\cup\{\alpha\})$ with $\operatorname{cf}(\bar\alpha)=\eta$. Then $\bar\alpha\in\Gamma\cap E^{\lambda^+}_\eta\cap E_i$, and $\min(c^i_{\bar\alpha})=c_{\bar\alpha}(i)=c_\alpha(i)=\beta\in D\subseteq D_i$. This is a contradiction.

(2) $\implies$ (3): Suppose $\langle C_\alpha\mid\alpha\in\Gamma\rangle$ is given and satisfies the hypotheses. We extend it to a sequence $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ as follows:

- Let $C_0:=\emptyset$.
- Let $C_{\alpha+1}:=\{\alpha\}$ for every $\alpha<\lambda^+$.
- For every $\alpha\in\operatorname{acc}(\lambda^+)\setminus\Gamma$, let C_α be a club subset of α of order-type $\operatorname{cf}(\alpha)$ with $\operatorname{nacc}(C_\alpha)\subseteq\operatorname{nacc}(\alpha)$.

It is clear that $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+,2,{\sqsubseteq_\chi},1,\{\lambda^+\},2,0,\mathcal{E}_\lambda)$.

(3) $\implies$ (1): Let $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ witness $\mathrm{P}^-(\lambda^+,2,{\sqsubseteq_\chi},1,\{\lambda^+\},2,0,\mathcal{E}_\lambda)$. To see that its restriction $\langle C_\alpha\mid\alpha\in E^{\lambda^+}_{\ge\chi}\rangle$ satisfies $\boxminus_{\lambda,\ge\chi}$, consider any $\alpha,\beta\in E^{\lambda^+}_{\ge\chi}$ and any $\gamma\in\operatorname{acc}(C_\alpha)\cap\operatorname{acc}(C_\beta)$. We must have $C_\gamma\sqsubseteq_\chi C_\alpha$ and $C_\gamma\sqsubseteq_\chi C_\beta$. But $\operatorname{otp}(C_\alpha)\ge\operatorname{cf}(\alpha)\ge\chi$, so that by definition of $\sqsubseteq_\chi$ we must have $C_\gamma\sqsubseteq C_\alpha$, and similarly $C_\gamma\sqsubseteq C_\beta$. Thus $C_\alpha\cap\gamma=C_\gamma=C_\beta\cap\gamma$, as required. □

Theorem 4.2. *Suppose that $\boxminus_{\lambda,\ge\chi}$ holds for some regular uncountable cardinal $\lambda=\lambda^{<\lambda}$ and some infinite regular cardinal $\chi\le\lambda$. Then:*

1. $V^{\mathrm{Add}(\lambda,1)}\models\mathrm{P}^-(\lambda^+,2,{\sqsubseteq_\chi},\lambda^+,\{S\subseteq E^{\lambda^+}_\lambda\mid S\text{ is stationary}\},2,\sigma,\mathcal{E}_\lambda)$ *for all* $\sigma<\lambda$*;*
2. CH_λ *entails* $V^{\mathrm{Add}(\lambda,1)}\models\mathrm{P}(\lambda^+,2,{\sqsubseteq_\chi},\lambda^+,\{S\subseteq E^{\lambda^+}_\lambda\mid S\text{ is stationary}\},2,\sigma,\mathcal{E}_\lambda)$ *for all* $\sigma<\lambda$*.*

Proof. Work in V. Using Lemma 4.1, fix a sequence $\langle C_\alpha\mid\alpha\in\Gamma\rangle$ satisfying:

- $E^{\lambda^+}_{\ge\chi}\subseteq\Gamma\subseteq\operatorname{acc}(\lambda^+)$;
- if $\alpha\in\Gamma$, then C_α is a club subset of α of order-type $\le\lambda$;
- if $\alpha\in\Gamma$ and $\bar\alpha\in\operatorname{acc}(C_\alpha)$, then $\bar\alpha\in\Gamma$ and $C_{\bar\alpha}=C_\alpha\cap\bar\alpha$.

We may also assume that $0 \notin C_\alpha$ for all $\alpha \in \Gamma$.

For every nonzero $\beta < \lambda^+$, let $\psi_\beta : \lambda \setminus \{0\} \rightarrow \beta$ be some surjection. Let $g : \lambda \rightarrow \lambda$ be $\mathrm{Add}(\lambda, 1)$-generic over V. Work in $V[g]$. For every $\alpha \in \Gamma$, we derive the following objects:

- $1^g_\alpha := \{j < \mathrm{otp}(C_\alpha) \mid g(j) \neq 0\}$;
- $g_\alpha : 1^g_\alpha \rightarrow \alpha$ by stipulating $g_\alpha(j) := \psi_{C_\alpha(j)}(g(j))$;
- $C^g_\alpha := \{C_\alpha(j) \mid j \in \mathrm{acc}^+(1^g_\alpha)\} \cup \{\max\{g_\alpha(j), C_\alpha(\sup(1^g_\alpha \cap j))\} \mid j \in \mathrm{nacc}(1^g_\alpha)\}$;
- $D^g_\alpha := C^g_\alpha$ whenever $\sup(C^g_\alpha) = \alpha$, and $D^g_\alpha := C_\alpha \setminus \sup(C^g_\alpha)$ otherwise.

Claim 4.2.1. *For every $\alpha \in \Gamma$:*

- *D^g_α is a club subset of α of order-type $\leq \lambda$;*
- *if $\bar\alpha \in \mathrm{acc}(D^g_\alpha)$, then $\bar\alpha \in \Gamma$ and $D^g_{\bar\alpha} = D^g_\alpha \cap \bar\alpha$.*

Proof. By the same proof of Claim 2.3.2 of [Rin15a]. □

1. As $\lambda^{<\lambda} = \lambda$, every cofinal subset of λ^+ from $V[g]$ covers a cofinal subset of λ^+ from V. Thus, a simple density argument (cf. Claims 2.3.1 and 2.3.3 of [Rin15a]) establishes that for every cofinal subset $A \subseteq \lambda^+$, there exists a club $D_A \subseteq \lambda^+$ such that for every $\alpha \in E^{\lambda^+}_\lambda \cap D_A$, we have
$$\sup\{\beta \in D^g_\alpha \mid \mathrm{succ}_\sigma(D^g_\alpha \setminus \beta) \subseteq A\} = \alpha.$$
It follows that for every sequence $\langle A_i \mid i < \lambda^+\rangle$ of cofinal subsets of λ^+, if we let $D := \triangle_{i<\lambda^+} D_{A_i}$, then for every $\alpha \in D \cap E^{\lambda^+}_\lambda$ and every $i < \alpha$:
$$\sup\{\beta \in D^g_\alpha \mid \mathrm{succ}_\sigma(D^g_\alpha \setminus \beta) \subseteq A_i\} = \alpha.$$
Let $D^g_0 := \emptyset$ and let $D^g_{\alpha+1} := \{\alpha\}$ for every $\alpha < \lambda^+$. For every $\alpha \in \mathrm{acc}(\lambda^+) \setminus \Gamma$, let D^g_α be some club in α of order-type $\mathrm{cf}(\alpha)$ with $\mathrm{nacc}(D^g_\alpha) \subseteq \mathrm{nacc}(\alpha)$. Then $\langle D^g_\alpha \mid \alpha < \lambda^+\rangle$ witnesses the validity of $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{S \subseteq E^{\lambda^+}_\lambda \mid S \text{ is stationary}\}, 2, \sigma, \mathcal{E}_\lambda)$.
2. By $\mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$, we have $V[g] \models \lambda > \aleph_0\ \&\ \mathrm{CH}_\lambda$. So, by Fact 8.2, $V[g] \models \diamondsuit(\lambda^+)$. Recalling the previous clause, we are done. □

Theorem 4.3. *Suppose that $\boxminus_{\lambda,\geq\chi} + \mathrm{CH}_\lambda$ holds for a given uncountable limit cardinal λ and some fixed infinite regular cardinal $\chi \leq \lambda$. Then:*

1. *$\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ holds for every $\theta, \sigma < \lambda$.*
2. *$\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$ holds.*
3. *If λ is singular, then $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \sigma, \mathcal{E}_\lambda)$ holds for every $\sigma < \lambda$.*

Proof. As λ is uncountable, Fact 8.2 entails $\diamondsuit(\lambda^+)$, and so we only need to establish the corresponding $\mathrm{P}^-(\dots)$ principles of Clauses (1)–(3).

The upcoming proof will invoke tools from [Rin14b] to establish Clauses (1),(2). Then, by going further and invoking tools from [Rin15b], we shall establish Clause (3).

Claim 4.3.1. *There exist sequences $\langle C_\alpha \mid \alpha \in \Gamma\rangle$ and $\langle (S_i, \gamma_i) \mid i \leq \mathrm{cf}(\lambda)\rangle$ such that:*

- $E^{\lambda^+}_{\geq\chi} \subseteq \Gamma \subseteq \mathrm{acc}(\lambda^+)$, *and* $\Gamma = \biguplus_{i\leq\mathrm{cf}(\lambda)} S_i$;
- *if* $\alpha \in \Gamma$, *then* C_α *is a club subset of* α *of order-type* $\leq \lambda$;
- *if* $\alpha \in S_i$ *and* $\bar\alpha \in \mathrm{acc}(C_\alpha)$, *then* $\bar\alpha \in S_i$ *and* $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- $\{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \gamma_i, C_\alpha \subseteq E\}$ *is stationary for every* $i < \mathrm{cf}(\lambda)$ *and every club* $E \subseteq \lambda^+$;
- $\{\gamma_i \mid i < \mathrm{cf}(\lambda)\}$ *is a cofinal subset of* λ.

Proof. By Lemma 4.1 and the proof of [Rin14b, Lemma 2.3]. That lemma builds on [Rin14b, Lemma 2.1] in case that λ is regular, and [Rin14b, Lemma 2.2] in case that λ is singular. The proof of the latter goes through as soon as one replaces there "$\lambda_0 = \mathrm{cf}(\lambda)$" with "$\lambda_0 = \max\{\mathrm{cf}(\lambda), \chi\}$"; the proof of the former goes through verbatim. □

Let $\langle C_\alpha \mid \alpha \in \Gamma\rangle$ and $\langle (S_i, \gamma_i) \mid i \leq \mathrm{cf}(\lambda)\rangle$ be given by the preceding claim. Note that given any club $E \subseteq \lambda^+$, any $i < \mathrm{cf}(\lambda)$ and any nonzero limit ordinal $\Theta < \gamma_i$, we can choose $\alpha \in S_i$ with $\mathrm{otp}(C_\alpha) = \gamma_i$ and $C_\alpha \subseteq E$, so that letting $\bar\alpha = C_\alpha(\Theta)$ we have $\bar\alpha \in \mathrm{acc}(C_\alpha)$, and it follows that $\bar\alpha \in S_i \cap E$, $\mathrm{otp}(C_{\bar\alpha}) = \Theta$, and $C_{\bar\alpha} \subseteq C_\alpha \subseteq E$. Therefore, we can fix a sequence $\langle \Theta_i \mid i < \mathrm{cf}(\lambda)\rangle$ such that:

- $\{\Theta_i \mid i < \mathrm{cf}(\lambda)\}$ is a set of regular cardinals, cofinal in the limit cardinal λ;
- $\Theta_i \leq \gamma_i$ for all $i < \mathrm{cf}(\lambda)$;
- $\{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \mathrm{cf}(\alpha) = \Theta_i, C_\alpha \subseteq E\}$ is stationary for every $i < \mathrm{cf}(\lambda)$ and every club $E \subseteq \lambda^+$.

By removing elements of $\{\Theta_i \mid i < \mathrm{cf}(\lambda)\}$ if necessary (and merging the corresponding sets S_i into $S_{\mathrm{cf}(\lambda)}$), and re-indexing, we may assume that if λ is singular, then $\Theta_i > \mathrm{cf}(\lambda)$ for all $i < \mathrm{cf}(\lambda)$.

For every $i < \mathrm{cf}(\lambda)$, denote $T_i := \{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \Theta_i\}$. Fix a sequence of injections $\langle \psi_\gamma : \gamma+1 \to \lambda \mid \gamma < \lambda^+\rangle$. For every $\alpha \in \Gamma$, define an injection $\varrho_\alpha : \alpha \to \lambda \times \lambda$ by stipulating $\varrho_\alpha(\delta) := (\mathrm{otp}(C_\alpha \cap \delta), \psi_{\min(C_\alpha\setminus\delta)}(\delta))$. Now, put $H^j_\alpha := (\varrho_\alpha^{-1}[\Theta_j \times \Theta_j])^2$ for all $j < \mathrm{cf}(\lambda)$. Then $\{H^j_\alpha \mid j < \mathrm{cf}(\lambda)\} \subseteq [\alpha \times \alpha]^{<\lambda}$ is an increasing chain,[22] converging to $\alpha \times \alpha$, and if $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $\varrho_{\bar\alpha} = \varrho_\alpha \restriction \bar\alpha$, so that $H^j_{\bar\alpha} = H^j_\alpha \cap (\bar\alpha \times \bar\alpha)$ for all $j < \mathrm{cf}(\lambda)$.

By CH_λ, let $\{X_\gamma \mid \gamma < \lambda^+\}$ be an enumeration of $[\lambda \times \lambda \times \lambda^+]^{\leq\lambda}$. For all $(j, \tau) \in \lambda \times \lambda$ and $X \subseteq \lambda \times \lambda \times \lambda^+$, let $\pi_{j,\tau}(X) := \{\varsigma < \lambda^+ \mid (j, \tau, \varsigma) \in X\}$.

Claim 4.3.2. *Suppose that* $i < \mathrm{cf}(\lambda)$.

There exist $(j, \tau) \in \mathrm{cf}(\lambda) \times \lambda$ *and* $Y \subseteq \lambda^+ \times \lambda^+$ *such that for every club* $D \subseteq \lambda^+$ *and every subset* $Z \subseteq \lambda^+$, *there exists some* $\alpha \in T_i$ *such that:*

[22] Recall that $\Theta_j < \lambda$ for all $j < \mathrm{cf}(\lambda)$.

1. $C_\alpha \subseteq D$;
2. $H^j_\alpha \setminus Y \subseteq \{(\eta,\gamma) \mid Z \cap \eta = \pi_{j,\tau}(X_\gamma)\}$;
3. $\sup(\mathrm{acc}^+(\{\eta < \alpha \mid (\eta,\gamma) \in H^j_\alpha \setminus Y$ *for some* $\gamma < \min(C_\alpha \setminus (\eta+1))\}) \cap \mathrm{acc}(C_\alpha)) = \alpha$.

Proof. This is Claim 2.5.2 of [Rin14b], and the proof is identical. □

Let $\langle (j_i, \tau_i, Y_i) \mid i < \mathrm{cf}(\lambda)\rangle$ be given by the previous claim.
For every $i < \mathrm{cf}(\lambda)$ and $\alpha \in S_i$, let:

$$f^i_\alpha := \{(\eta,\gamma) \in H^{j_i}_\alpha \setminus Y_i \mid \gamma = \min\{\gamma' < \min(C_\alpha \setminus (\eta+1)) \mid (\eta,\gamma') \in H^{j_i}_\alpha \setminus Y_i\}\}.$$

Then, let C^i_α be the set of all δ such that all of the following properties hold:

1. $\delta \in C_\alpha$;
2. $\sup(\mathrm{dom}(f^i_\alpha) \cap \delta) \geq \sup(C_\alpha \cap \delta)$;
3. if $\eta \in \mathrm{dom}(f^i_\alpha) \cap \delta$, then $\pi_{j_i,\tau_i}(X_{f^i_\alpha(\eta)}) \subseteq \eta$;
4. if $\eta' < \eta < \delta$ satisfy $\eta', \eta \in \mathrm{dom}(f^i_\alpha)$, then $\pi_{j_i,\tau_i}(X_{f^i_\alpha(\eta)}) \setminus \pi_{j_i,\tau_i}(X_{f^i_\alpha(\eta')}) \subseteq [\eta', \eta)$.

For every $\alpha \in S_{\mathrm{cf}(\lambda)}$, write $C^{\mathrm{cf}(\lambda)}_\alpha := \emptyset$.
Finally, for all $\alpha \in \Gamma$, put:

$$C^\bullet_\alpha := \begin{cases} C^i_\alpha, & \text{if } \alpha \in S_i, \sup(C^i_\alpha) = \alpha; \\ C_\alpha \setminus \sup(C^i_\alpha), & \text{if } \alpha \in S_i, \sup(C^i_\alpha) < \alpha. \end{cases}$$

Also, for all $\alpha < \lambda^+$, let

$$Z_\alpha := \begin{cases} \bigcup\{\pi_{j_i,\tau_i}(X_{f^i_\alpha(\eta)}) \mid \eta \in \mathrm{dom}(f^i_\alpha)\}, & \text{if } \alpha \in S_i, \sup(C^i_\alpha) = \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim 4.3.3. *All of the following properties hold for* $\langle (C^\bullet_\alpha, Z_\alpha) \mid \alpha \in \Gamma\rangle$*:*

1. $C^\bullet_\alpha$ *is a club subset of* α *(in fact a subclub of* C_α*) of order-type* $\leq \lambda$ *for all* $\alpha \in \Gamma$;
2. *if* $\alpha \in \Gamma$ *and* $\bar\alpha \in \mathrm{acc}(C^\bullet_\alpha)$, *then* $\bar\alpha \in \Gamma$, $C^\bullet_{\bar\alpha} = C^\bullet_\alpha \cap \bar\alpha$, *and* $Z_{\bar\alpha} = Z_\alpha \cap \bar\alpha$;
3. *for every club* $D \subseteq \lambda^+$, *every subset* $A \subseteq \lambda^+$, *and every* $i < \mathrm{cf}(\lambda)$, *there exists some* $\alpha \in \Gamma$ *such that:*
   (a) $C^\bullet_\alpha \subseteq D$;
   (b) $Z_\alpha = A \cap \alpha$;
   (c) $\mathrm{cf}(\alpha) = \Theta_i$;
   (d) $\sup(\mathrm{acc}(C^\bullet_\alpha)) = \alpha$.

Proof. This is the content of Claim 2.5.4 of [Rin14b]. □

Notice that $Z_\alpha \subseteq \alpha$ for all $\alpha < \lambda^+$, using property (3) of the definition of C^i_α. It then follows from the last claim that $\langle Z_\alpha \mid \alpha < \lambda^+\rangle$ is a $\diamondsuit(\lambda^+)$-sequence.

Fix a bijection $\psi : \lambda \times \lambda^+ \leftrightarrow \lambda^+$.

We define $\langle D_\alpha \mid \alpha < \lambda^+\rangle$ as follows:

- Let $D_0 := \emptyset$, and for every $\alpha < \lambda^+$, let $D_{\alpha+1} := \{\alpha\}$.
- For every $\alpha \in \mathrm{acc}(\lambda^+) \setminus \Gamma$, let D_α be a club subset of α of order-type $\mathrm{cf}(\alpha)$ with $\mathrm{nacc}(D_\alpha) \subseteq \mathrm{nacc}(\alpha)$.
- Let $\alpha \in \Gamma$ be arbitrary. Put $C'_\alpha := \mathrm{acc}(C^\bullet_\alpha)$ in case that $\sup(\mathrm{acc}(C^\bullet_\alpha)) = \alpha$, and let C'_α be some cofinal subset of α of order-type ω otherwise. Thus C'_α is a club subset of α of order-type $\leq \lambda$. Next, for $\beta \in \mathrm{nacc}(C'_\alpha)$, let:
  - $X^\beta_\alpha := \{\gamma < \lambda^+ \mid \psi(\mathrm{otp}(\mathrm{nacc}(C'_\alpha) \cap \beta), \gamma) \in Z_\beta\}$;
  - $Y^\beta_\alpha := X^\beta_\alpha \cap (\min(C^\bullet_\alpha \setminus (\sup(C'_\alpha \cap \beta) + 1)), \beta)$;
  - $\beta_\alpha := \min(Y^\beta_\alpha \cup \{\beta\})$;
  - $D_\alpha := \mathrm{acc}(C'_\alpha) \cup \{\beta_\alpha \mid \beta \in \mathrm{nacc}(C'_\alpha)\}$.

For all $\alpha \in \Gamma$ and all $\beta \in \mathrm{nacc}(C'_\alpha)$, we have $\sup(C'_\alpha \cap \beta) < \beta_\alpha \leq \beta$. Thus, for all $\alpha \in \Gamma$, $\mathrm{acc}(D_\alpha) = \mathrm{acc}(C'_\alpha) \subseteq \mathrm{acc}(C^\bullet_\alpha) \subseteq \Gamma$, so that $\mathrm{otp}(D_\alpha) = \mathrm{otp}(C'_\alpha) \leq \mathrm{otp}(C^\bullet_\alpha) \leq \lambda$, and D_α is a club in α.

Then, just as in the proof of [Rin14b, Claim 3.2.1], $\langle D_\alpha \mid \alpha < \lambda^+\rangle$ is a sequence of local clubs, each of order-type $\leq \lambda$, and if $\alpha \in \Gamma$ and $\bar\alpha \in \mathrm{acc}(D_\alpha)$, then $\bar\alpha \in \Gamma$ and $D_{\bar\alpha} = D_\alpha \cap \bar\alpha$. It then follows from the definition of D_α in case $\alpha \notin \Gamma$ that $D_{\bar\alpha} \sqsubseteq_\chi D_\alpha$ for all $\alpha < \lambda^+$ and all $\bar\alpha \in \mathrm{acc}(D_\alpha)$.

Claim 4.3.4. *For every nonzero limit ordinal $\Theta < \lambda$ and every sequence $\langle A_i \mid i < \Theta\rangle$ of cofinal subsets of λ^+, there exists some $\delta \in \Gamma$ such that:*

- $\mathrm{otp}(D_\delta) = \Theta$*; and*
- $D_\delta(i+1) \in A_i$ *for all* $i < \Theta$.

Proof. This is the content of Claim 3.2.2 from [Rin14b]. □

Then, the fact that $\langle D_\alpha \mid \alpha < \lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ follows from Lemma 3.8(1), and that it witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$ follows from Lemma 3.8(2).

Next, let us work towards establishing Clause (3). Thus, we assume that λ is a singular cardinal.

By removing the minimal element of D_α, and putting 0 instead, we may assume that $D_\alpha(0) = 0$ for all $\alpha \in \Gamma$. Next, fix an increasing and continuous sequence $\langle \lambda_j \mid j \leq \mathrm{cf}(\lambda)\rangle$ with $\lambda_0 = \mathrm{cf}(\lambda)$, $\mathrm{cf}(\lambda_{j+1}) = \lambda_{j+1}$ for all $j < \mathrm{cf}(\lambda)$, and $\lambda_{\mathrm{cf}(\lambda)} = \lambda$. Denote $\Lambda := \{\lambda_j \mid j < \mathrm{cf}(\lambda)\}$. For every limit $\epsilon \leq \lambda$, put

$$E_\epsilon := \begin{cases} \epsilon, & \text{if } \epsilon \leq \lambda_0; \\ \epsilon \setminus \lambda_j, & \text{if } \epsilon \in (\lambda_j, \lambda_{j+1}] \text{ for } j < \mathrm{cf}(\lambda); \\ \Lambda \cap \epsilon, & \text{otherwise.} \end{cases}$$

Then E_ϵ is a club subset of ϵ for all limit $\epsilon \leq \lambda$. In particular, $E_{\mathrm{otp}(D_\delta)}$ is a club subset of $\mathrm{otp}(D_\delta)$ for all limit $\delta < \lambda^+$.

As in the proof of [Rin15b], we let $\pi_\delta : \mathrm{otp}(D_\delta) \to D_\delta$ denote the order-preserving bijection, and then put $D'_\delta := \pi_\delta[E_{\mathrm{otp}(D_\delta)}]$ for every $\delta \in \Gamma$. Thus for every $\delta \in \Gamma$, $D'_\delta \subseteq D_\delta$ is a club subset of δ, and $\mathrm{otp}(D'_\delta) \leq \mathrm{otp}(D_\delta) \leq \lambda$.

Let $\varphi : \lambda^+ \to \lambda^+$ be a surjection such that for all $\alpha < \lambda^+$, $\varphi(\alpha) \leq \alpha$ and $\varphi^{-1}\{\alpha\}$ is stationary. Split Γ into three sets:

- $\Gamma_0 := \{\delta \in \Gamma \mid \mathrm{otp}(D_\delta) \leq \lambda_0\}$;
- $\Gamma_1 := \{\delta \in \Gamma \mid \mathrm{otp}(D_\delta) \in (\lambda_j, \lambda_{j+1}]$ for some $j < \mathrm{cf}(\lambda)\}$;
- $\Gamma_2 := \{\delta \in \Gamma \mid \mathrm{otp}(D_\delta) = \lambda_j$ for some nonzero limit $j \leq \mathrm{cf}(\lambda)\}$.

We shall define a sequence $\langle G_\delta \mid \delta < \lambda^+ \rangle$ by recursion over $\delta < \lambda^+$. Let $G_\delta = \emptyset$ for all $\delta \in \lambda^+ \setminus \Gamma$. Now, suppose that $\delta \in \Gamma$, and $\langle G_\alpha \mid \alpha < \delta \rangle$ has already been defined. The definition of G_δ splits into cases:

- If $\delta \in \Gamma_2$, then let $G_\delta := D'_\delta$.
- If $\delta \in \Gamma_1$, then consider the ordinal $\phi_\delta := \varphi(\pi_\delta(1))$:
  - If $\phi_\delta \in \Gamma$ and $|G_{\phi_\delta}| < \lambda$, then let $G_\delta := G_{\phi_\delta} \cup \{\phi_\delta\} \cup D'_\delta$.
  - Otherwise, let $G_\delta := D'_\delta$.
- If $\delta \in \Gamma_0$, then we shall try to define an increasing and continuous sequence of ordinals $\langle \delta^i \mid i < \mathrm{otp}(D_\delta) \rangle$, by recursion over $i < \mathrm{otp}(D_\delta)$. Let $\delta^0 := 0$. Suppose that $i < \mathrm{otp}(D_\delta)$ and δ^i has already been defined. If there exists an ordinal β such that $\pi_\delta(i) < \beta < \pi_\delta(i+1)$, $G_{\delta^i} \sqsubseteq G_\beta$, $\mathrm{nacc}(G_\beta) \subseteq Z_{\pi_\delta(i+1)}$, and $\mathrm{otp}(G_\beta) = \lambda_{i+1}$, then put $\delta^{i+1} := \beta$ for the least such β. If not, then we shall terminate the recursion and say that "the δ-process identified a failure at stage $i+1$".
  - If the δ-process identified a failure at stage $i+1$, then let $G_\delta := D_\delta \setminus \pi_\delta(i)$.
  - Otherwise, let $G_\delta := \bigcup\{G_{\delta^i} \mid i < \mathrm{otp}(D_\delta)\}$.

This concludes the definition of $\langle G_\delta \mid \delta < \lambda^+ \rangle$.

Claim 4.3.5. *The sequence $\langle (G_\delta, Z_\delta) \mid \delta \in \Gamma \rangle$ satisfies:*

1. *for every $\delta \in \Gamma$, G_δ is a club in δ of order-type $\leq \lambda$, and $Z_\delta \subseteq \delta$;*
2. *if $\delta \in \Gamma$ and $\bar\delta \in \mathrm{acc}(G_\delta)$, then $\bar\delta \in \Gamma$ and $G_\delta \cap \bar\delta = G_{\bar\delta}$;*
3. *for every subset $Z \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists $\delta \in \Gamma$ with $\mathrm{otp}(G_\delta) = \lambda$ such that $\mathrm{nacc}(G_\delta) \subseteq \{\gamma \in E \mid Z \cap \gamma = Z_\gamma\}$.*

Proof. (1) and (2) are just like the proof of Claim 1 of [Rin15b].

(3) Given Z and E as above, let $X := \{\gamma \in E \mid Z \cap \gamma = Z_\gamma\}$. By the fact that $\langle Z_\gamma \mid \gamma < \lambda^+\rangle$ is a $\diamondsuit(\lambda^+)$-sequence, we have $X \in [\lambda^+]^{\lambda^+}$. Then, by the proofs of Claims 2 and 3 of [Rin15b], there exists some $\delta \in \Gamma$ with $\operatorname{otp}(G_\delta) = \lambda$ such that $\operatorname{nacc}(G_\delta) \subseteq X$. $\square$

Let $\sigma < \lambda$ be an arbitrary infinite cardinal. Using CH_λ, fix a function $\pi : \lambda^+ \to {}^{\lambda^+}\lambda^+$ such that $\{\alpha < \lambda^+ \mid f \subseteq \pi(\alpha)\}$ is cofinal in λ^+ for all $f \in {}^{<\lambda^+}\lambda^+$. Also fix a function $\psi' : \lambda \to \lambda \times \lambda$ such that $\{k < \lambda \mid (i,j) = \psi'(k)\}$ has order-type λ for all $(i,j) \in \lambda \times \lambda$. We now relativize the proof of Theorem 3.6 to the sequence $\langle (G_\delta, Z_\delta) \mid \delta \in \Gamma\rangle$, as follows.

Let $\alpha \in \Gamma$ be arbitrary. Let $o_\alpha : G_\alpha \to \lambda$ be the unique function satisfying $\operatorname{otp}(G_\alpha \cap \beta) \in [\sigma \cdot o_\alpha(\beta), \sigma \cdot o_\alpha(\beta) + \omega)$ for each $\beta \in G_\alpha$. Define $\varphi_\alpha : G_\alpha \to \alpha$ by letting for all $\beta \in G_\alpha$:

$$\varphi_\alpha(\beta) := \begin{cases} \delta, & \text{if } \delta < \beta \ \&\ \psi'(o_\alpha(\beta)) = (\operatorname{otp}(G_\alpha \cap \delta), \psi_{\min(G_\alpha \setminus \delta)}(\delta)); \\ 0, & \text{otherwise.} \end{cases}$$

Define $d_\alpha : G_\alpha \to \lambda^+$ by letting for all $\beta \in G_\alpha$:

$$d_\alpha(\beta) := \begin{cases} \pi(\min(Z_{\min(G_\alpha \setminus (\beta+1))} \setminus (\beta+1)))(\varphi_\alpha(\beta)), & \text{if } Z_{\min(G_\alpha \setminus (\beta+1))} \nsubseteq \beta+1; \\ 0, & \text{otherwise.} \end{cases}$$

Define $c_\alpha : G_\alpha \to \lambda^+$ by letting for all $\beta \in G_\alpha$:

$$c_\alpha(\beta) := \begin{cases} d_\alpha(\beta), & \text{if } \beta < d_\alpha(\beta) < \min(G_\alpha \setminus (\beta+1)); \\ \min(G_\alpha \setminus (\beta+1)), & \text{otherwise.} \end{cases}$$

Finally, let:

$$G^\bullet_\alpha := \operatorname{acc}(G_\alpha) \cup \{c_\alpha(\beta) \mid \beta \in G_\alpha\}.$$

For all $\alpha \in \operatorname{acc}(\lambda^+) \setminus \Gamma$, let $G^\bullet_\alpha$ be a club in α with $\operatorname{otp}(G^\bullet_\alpha) = \operatorname{cf}(\alpha)$ and $\operatorname{nacc}(G^\bullet_\alpha) \subseteq \operatorname{nacc}(\alpha)$. Let $G^\bullet_0 := \emptyset$, and let $G^\bullet_{\alpha+1} := \{\alpha\}$ for all $\alpha < \lambda^+$. Then $\langle G^\bullet_\alpha \mid \alpha < \lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_{\operatorname{cf}(\lambda)}\}, 2, \sigma, \mathcal{E}_\lambda)$ just as in the proof of Theorem 3.6. $\square$

The preceding theorem was focused on limit cardinals. We now establish the same result for λ successor.

Theorem 4.4. *Suppose that* $\boxminus_{\lambda,\geq\chi} + \mathrm{CH}_\lambda$ *holds for a given successor cardinal* λ, *and for some fixed infinite regular cardinal* $\chi \leq \lambda$. *Then:*

1. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \operatorname{cf}(\eta) = \eta < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ *holds for every cardinal* $\theta < \lambda$ *and every ordinal* $\sigma < \lambda$;
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_\eta \mid \aleph_0 \leq \operatorname{cf}(\eta) = \eta < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$ *holds.*

Proof. As λ is uncountable, Fact 8.2 entails $\diamondsuit(\lambda^+)$, so that we only need to establish the corresponding $\mathrm{P}^-(\dots)$ principles of Clauses (1) and (2).

As in Claim 4.3.1, we find sequences $\langle C_\alpha \mid \alpha \in \Gamma\rangle$ and $\langle (S_i, \gamma_i) \mid i \leq \lambda\rangle$ such that:

- $E^{\lambda^+}_{\geq\chi} \subseteq \Gamma \subseteq \mathrm{acc}(\lambda^+)$, and $\Gamma = \biguplus_{i\leq\lambda} S_i$;
- if $\alpha \in \Gamma$, then C_α is a club subset of α of order-type $\leq \lambda$;
- if $\alpha \in S_i$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, then $\bar{\alpha} \in S_i$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- $\{\alpha \in S_i \mid \mathrm{otp}(C_\alpha) = \gamma_i, C_\alpha \subseteq E\}$ is stationary for every $i < \lambda$ and every club $E \subseteq \lambda^+$;
- $\{\gamma_i \mid i < \lambda\}$ is a cofinal subset of λ.

For every $i < \lambda$, write $T_i := \{\delta \in S_i \mid \mathrm{otp}(C_\delta) = \gamma_i\}$. We now go along the lines of the proof of Theorem 3.3 from [Rin14b]. By CH_λ, let $\{X_\gamma \mid \gamma < \lambda^+\}$ be an enumeration of $[\lambda \times \lambda \times \lambda^+]^{\leq\lambda}$. For all $(j, \tau) \in \lambda \times \lambda$ and $X \subseteq \lambda \times \lambda \times \lambda^+$, write $\pi_{j,\tau}(X) := \{\varsigma < \lambda^+ \mid (j, \tau, \varsigma) \in X\}$. Fix a sequence of surjections $\langle \psi_\xi : \lambda \to \xi \mid \xi < \lambda^+\rangle$.

For all $\delta \in \Gamma$ and $j < \lambda$, denote

$$H^j_\delta := \left\{(\eta, \psi_{\min(C_\delta\setminus(\eta+1))}(\iota)) \mid \eta \in C_\delta, \iota < j\right\}.$$

Notice that if $\bar{\delta} \in \mathrm{acc}(C_\delta)$, then $H^j_{\bar{\delta}} = \{(\eta, \gamma) \in H^j_\delta \mid \eta < \bar{\delta}\}$ for all $j < \lambda$.

Claim 4.4.1. *Suppose that $i < \lambda$.*

Then there exist $(j, \tau) \in \lambda \times \lambda$ and $Y \subseteq \lambda^+ \times \lambda^+$ such that for every club $D \subseteq \lambda^+$ and every subset $Z \subseteq \lambda^+$, there exists some $\delta \in T_i$ such that:

1. $\mathrm{dom}(H^j_\delta \setminus Y) = C_\delta \subseteq D$;
2. $H^j_\delta \setminus Y \subseteq \{(\eta, \gamma) \mid Z \cap \eta = \pi_{j,\tau}(X_\gamma)\}$.

Proof. This is Claim 3.3.1 of [Rin14b]. □

Let $\langle (j_i, \tau_i, Y_i) \mid i < \lambda\rangle$ be given by the previous claim. Let $(j_\lambda, \tau_\lambda, Y_\lambda)$ be an arbitrary element of $\lambda \times \lambda \times \mathcal{P}(\lambda^+ \times \lambda^+)$. Then, for all $\delta \in \Gamma$ and $\eta \in C_\delta$, find the unique $i \leq \lambda$ such that $\delta \in S_i$, and put:

$$X_{\eta,\delta} := \bigcup\{\pi_{j_i,\tau_i}(X_\gamma) \mid (\eta, \gamma) \in H^{j_i}_\delta \setminus Y_i\} \cap \eta.$$

Next, for $\delta \in \Gamma$, define $h_\delta : C_\delta \to \delta$ by setting, for all $\eta \in C_\delta$:

$$h_\delta(\eta) := \begin{cases} \min(X_{\eta,\delta} \setminus (\sup(C_\delta \cap \eta) + 1)), & \text{if } X_{\eta,\delta} \nsubseteq \sup(C_\delta \cap \eta) + 1; \\ \eta, & \text{otherwise.} \end{cases}$$

Then, for all $\delta \in \Gamma$, put $G_\delta := \mathrm{Im}(h_\delta)$. For consecutive points $\eta_1 < \eta_2$ in C_δ, notice that $\eta_1 < h_\delta(\eta_2) \leq \eta_2$.[23] Also, $h_\delta(\eta) = \eta$ for any $\eta \in \mathrm{acc}(C_\delta)$. Thus $\mathrm{acc}(G_\delta) = \mathrm{acc}(C_\delta)$ and $\mathrm{nacc}(G_\delta) = h_\delta[\mathrm{nacc}(C_\delta)]$.

[23] Recall that $X_{\eta,\delta} \subseteq \eta$.

Next, we shall use $\diamondsuit(\lambda^+)$ to guess subsets of $\lambda\times\lambda^+$ (rather than subsets of λ^+).[24] More specifically, we fix a matrix $\langle S^\iota_\gamma \mid \iota<\lambda, \gamma<\lambda^+\rangle$ with the property that for every sequence $\langle Z_\iota \mid \iota<\lambda\rangle$ of subsets of λ^+, the following set is stationary:

$$\{\gamma<\lambda^+ \mid \forall\iota<\lambda(Z_\iota\cap\gamma = S^\iota_\gamma)\}.$$

Of course, we may assume that $S^\iota_\gamma\subseteq\gamma$ for all ι,γ.

The next claim is analogous to Claim 4.3.5.

Claim 4.4.2. 1. *for every $\delta\in\Gamma$, G_δ is a club in δ of order-type $\le\lambda$;*

2. *if $\delta\in\Gamma$ and $\bar\delta\in\mathrm{acc}(G_\delta)$, then $\bar\delta\in\Gamma$ and $G_\delta\cap\bar\delta = G_{\bar\delta}$;*
3. *for every sequence $\langle Z_\iota \mid \iota<\lambda\rangle$ of subsets of λ^+, every club $E\subseteq\lambda^+$, and every nonzero limit $\Theta<\lambda$, there exists $\alpha\in\Gamma$ with $\mathrm{otp}(G_\alpha)=\Theta$ such that $\mathrm{nacc}(G_\alpha)\subseteq\{\gamma\in E \mid \forall\iota<\lambda(Z_\iota\cap\gamma=S^\iota_\gamma)\}$.*

Proof. (1) & (2): Just like the proof of Claim 3.3.2 from [Rin14b].

(3): Given $\langle Z_\iota \mid \iota<\lambda^+\rangle$ and E as above, consider the stationary set $Z:=\{\gamma\in E \mid \forall\iota<\lambda(Z_\iota\cap\gamma=S^\iota_\gamma)\}$. Denote $D:=\mathrm{acc}^+(Z)$, which is club in λ^+. Fix a large enough $i<\lambda$ so that $\gamma_i>\Theta$. Recalling that the triple (j_i,τ_i,Y_i) was given by Claim 4.4.1, we may now fix some $\delta\in T_i$ such that:

1. $\mathrm{dom}(H^{j_i}_\delta\setminus Y_i)=C_\delta\subseteq D$;
2. $H^{j_i}_\delta\setminus Y_i\subseteq\{(\eta,\gamma)\mid Z\cap\eta=\pi_{j_i,\tau_i}(X_\gamma)\}$.

Consider any $\eta\in C_\delta$. Since $\eta\in C_\delta\subseteq D=\mathrm{acc}^+(Z)$, we have $\sup(Z\cap\eta)=\eta$. Then, using Clause (2) and the fact that $\eta\in C_\delta=\mathrm{dom}(H^{j_i}_\delta\setminus Y_i)$, it follows that

$$X_{\eta,\delta}=\bigcup\{Z\cap\eta \mid (\eta,\gamma)\in H^{j_i}_\delta\setminus Y_i\}\cap\eta = Z\cap\eta.$$

In particular, if $\eta\in\mathrm{nacc}(C_\delta)$, then $\sup(C_\delta\cap\eta)<\eta=\sup(Z\cap\eta)=\sup(X_{\eta,\delta})$, so that $h_\delta(\eta)\in X_{\eta,\delta}\subseteq Z$. Altogether, it follows that $\mathrm{nacc}(G_\delta)=h_\delta[\mathrm{nacc}(C_\delta)]\subseteq Z$.

Let $\alpha:=G_\delta(\Theta)$. Then $\alpha\in\mathrm{acc}(G_\delta)$, $\mathrm{otp}(G_\alpha)=\Theta$ and $\mathrm{nacc}(G_\alpha)\subseteq\mathrm{nacc}(G_\delta)\subseteq Z$. □

Let $\alpha\in\Gamma$. Let $\pi_\alpha:\mathrm{otp}(G_\alpha)\to G_\alpha$ denote the order-preserving bijection. Define $g_\alpha:\mathrm{otp}(G_\alpha)\to\alpha$ by stipulating:

$$g_\alpha(j):=\begin{cases}\min(S^\iota_{\pi_\alpha(j)}\setminus(\pi_\alpha(\iota)+1)), & \text{if } j=\iota+1 \ \&\ S^\iota_{\pi_\alpha(j)}\not\subseteq\pi_\alpha(\iota)+1;\\ \pi_\alpha(j), & \text{otherwise.}\end{cases}$$

Let $G^\bullet_\alpha:=\mathrm{Im}(g_\alpha)$. For every $i<\mathrm{otp}(G_\alpha)$, $\pi_\alpha(i)<g_\alpha(i+1)\le\pi_\alpha(i+1)$, and for every limit i, $g_\alpha(i)=\pi_\alpha(i)$. Thus, for every $\alpha\in\Gamma$, $G^\bullet_\alpha$ is club in α, $\mathrm{acc}(G^\bullet_\alpha)=\mathrm{acc}(G_\alpha)$, and $\mathrm{otp}(G^\bullet_\alpha)=\mathrm{otp}(G_\alpha)\le\lambda$. Furthermore, $G^\bullet_{\bar\alpha}=G^\bullet_\alpha\cap\bar\alpha$ for every $\alpha\in\Gamma$ and every $\bar\alpha\in\mathrm{acc}(G^\bullet_\alpha)$.

Let $G^\bullet_0:=\emptyset$, and let $G^\bullet_{\alpha+1}:=\{\alpha\}$ for all $\alpha<\lambda^+$. For all $\alpha\in\mathrm{acc}(\lambda^+)\setminus\Gamma$, let $G^\bullet_\alpha$ be a club subset of α of order-type $\mathrm{cf}(\alpha)$ with $\mathrm{nacc}(G^\bullet_\alpha)\subseteq\mathrm{nacc}(\alpha)$.

[24] See Exercise II.51 of [Kun80]

Claim 4.4.3. *For every nonzero limit ordinal $\Theta < \lambda$ and every sequence $\langle B_\iota \mid \iota < \Theta\rangle$ of cofinal subsets of λ^+, there exists some $\alpha \in \Gamma$ such that:*

1. $\mathrm{otp}(G^\bullet_\alpha) = \Theta$*; and*
2. $G^\bullet_\alpha(\iota+1) \in B_\iota$ *for all* $\iota < \Theta$.

Proof. Given a sequence $\langle B_\iota \mid \iota < \Theta\rangle$ as in the hypothesis, let $E := \bigcap_{\iota<\Theta} \mathrm{acc}^+(B_\iota)$, which is club in λ^+. By Claim 4.4.2 (letting $Z_\iota = B_\iota$ for all $\iota < \Theta$), we now fix $\alpha \in \Gamma$ with $\mathrm{otp}(G_\alpha) = \Theta$ such that $\mathrm{nacc}(G_\alpha) \subseteq \{\gamma \in E \mid \forall\iota < \Theta(B_\iota \cap \gamma = S^\iota_\gamma)\}$. In particular, $\mathrm{otp}(G^\bullet_\alpha) = \mathrm{otp}(G_\alpha) = \Theta$. Now, let $\iota < \Theta$ be arbitrary. Denote $\gamma := \pi_\alpha(\iota+1)$, and $\gamma^- := \pi_\alpha(\iota)$. By definition, $g_\alpha(\iota+1)$ is equal to $\min(S^\iota_\gamma \setminus (\gamma^- + 1))$, provided that the latter is nonempty. As $\gamma \in \mathrm{nacc}(G_\alpha)$, we know that $\gamma \in E \subseteq \mathrm{acc}^+(B_\iota)$ and $B_\iota \cap \gamma = S^\iota_\gamma$. Consequently, $G^\bullet_\alpha(\iota+1) = g_\alpha(\iota+1) \in B_\iota$. □

Then, the fact that $\langle G^\bullet_\alpha \mid \alpha < \lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \le \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ and $\mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_\eta \mid \aleph_0 \le \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$ follows from Lemma 3.8. □

The following two corollaries are generalizations of Corollaries 3.9 and 3.10, respectively, which are the special cases of the following when $\chi = \aleph_0$.

Corollary 4.5. *For any uncountable cardinal λ and any infinite regular cardinal $\chi \le \lambda$, the following are equivalent:*

1. $\boxminus_{\lambda,\ge\chi} + \mathrm{CH}_\lambda$;
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \theta, \{E^{\lambda^+}_\eta \mid \aleph_0 \le \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \sigma, \mathcal{E}_\lambda)$ *for every cardinal $\theta < \lambda$ and every ordinal $\sigma < \lambda$;*
3. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, 1, \{E^{\lambda^+}_\eta \mid \aleph_0 \le \mathrm{cf}(\eta) = \eta < \lambda\}, 2, \lambda^+, \mathcal{E}_\lambda)$;
4. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda)$.

Proof. (2) $\implies$ (4) and (3) $\implies$ (4) are immediate by monotonicity of the parameters.

(4) $\implies$ (1): By $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda)$, we have $\diamondsuit(\lambda^+)$, and hence CH_λ holds. The validity of $\boxminus_{\lambda,\ge\chi}$ follows using Lemma 4.1.

(1) $\implies$ (2) & (3): By Theorem 4.3 if λ is a limit cardinal, and Theorem 4.4 if λ is a successor cardinal. □

Corollary 4.6. *For every singular cardinal λ and any infinite regular cardinal $\chi < \lambda$, the following are equivalent:*

1. $\boxminus_{\lambda,\ge\chi} + \mathrm{CH}_\lambda$;
2. $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \sigma, \mathcal{E}_\lambda)$ *for all $\sigma < \lambda$.*

Proof. The forward implication follows from Theorem 4.3(3). The backward implication follows from Corollary 4.5. □

By [She79, Claim 27], if χ is a supercompact cardinal, then for every singular cardinal λ of cofinality $<\chi$, very weak forms of $\square_\lambda$ (such as $\square^*_\lambda$ and even AP_λ) fail. In contrast, we have the following.

Corollary 4.7. *Relative to the existence of a supercompact cardinal, it is consistent that there exists a supercompact cardinal χ such that $\mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_\omega\}, 2, \sigma, \mathcal{E}_\lambda)$ holds for every $\sigma < \lambda$, where $\lambda := \chi^{+\omega}$.*

Proof. Start with a model where χ is a Laver-indestructible supercompact cardinal [Lav78], and CH_λ holds for $\lambda := \chi^{+\omega}$. Denote by $Q(\chi,\lambda)$ the collection of all partial functions $p : \lambda^+ \rightarrow [\lambda^+]^{\leq\lambda}$ satisfying:

- $\mathrm{dom}(p)$ is a bounded subset of λ^+ with some maximal element, which we denote by $m(p)$;
- $\mathrm{dom}(p) \supseteq E^{m(p)}_{\geq\chi}$;
- for all $\alpha \in \mathrm{dom}(p)$:
  - $p(\alpha)$ is a club subset of α of order-type $\leq \lambda$;
  - if $\bar\alpha \in \mathrm{acc}(p(\alpha))$, then $\bar\alpha \in \mathrm{dom}(p)$, and $p(\bar\alpha) = p(\alpha) \cap \bar\alpha$.

We consider $Q(\chi,\lambda)$ as a notion of forcing where for $p, q \in Q(\chi,\lambda)$, q extends p iff $q \sqsupseteq p$.

By CH_λ, we have $|Q(\chi,\lambda)| = \lambda^+$. By virtually the same proof of [CFM01, Lemma 6.1], $Q(\chi,\lambda)$ is $(\lambda+1)$-strategically closed. Thus, altogether $Q(\chi,\lambda)$ preserves cofinalities.

Claim 4.7.1. $V^{Q(\chi,\lambda)} \models \mathrm{P}(\lambda^+, 2, \sqsubseteq_\chi, \lambda^+, \{E^{\lambda^+}_\omega\}, 2, \sigma, \mathcal{E}_\lambda)$ *holds for every $\sigma < \lambda$.*

Proof. We have already noticed that $V^{Q(\chi,\lambda)}$ is a λ-distributive forcing extension of V, and so $V^{Q(\chi,\lambda)} \models \mathrm{CH}_\lambda$. Thus, in light of Theorem 4.3, it suffices to prove that $\Vdash_{Q(\chi,\lambda)} \boxminus_{\lambda,\geq\chi}$. That is, it suffices to prove that $D_\alpha := \{p \in Q(\chi,\lambda) \mid m(p) \geq \alpha\}$ is dense for all $\alpha < \lambda^+$. We do so by induction:

▶ $D_0 = Q(\chi,\lambda)$, which is clearly dense.

▶ Suppose that $\alpha < \lambda^+$ and D_α is dense. We shall show that $D_{\alpha+1}$ is dense. Given $p \in Q(\chi,\lambda)$, we may assume that $p \in D_\alpha$. Now, let $p' := p \cup \{(m(p)+\omega, (m(p), m(p)+\omega))\}$. Then $p' \in D_{\alpha+1}$.

▶ Suppose that $\alpha < \lambda^+$ is a nonzero limit ordinal and D_β is dense for all $\beta < \alpha$. Let $p \in Q(\chi,\lambda)$ be arbitrary. Fix a function $f : \mathrm{cf}(\alpha) \rightarrow \alpha$ whose image is cofinal in α. Clearly, $\mathrm{dom}(f) \leq \lambda$. Now, since $Q(\chi,\lambda)$ is $(\lambda+1)$-strategically closed, use the winning strategy of player II to play a game of length $\mathrm{dom}(f)+1$, producing an increasing sequence of conditions $\langle p_j \mid j < \mathrm{dom}(f)+1\rangle$ so that $p_1 \geq p$ and $p_{2j+1} \in D_{f(j)}$ for all $j < \mathrm{dom}(f)$. Then $p_{\mathrm{dom}(f)}$ is an extension of p that belongs to D_α, showing that D_α is dense. □

Claim 4.7.2. $Q(\chi,\lambda)$ *is $(<\chi)$-directed-closed.*

Proof. Suppose that $D \subseteq Q(\chi,\lambda)$ is a directed family of size $<\chi$. So, for all $p,q\in D$, we know that $p\cup q$ is a condition. Let $d := \bigcup D$. If $\mathrm{dom}(d)$ has a maximal element, then $d\in Q(\chi,\lambda)$, and we are done. Otherwise, for all $\alpha<\delta := \sup(\mathrm{dom}(d))$, we may pick $p_\alpha \in D$ such that $m(p_\alpha)>\alpha$, which must mean that $\mathrm{cf}(\delta)\le |D|<\chi$. So $d\cup\{(\delta+\omega,(\delta,\delta+\omega))\}$ is a legitimate condition that serves as a bound to all elements in D. □

So χ remains supercompact in the extension. □

Lemma 4.8. *Suppose that $\chi<\mathrm{cf}(\kappa)=\kappa$ are infinite cardinals, and $\mathrm{P}^-(\kappa,2,\sqsubseteq_\chi)$ holds.*

Then every stationary subset of $E^\kappa_{\ge\chi}$ may be partitioned into κ-many pairwise disjoint stationary sets such that no two of them reflect simultaneously.

Proof. Let $\langle C_\alpha \mid \alpha<\kappa\rangle$ be a witness to $\mathrm{P}^-(\kappa,2,\sqsubseteq_\chi)$. Suppose that Γ is some stationary subset of $E^\kappa_{\ge\chi}$. Following the proof of [Rin14a, Lemma 3.2], write $G^\tau_i := \{\beta\in\Gamma \mid \mathrm{otp}(C_\beta)>i \ \&\ C_\beta(i)\ge\tau\}$ for all $i,\tau<\kappa$.

Claim 4.8.1. *There exists $i<\kappa$ such that G^τ_i is stationary for all $\tau<\kappa$.*

Proof. Suppose not. Then there exists a function $f:\kappa\to\kappa$ such that $G^{f(i)}_i$ is nonstationary for all $i<\kappa$. For each $i<\kappa$, let D_i be a club subset of $\kappa\setminus G^{f(i)}_i$. Let $D := \{\delta\in\triangle_{i<\kappa}D_i \mid f[\delta]\subseteq\delta\}$, which is club in κ, and put $S := \{\beta\in\Gamma\mid \mathrm{otp}(C_\beta)=\beta\}$. By the first and third triangular bullets in the proof of [Rin14a, Claim 3.2.1], the set $B := \{\beta\in\mathrm{acc}(D)\cap S \mid D\cap\beta\nsubseteq C_\beta\}$ must be empty, and $\mathrm{acc}(D)\cap S$ must be cofinal in κ.

By $B=\emptyset$, for every $\alpha<\beta$ both in $\mathrm{acc}(D)\cap S$, we have $D\cap\beta\subseteq C_\beta$, so that $\alpha\in\mathrm{acc}(D)\cap\beta\subseteq\mathrm{acc}(C_\beta)$ and hence $C_\alpha\sqsubseteq_\chi C_\beta$. As $\mathrm{otp}(C_\beta)=\beta\ge\chi$ for all $\beta\in S$, we infer that $\{C_\delta\mid\delta\in\mathrm{acc}(D)\cap S\}$ is a $\sqsubseteq$-chain, converging to the club $C := \bigcup\{C_\delta\mid\delta\in\mathrm{acc}(D)\cap S\}$. Write $A := \mathrm{acc}(C)$, which is club in κ. As $\langle C_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-(\kappa,2,\sqsubseteq_\chi)$, let us pick some $\beta\in A$ such that $\sup(A\cap\mathrm{nacc}(C_\beta))=\beta$.

By $\beta\in\mathrm{acc}(C)$, we know that $\beta\in\mathrm{acc}(C_\delta)$ for some $\delta\in\mathrm{acc}(D)\cap S$, and then $C\cap\beta=(C\cap\delta)\cap\beta=C_\delta\cap\beta=C_\beta$. So $A\cap\mathrm{nacc}(C_\beta)=\mathrm{acc}(C)\cap\mathrm{nacc}(C\cap\beta)=\emptyset$, contradicting the choice of β. □

Let $i<\kappa$ be given by the preceding claim. Denote $H^\tau := \{\beta\in\Gamma\mid\mathrm{otp}(C_\beta)>i \ \&\ C_\beta(i)=\tau\}$, and $\Theta := \{\tau<\kappa\mid H^\tau \text{ is stationary}\}$. Note that if $\sup(\Theta)<\kappa$, then a pressing-down argument would contradict the choice of i. Thus, $\{H^\tau\mid\tau\in\Theta\}$ is a partition of a subset of Γ into κ-many pairwise disjoint stationary sets. Let $\{S_j\mid j<\kappa\}$ be a partition of Γ such that $|S_j\mathbin{\Delta}H^{\Theta(j)}|\le 1$ for all $j<\kappa$.

Let $j_0<j_1<\kappa$ be arbitrary. Towards a contradiction, suppose that there exists some $\delta<\kappa$ such that $S_{j_0}\cap\delta$ and $S_{j_1}\cap\delta$ are both stationary. Let $\ell<2$. Write $\tau_\ell := \Theta(j_\ell)$. Then $H^{\tau_\ell}\cap\delta$ is a stationary subset of $E^\delta_{\ge\chi}$. In particular, $\mathrm{cf}(\delta)>\chi$ and $\mathrm{acc}(C_\delta)$ is a club in δ. Pick $\beta_\ell\in H^{\tau_\ell}\cap\mathrm{acc}(C_\delta)$. Then $C_{\beta_\ell}\sqsubseteq_\chi C_\delta$. By $\mathrm{otp}(C_\delta)\ge\mathrm{cf}(\delta)>\chi$, we have $C_{\beta_\ell}\sqsubseteq C_\delta$, and hence $C_\delta(i)=C_{\beta_\ell}(i)=\tau_\ell$.

Altogether, $\tau_0=C_\delta(i)=\tau_1$, contradicting the fact that $\tau_0<\tau_1$. □

5. The coherence relation ${}_\lambda\sqsubseteq$

In this section, we deal with the coherence relation ${}_\lambda\sqsubseteq$. Of course, this relation is of particular interest in the context of $\mathrm{P}(\lambda^+,.,{}_\lambda\sqsubseteq,\dots,\mathcal{E}_\lambda)$, because $\mathcal{E}_\lambda$ ensures that all accumulation points of all involved clubs have cofinality $<\lambda$, thereby yielding ${}_\lambda\sqsubseteq$-coherence for free.

However, in applications, ${}_\lambda\sqsubseteq$-coherence is useful even in the absence of $\mathcal{E}_\lambda$. For instance, by [BR17c, Theorem 6.7], $\mathrm{P}(\lambda^+,\lambda^+,{}_\lambda\sqsubseteq,\lambda^+,\{E^{\lambda^+}_\lambda\},2)$ together with $\lambda^{<\lambda}=\lambda$ entails the existence of a λ-complete λ-free λ^+-Souslin tree.

Thus, we mention that in [BR17b] we will show how to obtain $\mathrm{P}(\lambda^+,\lambda^+,{}_\lambda\sqsubseteq,\dots)$ in various scenarios where we cannot guarantee a bound on the order-type of the witnessing clubs.

Theorem 5.1. *Suppose that λ is any infinite cardinal, and $S\subseteq\lambda^+$ is stationary.*

1. *The following are equivalent:*
   (a) $\clubsuit_w(S)$;
   (b) $\mathrm{P}^-(\lambda^+,2,{}_\lambda\sqsubseteq,1,\{S\},2,\lambda,\mathcal{E}_\lambda)$;
   (c) $\mathrm{P}^-(\lambda^+,2,{}_\lambda\sqsubseteq,1,\{S\},2,\lambda^+,\mathcal{E}_\lambda)$;
   (d) $\mathrm{P}^-(\lambda^+,2,\mathcal{R},1,\{S\},2,\lambda^+)$ *for some* $\mathcal{R}$;
   (e) $\mathrm{P}^-(\lambda^+,2,{}_{\lambda^+}\sqsubseteq,1,\{S\},2,\lambda^+)$.
2. *The following are equivalent:*
   (a) $\diamondsuit(S)$;
   (b) $\mathrm{P}(\lambda^+,2,{}_\lambda\sqsubseteq,1,\{S\},2,\lambda,\mathcal{E}_\lambda)$;
   (c) $\mathrm{P}(\lambda^+,2,{}_\lambda\sqsubseteq,1,\{S\},2,\lambda^+,\mathcal{E}_\lambda)$;
   (d) $\mathrm{P}(\lambda^+,2,\mathcal{R},1,\{S\},2,\lambda^+)$ *for some* $\mathcal{R}$;
   (e) $\mathrm{P}(\lambda^+,2,{}_{\lambda^+}\sqsubseteq,1,\{S\},2,\lambda^+)$.

Proof. Clause (2) follows from Clause (1) together with the equivalence (1) $\iff$ (2) of Fact 8.5. Thus, let us prove Clause (1).

(a) $\implies$ (b) Let $\langle X_\alpha\mid\alpha\in S\rangle$ be as in Definition 8.4. Let $C_\alpha:=X_\alpha\cup\mathrm{acc}^+(X_\alpha)$ for all limit $\alpha\in S$. Let C_α be a club subset of α of order-type $\mathrm{cf}(\alpha)$ for all limit $\alpha\in\lambda^+\setminus S$. Let $C_{\alpha+1}:=\{\alpha\}$ for all $\alpha<\lambda^+$. For all limit $\alpha<\lambda^+$, it is clear that C_α is a club subset of α of order-type $\mathrm{cf}(\alpha)\le\lambda$. If $\alpha<\lambda^+$ and $\bar\alpha\in\mathrm{acc}(C_\alpha)$, then $\mathrm{cf}(\sup(C_{\bar\alpha}))=\mathrm{cf}(\bar\alpha)\le\mathrm{otp}(C_\alpha\cap\bar\alpha)<\mathrm{otp}(C_\alpha)=\mathrm{cf}(\alpha)\le\lambda$, so that automatically $C_{\bar\alpha}\ {}_\lambda\sqsubseteq C_\alpha$.

To see that $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+,2,{}_\lambda\sqsubseteq,1,\{S\},2,\lambda,\mathcal{E}_\lambda)$, fix an arbitrary cofinal subset A_0 of λ^+ and some club $D\subseteq\lambda^+$. Define $f:\lambda^+\to\lambda^+$ by recursion over $\alpha<\lambda^+$:

- $f(0):=\min(A_0)$; and
- for nonzero $\alpha<\lambda^+$, $f(\alpha):=\min(A_0\setminus(\min(D\setminus(\sup(f[\alpha])+1))+1))$.

Write $X := \mathrm{Im}(f)$. Then X is a cofinal subset of A_0 (thus also of λ^+) and has the property that for all $\beta < \gamma$ in X, the ordinal-interval (β, γ) contains an element from D. Pick a limit $\alpha \in S$ such that $\sup(X_\alpha \setminus X) < \alpha$. In particular, $\alpha \in \mathrm{acc}^+(X) \subseteq \mathrm{acc}^+(D) \subseteq D$. Let $\gamma = \sup(X_\alpha \setminus X) + 1$. Then $\mathrm{nacc}(C_\alpha) \setminus \gamma = \mathrm{nacc}(X_\alpha) \setminus \gamma \subseteq X_\alpha \setminus \gamma \subseteq X \subseteq A_0$. Thus, $\mathrm{succ}_\lambda(C_\alpha \setminus \beta) \subseteq A_0$ whenever $\gamma \leq \beta < \alpha$.

(b) $\implies$ (c) Any witness to $\mathrm{P}^-(\dots, \lambda, \mathcal{E}_\lambda)$ forms a witness $\mathrm{P}^-(\dots, \lambda^+, \mathcal{E}_\lambda)$, by virtue of the eighth parameter.

(c) $\implies$ (d) Immediate, by taking $\mathcal{R} := {}_\lambda\sqsubseteq$.

(d) $\implies$ (e) The relation ${}_{\lambda^+}\sqsubseteq$ provides no coherence information whatsoever.

(e) $\implies$ (a) Let $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ be a witness to $\mathrm{P}^-(\lambda^+, 2, {}_{\lambda^+}\sqsubseteq, 1, \{S\}, 2, \lambda^+)$. For all limit $\alpha \in S$, let X_α be a cofinal subset of $\mathrm{nacc}(C_\alpha)$ of order-type $\mathrm{cf}(\alpha)$. For successor $\alpha \in S$, choose X_α arbitrarily. Then $\langle X_\alpha \mid \alpha \in S\rangle$ witnesses $\clubsuit_w(S)$. □

Notice that likewise, if S is a stationary subset of an inaccessible cardinal κ, then $\clubsuit_w(S) \iff \mathrm{P}^-(\kappa, 2, {}_\kappa\sqsubseteq, 1, \{S\}, 2, \kappa)$, and $\diamondsuit(S) \iff \mathrm{P}(\kappa, 2, {}_\kappa\sqsubseteq, 1, \{S\}, 2, \kappa)$.

Lemma 5.2. *Suppose that λ is an uncountable cardinal, and S is a stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$. Then the following are equivalent:*

1. *$\langle\lambda\rangle^-_S$ holds;*
2. *There exist sequences $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A_\beta \mid \beta < \lambda^+\rangle$ such that:*
   (a) *for all $\delta \in S$, C_δ is a club subset of δ of order-type λ;*
   (b) *for every club $D \subseteq \lambda^+$, every subset $A \subseteq \lambda^+$, and every infinite regular $\sigma < \lambda$ with $\sigma \neq \mathrm{cf}(\lambda)$, there exist stationarily many $\delta \in S$ for which:*

$$\mathrm{otp}(\{\beta \in \mathrm{nacc}(C_\delta) \cap D \cap E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}) = \lambda.$$

In particular, $\langle\lambda\rangle^-_S$ entails $\diamondsuit(\lambda^+)$.

Proof. (2) $\implies$ (1): Let $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A_\beta \mid \beta < \lambda^+\rangle$ be as in (2). Let $\delta \in S$ be arbitrary. Put $A^\delta_i := A_{C_\delta(i)}$ for all $i < \lambda$. Evidently, $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A^\delta_i \mid \delta \in S, i < \lambda\rangle$ together witness $\langle\lambda\rangle^-_S$.

(1) $\implies$ (2): Fix $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A^\delta_i \mid \delta \in S, i < \lambda\rangle$ witnessing $\langle\lambda\rangle^-_S$. We commence by proving that $\mathcal{P}(\lambda) \subseteq \{A^\delta_{i+1} \mid \delta \in S, i < \lambda\}$, thus establishing that CH_λ holds.

Let $A \subseteq \lambda$ be arbitrary. In particular, $A \subseteq \lambda^+$, so we can fix $\delta \in S$ above λ such that $B := \{i < \lambda \mid A \cap (C_\delta(i+1)) = A^\delta_{i+1}\}$ is cofinal in λ. Then $\{C_\delta(i+1) \mid i \in B\}$ is cofinal in C_δ and therefore in δ, so we can choose $i \in B$ such that $\lambda < C_\delta(i+1) < \delta$. Since $A \subseteq \lambda$ and $i \in B$, it follows that $A = A \cap (C_\delta(i+1)) = A^\delta_{i+1}$.

By CH_λ, $\lambda > \aleph_0$ and Fact 8.2, let us fix a sequence $\langle A_\beta \mid \beta < \lambda^+\rangle$ such that for every $A \subseteq \lambda^+$ and every infinite regular $\sigma < \lambda$ with $\sigma \neq \mathrm{cf}(\lambda)$, the set $\{\beta \in E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}$ is stationary.

Let $\delta \in S$ be arbitrary. Define $d_\delta : C_\delta \to \delta$ by setting, for every $\beta \in C_\delta$:

$$d_\delta(\beta) := \min(\{\beta\} \cup (A^\delta_{\mathrm{otp}(C_\delta \cap \beta)} \setminus (\sup(C_\delta \cap \beta) + 1))).$$

Then define

$$C^\bullet_\delta := \{d_\delta(\beta) \mid \beta \in C_\delta\}.$$

Clearly $C^\bullet_\delta$ is a club subset of δ of order-type λ, and $\mathrm{acc}(C^\bullet_\delta) = \mathrm{acc}(C_\delta)$. Furthermore:

Claim 5.2.1. *For every cofinal subset $Z \subseteq \lambda^+$, there exist stationarily many $\delta \in S$ such that*

$$\mathrm{otp}(\mathrm{nacc}(C^\bullet_\delta) \cap Z) = \lambda.$$

Proof. Let $Z \subseteq \lambda^+$ be an arbitrary cofinal set. Let $D := \mathrm{acc}^+(Z)$, which is club in λ^+, and let $A := Z$. From the fact that $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A^\delta_i \mid \delta \in S, i < \lambda\rangle$ witness $\langle\lambda\rangle^-_S$, we obtain stationarily many $\delta \in S$ such that $\mathrm{otp}(H) = \lambda$, where

$$H := \{\beta \in \mathrm{nacc}(C_\delta) \cap D \mid A \cap \beta = A^\delta_{\mathrm{otp}(C_\delta \cap \beta)}\}.$$

Let $\beta \in H$ be arbitrary. Then $Z \cap \beta = A^\delta_{\mathrm{otp}(C_\delta \cap \beta)}$. Since $\beta \in \mathrm{nacc}(C_\delta)$ and $\beta \in D = \mathrm{acc}^+(Z)$, the set $A^\delta_{\mathrm{otp}(C_\delta \cap \beta)} \setminus (\sup(C_\delta \cap \beta) + 1) = Z \cap (\sup(C_\delta \cap \beta), \beta)$ is nonempty, and thus it contains $d_\delta(\beta)$. It follows that $d_\delta[H] \subseteq \mathrm{nacc}(C^\bullet_\delta) \cap Z$, showing that $\mathrm{otp}(\mathrm{nacc}(C^\bullet_\delta) \cap Z) = \lambda$, as required. □

Given a club $D \subseteq \lambda^+$, a subset $A \subseteq \lambda^+$, and an infinite regular $\sigma < \lambda$ with $\sigma \neq \mathrm{cf}(\lambda)$, the set $Z := \{\beta \in D \cap E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}$ is cofinal in λ^+ (indeed, even stationary), and hence by the previous claim there exist stationarily many $\delta \in S$ such that $\mathrm{otp}(\mathrm{nacc}(C^\bullet_\delta) \cap Z) = \lambda$, as sought. □

Theorem 5.3. *Suppose that CH_λ holds for a given regular uncountable cardinal λ, and $S \subseteq E^{\lambda^+}_\lambda$ is stationary. Then for any uncountable cardinal $\chi \leq \lambda$, $\curlywedge^-(\chi, S)$ entails $\langle\lambda\rangle^-_S$.*

Note. Recall that Fact 8.9 already dealt with the case where $\chi < \lambda$ or λ is a successor cardinal. Here, we give a different proof that covers also the hardest case where $\chi = \lambda$ is inaccessible.

Proof. Let $\langle \mathcal{C}^i_\delta \mid \delta \in S, i < \lambda\rangle$ be a witness to $\curlywedge^-(\chi, S)$. Let $\delta \in S$ and $i < \lambda$ be arbitrary. Fix $f^i_\delta : \delta \to [\delta]^{<\chi}$ with the property that $\{\alpha, \alpha + 1\} \subseteq f^i_\delta(\alpha) \in \mathcal{C}^i_\delta$ for all $\alpha < \delta$. Let $\langle e^j_\delta \mid j < \lambda\rangle$ be the increasing enumeration of some club subset of δ. Define a function $c^i_\delta : \lambda \to [\delta]^{<\chi}$ by recursion:

- $c^i_\delta(0) := \emptyset$;
- $c^i_\delta(j + 1) := f^i_\delta(\sup(c^i_\delta(j))) \setminus \sup(c^i_\delta(j))$;
- $c^i_\delta(j) := \{\max\{e^j_\delta, \sup(\bigcup c^i_\delta[j])\}\}$ for nonzero limit $j < \lambda$.

Finally, let C^i_δ denote the closure in δ of $\bigcup \mathrm{Im}(c^i_\delta)$. Then C^i_δ is a club in δ of order-type λ.

Claim 5.3.1. *For every function $f : \lambda^+ \to \lambda^+$ and every club $D \subseteq \lambda^+$, there exist some $\delta \in S$ and $i < \lambda$ with*

$$\sup\{\alpha \in C^i_\delta \cap D \cap E^{\lambda^+}_\lambda \mid f(\alpha) \in f^i_\delta(\alpha)\} = \delta.$$

Proof. Given f and D as in the hypothesis, define $g : [\lambda^+]^{<\omega} \to \lambda^+$ by stipulating

$$g(\sigma) := \begin{cases} 0, & \text{if } \sigma = \emptyset; \\ \min(D \cap E^{\lambda^+}_\lambda \setminus (\beta+1)), & \text{if } \sigma = \{\beta\}; \\ f(\min(\sigma)), & \text{otherwise.} \end{cases}$$

Since χ is uncountable, $\mathcal{D} := \{x \in [\lambda^+]^{<\chi} \mid g``[x]^{<\omega} \subseteq x\}$ is a club subset of $[\lambda^+]^{<\chi}$, so (using the fact that $\langle \mathcal{C}^i_\delta \mid \delta \in S, i < \lambda\rangle$ witnesses $\curlywedge^-(\chi, S)$) let us pick $\delta \in S$ and $i < \lambda$ with $\mathcal{C}^i_\delta \subseteq \mathcal{D}$. Let $\beta < \delta$ be arbitrary. We shall find $\alpha \in C^i_\delta \cap D \cap E^{\lambda^+}_\lambda$ above β such that $f(\alpha) \in f^i_\delta(\alpha)$. Fix some nonzero limit $j < \lambda$ such that $e^j_\delta > \beta$. Then $\beta' := \sup(c^i_\delta(j)) \geq e^j_\delta > \beta$, and $f^i_\delta(\beta') \setminus \beta' = c^i_\delta(j+1) \subseteq C^i_\delta$.

By $\beta' \in f^i_\delta(\beta') \in \mathcal{C}^i_\delta \subseteq \mathcal{D}$, we have that $\alpha := g(\{\beta'\}) = \min(D \cap E^{\lambda^+}_\lambda \setminus (\beta'+1))$ is in $f^i_\delta(\beta') \setminus \beta'$, and therefore in C^i_δ. Thus, we have found an $\alpha \in D \cap E^{\lambda^+}_\lambda \cap C^i_\delta$ above β. Finally, since $\{\alpha, \alpha+1\} \subseteq f^i_\delta(\alpha) \in \mathcal{C}^i_\delta \subseteq \mathcal{D}$, we get $f(\alpha) = g(\{\alpha, \alpha+1\}) \in g``[f^i_\delta(\alpha)]^{<\omega} \subseteq f^i_\delta(\alpha)$, as sought. □

Invoking CH_λ, let $\{h_\beta \mid \beta < \lambda^+\}$ be some enumeration of ${}^\lambda\lambda^+$. For all $\delta \in S$ and $i < \lambda$, define $g^i_\delta : \delta \to [\delta]^{<\chi}$ by stipulating:

$$g^i_\delta(\alpha) := \{h_\beta(i) \mid \beta \in f^i_\delta(\alpha)\} \cap \delta.$$

Claim 5.3.2. *There exists $i < \lambda$ such that for every function $f : \lambda^+ \to \lambda^+$ and every club $D \subseteq \lambda^+$, there exist some $\delta \in S$ with*

$$\sup\{\alpha \in C^i_\delta \cap D \cap E^{\lambda^+}_\lambda \mid f(\alpha) \in g^i_\delta(\alpha)\} = \delta.$$

Proof. Suppose not, and pick, for every $i < \lambda$, a counterexample (f_i, D_i). Define $f : \lambda^+ \to \lambda^+$ by letting for all $\alpha < \lambda^+$:

$$f(\alpha) := \min\{\beta < \lambda^+ \mid h_\beta = \langle f_i(\alpha) \mid i < \lambda\rangle\}.$$

Let $D := \bigcap_{i<\lambda}\{\delta \in D_i \mid f_i[\delta] \subseteq \delta\}$, which is club in λ^+. Using Claim 5.3.1, pick $\delta \in S$ and $i < \lambda$ such that

$$\Delta := \{\alpha \in C^i_\delta \cap D \cap E^{\lambda^+}_\lambda \mid f(\alpha) \in f^i_\delta(\alpha)\}$$

is cofinal in δ. In particular, $\delta = \sup(\Delta) \in \mathrm{acc}^+(D) \subseteq D$, so that $f_i[\Delta] \subseteq f_i[\delta] \subseteq \delta$. Consider an arbitrary $\alpha \in \Delta$, and let $\beta := f(\alpha)$. Then $\alpha \in D \subseteq D_i$ and $\beta \in f^i_\delta(\alpha)$, so that $f_i(\alpha) = h_\beta(i) \in g^i_\delta(\alpha)$. Altogether,

$$\{\alpha \in C^i_\delta \cap D_i \cap E^{\lambda^+}_\lambda \mid f_i(\alpha) \in g^i_\delta(\alpha)\}$$

contains the cofinal subset Δ of δ, contradicting the choice of the pair (f_i, D_i). □

Let $i < \lambda$ be given by the previous claim. For notational simplicity, denote C^i_δ by C_δ. For every $\alpha < \lambda^+$, again invoking CH_λ, let $\{X^\beta_\alpha \mid \beta < \lambda^+\}$ be some enumeration (possibly with repetition) of all subsets of α.

Consider an arbitrary $\delta \in S$. Let $g_\delta : \delta \to {}^{<\chi}\delta$ be such that for every $\alpha < \delta$, $g_\delta(\alpha)$ is a surjection from some cardinal $< \chi$ to the set $\{\beta \in g^i_\delta(\alpha) \mid \sup(X^\beta_\alpha) = \alpha\}$. As $\mathrm{otp}(C_\delta) = \lambda$, we have $C_\delta \cap E^{\lambda^+}_\lambda \subseteq \mathrm{nacc}(C_\delta)$, so, for all $\alpha \in C_\delta \cap E^{\lambda^+}_\lambda$, let us define a strictly increasing and continuous function $\varphi_{\delta,\alpha} : (\mathrm{dom}(g_\delta(\alpha)) + 1) \to \alpha$ such that:

- $\varphi_{\delta,\alpha}(0) := \sup(C_\delta \cap \alpha)$, and
- $\varphi_{\delta,\alpha}(\xi + 1) := \min(X^{g_\delta(\alpha)(\xi)}_\alpha \setminus (\varphi_{\delta,\alpha}(\xi) + 1))$ for all $\xi \in \mathrm{dom}(g_\delta(\alpha))$.

Finally, put $D_\delta := C_\delta \cup \bigcup\{\mathrm{Im}(\varphi_{\delta,\alpha}) \mid \alpha \in C_\delta \cap E^{\lambda^+}_\lambda\}$. Clearly, D_δ is a club in δ of order-type λ.

By CH_λ, $\lambda > \aleph_0$ and Fact 8.2, let us fix a sequence $\langle A_\gamma \mid \gamma < \lambda^+\rangle$ such that for every $A \subseteq \lambda^+$ and every infinite regular $\sigma < \lambda$, the set $\{\gamma \in E^{\lambda^+}_\sigma \mid A \cap \gamma = A_\gamma\}$ is stationary.

Claim 5.3.3. *For every club $D \subseteq \lambda^+$, every subset $A \subseteq \lambda^+$, and every infinite regular $\sigma < \lambda$, there exist stationarily many $\delta \in S$ for which:*

$$\sup\{\gamma \in \mathrm{nacc}(D_\delta) \cap D \cap E^{\lambda^+}_\sigma \mid A \cap \gamma = A_\gamma\} = \delta.$$

Proof. Given arbitrary $A \subseteq \lambda^+$, clubs $D, E \subseteq \lambda^+$ and infinite regular $\sigma < \lambda$, we shall find $\delta \in E \cap S$ such that $\sup\{\gamma \in \mathrm{nacc}(D_\delta) \cap D \cap E^{\lambda^+}_\sigma \mid A \cap \gamma = A_\gamma\} = \delta$. Let $G := \{\gamma \in D \cap E^{\lambda^+}_\sigma \mid A \cap \gamma = A_\gamma\}$. By our choice of $\langle A_\gamma \mid \gamma < \lambda^+\rangle$, G is stationary. Choose $f : \lambda^+ \to \lambda^+$ so that for all $\alpha < \lambda^+$, $f(\alpha)$ is some $\beta < \lambda^+$ such that $G \cap \alpha = X^\beta_\alpha$. Put $D' := \mathrm{acc}^+(G) \cap E$, which is club. By the choice of i, we may find some $\delta \in S$ such that

$$\Delta := \{\alpha \in C_\delta \cap D' \cap E^{\lambda^+}_\lambda \mid f(\alpha) \in g^i_\delta(\alpha)\}$$

is cofinal in δ. In particular, $\delta = \sup(\Delta) \in \mathrm{acc}^+(D') \subseteq D' \subseteq E$. Consider arbitrary $\alpha \in \Delta$. By $\alpha \in D' \subseteq \mathrm{acc}^+(G)$, $\alpha = \sup(G \cap \alpha) = \sup(X^{f(\alpha)}_\alpha)$, and since $\alpha \in \Delta$, we have $f(\alpha) \in g^i_\delta(\alpha)$. Altogether, $f(\alpha) \in \mathrm{Im}(g_\delta(\alpha))$. Thus, there exists some $\xi \in \mathrm{dom}(g_\delta(\alpha))$ such that $g_\delta(\alpha)(\xi) = f(\alpha)$. Then $\varphi_{\delta,\alpha}(\xi + 1) \in \mathrm{nacc}(D_\delta) \cap G \setminus \sup(C_\delta \cap \alpha)$. Consequently, $\sup(\mathrm{nacc}(D_\delta) \cap G) = \delta$. □

Since $\mathrm{cf}(\delta) = \lambda$, it now follows from Lemma 5.2 that $\langle\lambda\rangle^-_S$ holds. □

Theorem 5.4. *Suppose that λ is an uncountable cardinal, and $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is stationary.*
Then $\langle\lambda\rangle^-_S$ entails $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$ for any regular $\sigma < \lambda$.

Proof. By Lemma 5.2, $\diamondsuit(\lambda^+)$ holds, and so it suffices to establish $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$. Let $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A_\beta \mid \beta < \lambda^+\rangle$ be given by Lemma 5.2(2). Let $\sigma < \lambda$ be an arbitrary infinite regular cardinal. By replacing σ with σ^+ if necessary, we may assume that $\sigma \neq \mathrm{cf}(\lambda)$ (and still $\sigma < \lambda$).

Fix a bijection $\pi : \lambda^+ \times \lambda^+ \leftrightarrow \lambda^+$, and let $E := \{\gamma < \lambda^+ \mid \pi[\gamma \times \gamma] = \gamma\}$ denote its club of closure points.

Let $D_0 := \emptyset$, and for every $\delta < \lambda^+$, let $D_{\delta+1} := \{\delta\}$. For every $\delta \in \mathrm{acc}(\lambda^+) \setminus S$, let D_δ be an arbitrary club subset of δ of order-type $\mathrm{cf}(\delta)$.

Next, let $\delta \in S$ be arbitrary. Define

$$N_\delta := \{\beta \in \mathrm{nacc}(C_\delta) \cap E \mid \text{for all } i, \gamma < \beta, \text{ there exists } \tau \in \beta \setminus \gamma \text{ with } \pi(i, \tau) \in A_\beta\}.$$

Define $o_\delta : \mathrm{nacc}(C_\delta) \to \lambda$ by letting for all $\gamma \in \mathrm{nacc}(C_\delta)$:

$$o_\delta(\gamma) := \mathrm{otp}(\{\beta \in N_\delta \cap E^\gamma_\sigma \mid A_\beta = A_\gamma \cap \beta\}).$$

Fix a surjection $f_\delta : \lambda \to \delta$ such that the preimage of any singleton is cofinal in λ, and then define $g_\delta : \mathrm{nacc}(C_\delta) \to \delta$ by letting for all $\beta \in \mathrm{nacc}(C_\delta)$:

$$g_\delta(\beta) := \begin{cases} f_\delta(o_\delta(\beta)), & \text{if } f_\delta(o_\delta(\beta)) < \beta; \\ 0, & \text{otherwise.} \end{cases}$$

For all $\beta \in \mathrm{nacc}(C_\delta)$, let $H^\delta_\beta := \{\tau \mid \sup(C_\delta \cap \beta) < \tau < \beta \ \&\ \pi(g_\delta(\beta), \tau) \in A_\beta\}$. Then let F^δ_β be the closure of $\mathrm{succ}_\sigma(H^\delta_\beta)$. Clearly, F^δ_β is a closed subset of $(\sup(C_\delta \cap \beta), \beta]$ of order-type $\leq \sigma + 1$. Finally, let

$$D_\delta := C_\delta \cup \bigcup\{F^\delta_\beta \mid \beta \in \mathrm{nacc}(C_\delta)\}.$$

As $\sigma < \lambda$, $(\sigma + 1) \cdot \lambda = \lambda$, and so D_δ is a club subset of δ of order-type λ.

Claim 5.4.1. *$\langle D_\delta \mid \delta < \lambda^+\rangle$ witnesses* $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$.

Proof. As $\mathrm{otp}(D_\delta) \leq \lambda$ for all $\delta < \lambda^+$, the verification of ${}_\lambda\sqsubseteq$ becomes trivial. Now, given a sequence $\langle X_i \mid i < \lambda^+\rangle$ of cofinal subsets of λ^+, we shall seek stationarily many $\delta \in S$ such that for every $i < \delta$,

$$\sup\{\beta \in D_\delta \mid \mathrm{succ}_\sigma(D_\delta \setminus \beta) \subseteq X_i\} = \delta.$$

Consider the club $D := E \cap \bigtriangleup_{i<\lambda^+}(\mathrm{acc}^+(X_i))$, and the set $A := \{\pi(i, \tau) \mid i < \lambda^+, \tau \in X_i\}$. Then, the set G of all $\delta \in S$ such that

$$M_\delta := \{\beta \in \mathrm{nacc}(C_\delta) \cap D \cap E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\} \text{ has order-type } \lambda,$$

is stationary.

Let δ be an arbitrary element of the stationary set G. Let us first show that

$$M_\delta = \{\beta \in N_\delta \cap E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}:$$

Let $\beta \in M_\delta$ be arbitrary. Then $\beta \in D \subseteq E$ and $A_\beta = A \cap \beta$. By $\beta \in D$, we also have $\beta \in \bigcap_{i<\beta} \mathrm{acc}^+(X_i)$. Thus, for all $i, \gamma < \beta$, there is some $\tau \in X_i \cap (\beta \setminus \gamma)$, so that $\pi(i, \tau) \in A$, and (since $\beta \in E$) $\pi(i, \tau) < \beta$, giving $\pi(i, \tau) \in A \cap \beta = A_\beta$. Thus $\beta \in N_\delta$.

Conversely, suppose $\beta \in N_\delta \cap E^{\lambda^+}_\sigma$ satisfies $A \cap \beta = A_\beta$. We have $\beta \in N_\delta \subseteq \mathrm{nacc}(C_\delta) \cap E$, so it remains to show that $\beta \in \bigtriangleup_{i<\lambda^+}(\mathrm{acc}^+(X_i))$. Consider any $i, \gamma < \beta$. Since $\beta \in N_\delta$, we can fix $\tau \in \beta \setminus \gamma$ such that $\pi(i,\tau) \in A_\beta = A \cap \beta$. Then $\tau \in X_i \cap (\beta \setminus \gamma)$, as required. Thus $\beta \in M_\delta$.

For any $\gamma \in M_\delta$,

$$\begin{aligned} o_\delta(\gamma) &= \mathrm{otp}\left(\{\beta \in N_\delta \cap E^\gamma_\sigma \mid A_\beta = A_\gamma \cap \beta\}\right) \\ &= \mathrm{otp}\left(\{\beta \in N_\delta \cap E^\gamma_\sigma \mid A_\beta = (A \cap \gamma) \cap \beta\}\right) \\ &= \mathrm{otp}(M_\delta \cap \gamma). \end{aligned}$$

Since $\mathrm{otp}(M_\delta) = \lambda$, it follows that $o_\delta[M_\delta] = \lambda$. Also, $\sup(M_\delta) = \sup(C_\delta) = \delta$.

Finally, let $i, \eta < \delta$ be arbitrary. We shall find $\beta' \in D_\delta \setminus \eta$ such that $\mathrm{succ}_\sigma(D_\delta \setminus \beta') \subseteq X_i$. Fix a large enough $\beta \in M_\delta$ such that $i, \eta < \sup(C_\delta \cap \beta)$ and $f_\delta(o_\delta(\beta)) = i$. Then $g_\delta(\beta) = i$. since $\beta \in M_\delta$, we get that $H^\delta_\beta = \{\tau \mid \sup(C_\delta \cap \beta) < \tau < \beta, \tau \in X_i \cap \beta\}$ is cofinal in β, so that $\mathrm{otp}(D_\delta \cap (\sup(C_\delta \cap \beta), \beta)) \geq \mathrm{cf}(\beta) = \sigma$. Set $\beta' := \sup(C_\delta \cap \beta)$. Then $\beta' \in D_\delta \setminus \eta$ and $\mathrm{succ}_\sigma(D_\delta \setminus \beta') \subseteq H^\delta_\beta \subseteq X_i$, as sought. □

□

Corollary 5.5. *For every regular uncountable cardinal λ and stationary $S \subseteq E^{\lambda^+}_\lambda$, the following are equivalent:*

1. $\langle\lambda\rangle^-_S$;
2. $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$ *for every regular cardinal $\sigma < \lambda$;*
3. $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, 1, \{S\}, 2, 1, \mathcal{E}_\lambda)$.

Proof. (1) ⇒ (2) By Theorem 5.4.

(3) ⇒ (1) By the hypothesis $\mathrm{P}(\dots)$, $\diamondsuit(\lambda^+)$ holds, so by Fact 8.2, let us fix a sequence $\langle A_\beta \mid \beta < \lambda^+\rangle$ such that for every $A \subseteq \lambda^+$ and every infinite regular $\sigma < \lambda$, the set $\{\beta \in E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}$ is stationary. Let $\langle C_\delta \mid \delta < \lambda^+\rangle$ be a witness to $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\sqsubseteq, 1, \{S\}, 2, 1, \mathcal{E}_\lambda)$. We verify that $\langle C_\delta \mid \delta \in S\rangle$ and $\langle A_\beta \mid \beta < \lambda^+\rangle$ satisfy Clause (2) of Lemma 5.2:

Given a club $D \subseteq \lambda^+$, a subset $A \subseteq \lambda^+$, and an infinite regular $\sigma < \lambda$, the set $Z := \{\beta \in D \cap E^{\lambda^+}_\sigma \mid A \cap \beta = A_\beta\}$ is cofinal in λ^+ (indeed, even stationary). Hence, there exist stationarily many $\delta \in S$ such that $\mathrm{otp}(\mathrm{nacc}(C_\delta) \cap Z) = \lambda$, as sought. □

Theorem 5.6. *Suppose that λ is an infinite cardinal, and $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is stationary.*

If $\diamondsuit(S)$ holds, then so does $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$ for every ordinal $\sigma < \lambda$.

Proof. Recalling Theorem 3.7, we may assume that λ is uncountable.

For regular cardinals $\sigma < \lambda$, we could have simply used Fact 8.9 together with Theorem 5.4, but let us a give a proof that works for all cases.

Let λ be an arbitrary uncountable cardinal, and let $\sigma < \lambda$ be some nonzero ordinal. By $\diamondsuit(S)$, we have $\diamondsuit(\lambda^+)$, and it remains to establish $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$.

Using $\diamondsuit(S)$, fix a sequence $\langle (X_\alpha, Y_\alpha) \mid \alpha < \lambda^+\rangle$ such that for all $X, Y \subseteq \lambda^+$ the set $\{\alpha \in S \mid X \cap \alpha = X_\alpha \ \&\ Y \cap \alpha = Y_\alpha\}$ is stationary. Of course, we may assume that X_α and Y_α are subsets of α. Denote

$$H_\alpha := \{\gamma \in Y_\alpha \mid X_\alpha \cap \gamma = X_\gamma \ \&\ Y_\alpha \cap \gamma = Y_\gamma\}.$$

Let $D_0 := \emptyset$ and $D_{\alpha+1} := \{\alpha\}$ for all $\alpha < \lambda^+$. Next, for every nonzero limit $\alpha < \lambda^+$, we do the following. If there exists a club D_α in α of order-type λ such that $\mathrm{nacc}(D_\alpha) \subseteq H_\alpha$, we let D_α be such a club. Otherwise, we let D_α be any club subset of α of order-type $\mathrm{cf}(\alpha)$. Just as in Theorem 3.6, write $\chi := \omega \cdot \lambda$. Of course, since we have assumed λ is uncountable, it follows that $\chi = \lambda$.

Claim 5.6.1. 1. *for every limit $\alpha < \lambda^+$, D_α is a club in α of order-type $\leq \chi$;*

2. *if $\bar\alpha \in \mathrm{acc}(D_\alpha)$, then $D_{\bar\alpha}\, {}_\lambda\!\sqsubseteq D_\alpha$;*
3. *for every subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists a limit $\alpha \in S$ such that $\mathrm{otp}(D_\alpha) = \chi$ and $\mathrm{nacc}(D_\alpha) \subseteq \{\gamma \in E \cap S \mid X \cap \gamma = X_\gamma\}$.*

Proof. (1) is trivial.

(2) follows from (1) and the fact that $\chi = \lambda$.

(3) Fix $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$. Define

$$G := \{\gamma \in E \cap S \mid X \cap \gamma = X_\gamma \ \&\ E \cap S \cap \gamma = Y_\gamma\}.$$

By applying our diamond sequence to X and $Y := E \cap S$, we find that G is a stationary subset of λ^+, being the intersection of the club set E with a stationary set. Thus, in particular, $Z := \{\alpha < \lambda^+ \mid \mathrm{otp}(G \cap \alpha) = \alpha \text{ is divisible by } \lambda\}$ is club in λ^+, and it follows that $G \cap Z$ is stationary in λ^+. Choose $\alpha \in G \cap Z$. Clearly, $\alpha \in G \subseteq S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ and $G \cap \alpha = H_\alpha$. Since $\mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)$ and $\alpha = \mathrm{otp}(H_\alpha)$ is divisible by λ, it follows that D_α is a club of order-type λ such that $\mathrm{nacc}(D_\alpha) \subseteq H_\alpha \subseteq G$. Then, as in Claim 3.5.2, it follows that $\mathrm{nacc}(D_\alpha) \subseteq G$, as well as $\mathrm{otp}(D_\alpha) = \lambda$, giving the required result. □

Now, continue with the very same construction of Theorem 3.6 to get a sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that C_α is a club in α of order-type $\leq \lambda$, and for every sequence $\langle A_\delta \mid \delta < \lambda^+\rangle$ of cofinal subsets of λ^+ and every club $D \subseteq \lambda^+$, there exists $\alpha \in S \cap D$ such that

$$\mathrm{otp}(\{\beta \in \mathrm{acc}(C_\alpha) \mid \mathrm{succ}_\sigma(C_\alpha \setminus \beta) \subseteq A_\delta\}) = \lambda \text{ for every } \delta < \alpha.$$

Then $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\!\sqsubseteq, \lambda^+, \{S\}, 2, \sigma, \mathcal{E}_\lambda)$. □

Theorem 5.7. *Suppose that λ is a given uncountable cardinal.*

If $\mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$, then $V^{\mathrm{Add}(\lambda,1)} \models \mathrm{P}(\lambda^+, 2, {}_\lambda\!\sqsubseteq, \lambda^+, \{S \subseteq E^{\lambda^+}_\lambda \mid S \text{ is stationary}\}, 2, \sigma, \mathcal{E}_\lambda)$ for every cardinal $\sigma < \lambda$.

Proof. This is the same proof as of Theorem 4.2, letting $\Gamma := \mathrm{acc}(\lambda^+)$. Only this time, we do not have to care about coherence. □

6. The coherence relation ${}_\chi\!\sqsubseteq^*$

A construction of a Souslin tree using the relation ${}_\chi\!\sqsubseteq^*$ may be found in [BR16]. Note that $\mathrm{P}(\kappa, \kappa, {}_\chi\!\sqsubseteq^*, 1, \{E^\kappa_\chi\})$ is equivalent to $\boxtimes^*(E^\kappa_\chi)$ for all regular cardinals $\chi < \kappa$.

Lemma 6.1. *Suppose that $\boxminus_{\lambda,\geq\chi} + \mathrm{CH}_\lambda$ holds for given infinite regular cardinals $\chi < \lambda$.*

Then, there exist $S \subseteq E^{\lambda^+}_\chi \subseteq E^{\lambda^+}_{\geq\chi} \subseteq \Gamma \subseteq \mathrm{acc}(\lambda^+)$, and sequences $\langle c_\gamma \mid \gamma \in \Gamma\rangle$, $\langle C_\gamma \mid \gamma < \lambda^+\rangle$, and $\langle X_\beta \mid \beta < \lambda^+\rangle$ that satisfy the following:

1. *if $\gamma \in \mathrm{acc}(\lambda^+)$, then C_γ is a club in γ;*
2. *if $\gamma \in \mathrm{acc}(\lambda^+)$ and $\bar\gamma \in \mathrm{acc}(C_\gamma)$, then $\bar\gamma \notin S$ and $C_{\bar\gamma} \ {}_\chi{\sqsubseteq^*} \ C_\gamma$;*
3. *if $\gamma \in \Gamma$, then c_γ is a club in γ with $\mathrm{otp}(c_\gamma) \leq \lambda$;*
4. *if $\gamma \in \Gamma$ and $\bar\gamma \in \mathrm{acc}(c_\gamma)$, then $\bar\gamma \in \Gamma \setminus S$, $c_{\bar\gamma} = c_\gamma \cap \bar\gamma$, and $C_{\bar\gamma} = C_\gamma \cap \bar\gamma$;*
5. *for every subset $X \subseteq \lambda^+$ and every club $D \subseteq \lambda^+$, there exists some $\gamma \in E^{\lambda^+}_\lambda$ such that $\min(C_\gamma) \in D$, and*

$$\sup(\{\beta \in \mathrm{nacc}(C_\gamma) \cap D \cap S \mid X_\beta = X \cap \beta\}) = \gamma.$$

Proof. This is the argument of [KS93, Theorem 3], modulo various adjustments.

By applying Lemma 4.1 with $S = E^{\lambda^+}_\chi$ and $\eta = \lambda$, let us fix a sequence $\langle c_\gamma \mid \gamma \in \Gamma\rangle$ and a stationary subset $S' \subseteq E^{\lambda^+}_\chi$ such that:

- $E^{\lambda^+}_{\geq\chi} \subseteq \Gamma \subseteq \mathrm{acc}(\lambda^+)$;
- if $\gamma \in \Gamma$, then c_γ is a club subset of γ of order-type $\leq \lambda$;
- if $\gamma \in \Gamma$ and $\bar\gamma \in \mathrm{acc}(c_\gamma)$, then $\bar\gamma \in \Gamma \setminus S'$ and $c_{\bar\gamma} = c_\gamma \cap \bar\gamma$;
- for every club $D \subseteq \lambda^+$, there exist stationarily many $\gamma \in E^{\lambda^+}_\lambda$ such that $\min(c_\gamma) \in D$.

Next, by CH_λ and the fact that S' is a stationary subset of $E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, we get from Fact 8.2 that $\diamondsuit(S')$ holds. We shall use $\diamondsuit(S')$ to guess subsets of $\omega \times \lambda^+$ rather than subsets of λ^+. More specifically, we fix a matrix $\mathbb{X} = \langle X^n_\beta \mid n < \omega, \beta < \lambda^+\rangle$ such that for every sequence $\langle X^n \mid n < \omega\rangle$ of subsets of λ^+, there exist stationarily many $\beta \in S'$ such that $\bigwedge_{n<\omega} X^n_\beta = X^n \cap \beta$.

We now attempt to construct, recursively, sequences $\langle S^n \mid n < \omega\rangle$ and $\langle\langle C^n_\gamma \mid \gamma < \lambda^+\rangle \mid n < \omega\rangle$ satisfying the following properties for all $n < \omega$ and all nonzero limit ordinals $\gamma < \lambda^+$:

(a) $S^{n+1} \subseteq S^n \subseteq S'$;
(b) $C^n_\gamma \subseteq C^{n+1}_\gamma \subseteq \gamma$;
(c) C^n_γ is a club in γ, and if $\gamma \in \Gamma$ then $\min(C^n_\gamma) = \min(c_\gamma)$;
(d) if $\gamma \in \Gamma$ and $\bar\gamma \in \mathrm{acc}(c_\gamma)$, then $C^n_{\bar\gamma} = C^n_\gamma \cap \bar\gamma$;
(e) if $\bar\gamma \in \mathrm{acc}(C^n_\gamma)$, then $\bar\gamma \notin S^n$ and $C^n_{\bar\gamma} \ {}_\chi{\sqsubseteq^*} \ C^n_\gamma$.

Let $S^0 := S'$, and let $C^0_\gamma := c_\gamma$ for all $\gamma \in \Gamma$. Let $C^0_0 = \emptyset$ and let $C^0_{\gamma+1} := \{\gamma\}$ for all $\gamma < \lambda^+$. For all $\gamma \in \mathrm{acc}(\lambda^+) \setminus \Gamma$, let C^0_γ be some club subset of γ of order-type $\mathrm{cf}(\gamma)$. Then $\mathrm{acc}(C^0_\gamma) \cap E^{\lambda^+}_{\geq\chi} = \emptyset$ for all $\gamma \in \lambda^+ \setminus \Gamma$. In particular, $S' \cap \mathrm{acc}(C^0_\gamma) = \emptyset$ for all $\gamma < \lambda^+$. Thus, it is clear that S^0 and $\langle C^0_\gamma \mid \gamma < \lambda^+\rangle$ satisfy the above properties.

Now, fix $n < \omega$, and suppose that S^n and $\langle C^n_\gamma \mid \gamma < \lambda^+\rangle$ have been constructed to satisfy the above properties. It is clear that S^n, $\langle c_\gamma \mid \gamma \in \Gamma\rangle$, $\langle C^n_\gamma \mid \gamma < \lambda^+\rangle$, and $\langle X^n_\beta \mid \beta < \lambda^+\rangle$

satisfy properties (1)–(4) of the conclusion of the lemma. If property (5) is satisfied as well, then the lemma is proven and we abandon the recursive construction at this point.

Otherwise, let us pick a subset $X^n \subseteq \lambda^+$ and a club $D^n \subseteq \lambda^+$ such that for all $\gamma \in E^{\lambda^+}_\lambda$, either $\min(C^n_\gamma) \notin D^n$, or $\sup(\mathrm{nacc}(C^n_\gamma) \cap S^{n+1}) < \gamma$, where we define:

$$S^{n+1} := \{\beta \in S^n \cap D^n \mid X^n_\beta = X^n \cap \beta\}.$$

Define C^{n+1}_γ by recursion over $\gamma < \lambda^+$, as follows. Let $C^{n+1}_0 = \emptyset$, and $C^{n+1}_{\gamma+1} := \{\gamma\}$. Now, if $\gamma < \lambda^+$ is a nonzero limit ordinal, and $\langle C^{n+1}_{\bar\gamma} \mid \bar\gamma < \gamma\rangle$ has already been defined, we let

$$C^{n+1}_\gamma := C^n_\gamma \cup \bigcup \{C^{n+1}_\beta \setminus \sup(C^n_\gamma \cap \beta) \mid \beta \in \mathrm{nacc}(C^n_\gamma) \setminus S^{n+1}, \beta > \min(C^n_\gamma)\}.$$

Claim 6.1.1. *S^{n+1} and $\langle C^{n+1}_\gamma \mid \gamma < \lambda^+\rangle$ satisfy properties (a)–(e) of the recursion.*

Proof. (a)–(d) are easily verified.

(e) Suppose not, and let γ denote the least counterexample. Let $\bar\gamma \in \mathrm{acc}(C^{n+1}_\gamma)$ be arbitrary. We address the two possible cases.

▶ Suppose that $\bar\gamma \in S^{n+1}$. Since $S^{n+1} \subseteq S^n$ and (by the previous step of the recursion) $\mathrm{acc}(C^n_\gamma) \cap S^n = \emptyset$, we get from $\bar\gamma \in \mathrm{acc}(C^{n+1}_\gamma)$ and the definition of C^{n+1}_γ, the existence of $\beta \in \mathrm{nacc}(C^n_\gamma) \setminus S^{n+1}$ such that either $\bar\gamma \in \mathrm{acc}(C^{n+1}_\beta \setminus \sup(C^n_\gamma \cap \beta))$ or $\beta = \bar\gamma$. By minimality of γ, we have $\mathrm{acc}(C^{n+1}_\beta) \cap S^{n+1} = \emptyset$ for all $\beta < \gamma$, and hence it must be the case that $\bar\gamma = \beta$. So $\bar\gamma \in \mathrm{nacc}(C^n_\gamma) \setminus S^{n+1}$, contradicting the fact that $\bar\gamma \in S^{n+1}$.

▶ Suppose that $C^{n+1}_{\bar\gamma} \ {}_\chi\!\not\sqsubseteq^* C^{n+1}_\gamma$. Then $\mathrm{cf}(\bar\gamma) \geq \chi$ and $C^{n+1}_{\bar\gamma} \not\sqsubseteq^* C^{n+1}_\gamma$, so that for no $\alpha < \bar\gamma$ do we have $C^{n+1}_{\bar\gamma} \cap (\alpha, \bar\gamma) = C^{n+1}_\gamma \cap (\alpha, \bar\gamma)$. Clearly, $\bar\gamma$ cannot be in $\mathrm{acc}(C^n_\gamma)$. Let $\beta \in \mathrm{nacc}(C^n_\gamma) \setminus S^{n+1}$ be such that either $\bar\gamma \in \mathrm{acc}(C^{n+1}_\beta \setminus \sup(C^n_\gamma \cap \beta))$ or $\beta = \bar\gamma$. If $\beta = \bar\gamma$, then $C^{n+1}_{\bar\gamma} \cap (\alpha, \bar\gamma) = C^{n+1}_\gamma \cap (\alpha, \bar\gamma)$ for $\alpha := \sup(C^n_\gamma \cap \beta)$. So $\bar\gamma < \beta$, and β contradicts the minimality of γ. □

If we reach the end of the above recursive process, then we are equipped with a sequence $\langle(\langle C^n_\gamma \mid \gamma < \lambda^+\rangle, S^n, D^n, X^n) \mid n < \omega\rangle$, from which we shall derive a contradiction. The set $\bigcap_{n<\omega} D^n$ is club in λ^+. Thus, by the choice of $\mathbb{X}$, the following set must be stationary:

$$S'' := \bigcap_{n<\omega} S^n = \left\{\beta \in S' \cap \bigcap_{n<\omega} D^n \mid \bigwedge_{n<\omega} X^n_\beta = X^n \cap \beta\right\}.$$

Thus, $\mathrm{acc}^+(S'')$ is a club in λ^+, and we may pick $\gamma \in E^{\lambda^+}_\lambda \cap \mathrm{acc}^+(S'')$ such that $\min(c_\gamma) \in \bigcap_{n<\omega} D^n$. For all $n < \omega$, put $\gamma_n := \sup(\mathrm{nacc}(C^n_\gamma) \cap S^{n+1})$. By $\min(C^n_\gamma) \in D^n$, we have $\gamma_n < \gamma$. By $\mathrm{cf}(\gamma) = \lambda > \aleph_0$, we get $\gamma^* < \gamma$, where $\gamma^* := \sup_{n<\omega} \gamma_n$.

Claim 6.1.2. *$C^n_\gamma \cap S^{n+1} \cap (\gamma^*, \gamma) = \emptyset$ for all $n < \omega$.*

Proof. Fix any $n < \omega$, and suppose $\beta \in S^{n+1} \cap (\gamma^*, \gamma)$. Then $\beta \in S^{n+1} \subseteq S^n$. By Clause (e) of the recursion, it follows that $\beta \notin \mathrm{acc}(C^n_\gamma)$. But also $\beta > \gamma^* \geq \gamma_n$, so that $\beta \notin \mathrm{nacc}(C^n_\gamma) \cap S^{n+1}$, and it follows that $\beta \notin \mathrm{nacc}(C^n_\gamma)$. Thus, altogether, $\beta \notin C^n_\gamma$. □

Since $S'' \cap \gamma$ is cofinal in γ, let us pick $\beta \in S'' \cap (\gamma^*, \gamma)$ above $\min(c_\gamma)$.

For all $n < \omega$, let $\beta_n := \min(C^n_\gamma \setminus \beta)$. As $\{C^n_\gamma \mid n < \omega\}$ is an increasing chain, $\langle \beta_n \mid n < \omega\rangle$ is weakly decreasing, and hence stabilizes. Fix $n < \omega$ such that $\beta_n = \beta_{n+1}$. Since $\beta \in S'' \subseteq S^{n+1}$, Claim 6.1.2 gives $\beta \notin C^n_\gamma$, so that $\beta_n > \beta$. In particular, $\beta_n = \min(C^n_\gamma \setminus (\beta+1))$, so that $\beta_n \in \mathrm{nacc}(C^n_\gamma)$. By Claim 6.1.2 again and $\gamma > \beta_n > \beta > \gamma^*$, it follows that $\beta_n \in \mathrm{nacc}(C^n_\gamma) \setminus S^{n+1}$. Recalling that also $\beta_n > \beta > \min(c_\gamma) = \min(C^n_\gamma)$, we infer from the definition of C^{n+1}_γ that

$$C^{n+1}_\gamma \cap [\sup(C^n_\gamma \cap \beta_n), \beta_n) = C^{n+1}_{\beta_n} \cap [\sup(C^n_\gamma \cap \beta_n), \beta_n),$$

and by $\beta_n = \min(C^n_\gamma \setminus \beta)$, we have $\sup(C^n_\gamma \cap \beta_n) \leq \beta$, and hence

$$C^{n+1}_\gamma \cap [\beta, \beta_n) = C^{n+1}_{\beta_n} \cap [\beta, \beta_n).$$

Since $\beta < \beta_n$, it follows that $\beta_{n+1} = \min(C^{n+1}_\gamma \setminus \beta) = \min(C^{n+1}_{\beta_n} \setminus \beta) < \beta_n$, contradicting the choice of n. □

Corollary 6.2. *Suppose that $\boxminus_{\lambda,\geq\chi} + \mathrm{CH}_\lambda$ holds for given infinite regular cardinals $\chi < \lambda$. Then both of the following hold:*

1. $\mathrm{P}(\lambda^+, 2, {}_\chi{\sqsubseteq^*}, 1, \{E^{\lambda^+}_\lambda\}, 2, \chi)$;
2. $\mathrm{P}(\lambda^+, 2, {}_\chi{\sqsubseteq^*}, \chi, \{E^{\lambda^+}_\lambda\}, 2, 1)$.

Proof. As λ is uncountable, Fact 8.2 entails $\diamondsuit(\lambda^+)$, so that we only need to establish the corresponding $\mathrm{P}^-(\dots)$ principles of Clauses (1) and (2).

Let $S \subseteq E^{\lambda^+}_\chi$ along with sequences $\langle C_\beta \mid \beta < \lambda^+\rangle$ and $\langle X_\beta \mid \beta < \lambda^+\rangle$ be given by Lemma 6.1. We may assume that $X_\beta \subseteq \beta$ for all $\beta < \lambda^+$.

(1) We shall define a sequence $\langle C^\bullet_\alpha \mid \alpha < \lambda^+\rangle$ that witnesses $\mathrm{P}^-(\lambda^+, 2, {}_\chi{\sqsubseteq^*}, 1, \{E^{\lambda^+}_\lambda\}, 2, \chi)$. First, for all limit $\beta < \lambda^+$ let D_β be some club in β of order-type $\mathrm{cf}(\beta)$, with the additional constraint that if $\sup(X_\beta) = \beta$, then $\mathrm{nacc}(D_\beta) \subseteq X_\beta$. Then, for all limit $\alpha < \lambda^+$, let

$$C^\bullet_\alpha := \begin{cases} C_\alpha \cup \bigcup\{D_\beta \setminus \sup(C_\alpha \cap \beta) \mid \beta \in \mathrm{nacc}(C_\alpha) \cap S\}, & \text{if } \alpha \in E^{\lambda^+}_{\geq\chi} \setminus S;\\ D_\alpha, & \text{otherwise.}\end{cases}$$

It is clear that $C^\bullet_\alpha$ is a club subset of α. Define $C^\bullet_{\alpha+1} := \{\alpha\}$ for all $\alpha < \lambda^+$.

Fix $\alpha < \lambda^+$ and $\bar\alpha \in \mathrm{acc}(C^\bullet_\alpha)$. To show that $C^\bullet_{\bar\alpha} \mathrel{{}_\chi{\sqsubseteq^*}} C^\bullet_\alpha$, we consider two possibilities:

- If $\alpha \notin E^{\lambda^+}_{\geq\chi} \setminus S$, then $C^\bullet_\alpha = D_\alpha$, and in particular (since $S \subseteq E^{\lambda^+}_\chi$) $\alpha \in E^{\lambda^+}_{\leq\chi}$, so that $\mathrm{cf}(\sup(C^\bullet_{\bar\alpha})) = \mathrm{cf}(\bar\alpha) \leq \mathrm{otp}(C^\bullet_\alpha \cap \bar\alpha) < \mathrm{otp}(C^\bullet_\alpha) = \mathrm{otp}(D_\alpha) = \mathrm{cf}(\alpha) \leq \chi$, and it follows that $C^\bullet_{\bar\alpha} \mathrel{{}_\chi{\sqsubseteq^*}} C^\bullet_\alpha$.

- If $\alpha \in E^{\lambda^+}_{\geq\chi} \setminus S$, then there are three cases to consider:

  - If $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then by Clause (2) of Lemma 6.1 we have $\bar\alpha \notin S$ and $C_{\bar\alpha} \mathrel{{}_\chi{\sqsubseteq^*}} C_\alpha$. If $\mathrm{cf}(\bar\alpha) < \chi$ then we automatically have $C^\bullet_{\bar\alpha} \mathrel{{}_\chi{\sqsubseteq^*}} C^\bullet_\alpha$. Thus we assume $\mathrm{cf}(\bar\alpha) \geq \chi$, so that $C_{\bar\alpha} \sqsubseteq^* C_\alpha$. Fix $\gamma < \bar\alpha$ such that $C_{\bar\alpha} \setminus \gamma \sqsubseteq C_\alpha \setminus \gamma$. Since $\alpha, \bar\alpha \in E^{\lambda^+}_{\geq\chi} \setminus S$, it is clear from the definition of $C^\bullet_\alpha$ in this case that $C^\bullet_{\bar\alpha} \setminus \gamma \sqsubseteq C^\bullet_\alpha \setminus \gamma$, so that $C^\bullet_{\bar\alpha} \sqsubseteq^* C^\bullet_\alpha$.

– If $\bar\alpha \in \mathrm{nacc}(C_\alpha)\cap S$, then $C^\bullet_{\bar\alpha} = D_{\bar\alpha}$, so that, letting $\gamma = \sup(C_\alpha\cap\bar\alpha)+1$, we have $C^\bullet_{\bar\alpha}\setminus\gamma = D_{\bar\alpha}\setminus\gamma \sqsubseteq C^\bullet_\alpha\setminus\gamma$, so that again $C^\bullet_{\bar\alpha}\sqsubseteq^* C^\bullet_\alpha$.
– If $\bar\alpha\in\mathrm{acc}(D_\beta)$ for some $\beta\in\mathrm{nacc}(C_\alpha)\cap S$, then in particular $\mathrm{cf}(\beta)=\chi$, so that $\mathrm{cf}(\sup(C^\bullet_{\bar\alpha})) = \mathrm{cf}(\bar\alpha)\le \mathrm{otp}(D_\beta\cap\bar\alpha) < \mathrm{otp}(D_\beta) = \mathrm{cf}(\beta)=\chi$, and it follows that $C^\bullet_{\bar\alpha}\ {}_\chi{\sqsubseteq^*}\ C^\bullet_\alpha$.

Thus, to verify that $\langle C^\bullet_\alpha \mid \alpha<\lambda^+\rangle$ witnesses $\mathrm{P}^-(\lambda^+,2,{}_\chi{\sqsubseteq^*},1,\{E^{\lambda^+}_\lambda\},2,\chi)$, consider any cofinal subset $A_0\subseteq\lambda^+$ and any club $E\subseteq\lambda^+$. Let $\alpha\in E^{\lambda^+}_\lambda$ be given by Clause (5) of Lemma 6.1 applied with $X := A_0$ and $D := E\cap\mathrm{acc}^+(A_0)$. Define

$$Z := \{\beta\in\mathrm{nacc}(C_\alpha)\cap E\cap\mathrm{acc}^+(A_0)\cap S \mid X_\beta = A_0\cap\beta\},$$

so that $\sup(Z)=\alpha$. Since $Z\subseteq E$ and E is club, it follows that $\alpha\in\mathrm{acc}^+(E)\subseteq E$.

Since $S\subseteq E^{\lambda^+}_\chi$ and $\chi<\lambda=\mathrm{cf}(\alpha)$, we clearly have $\alpha\in E^{\lambda^+}_{\ge\chi}\setminus S$. Now, consider any $\beta\in Z$, and let $\beta^- := \sup(C_\alpha\cap\beta)$. Since $\beta\in\mathrm{nacc}(C_\alpha)\cap S$, it follows that $\beta^-\in C_\alpha\cap\beta$ and $D_\beta\cap(\beta^-,\beta) = C^\bullet_\alpha\cap(\beta^-,\beta)$. Furthermore, since $\beta\in S\subseteq E^{\lambda^+}_\chi$, we have $\mathrm{otp}(D_\beta)=\mathrm{cf}(\beta)=\chi$. Thus also $\mathrm{otp}(D_\beta\cap(\beta^-,\beta))=\chi$, and it follows that $\mathrm{succ}_\chi(C^\bullet_\alpha\setminus\beta^-)\subseteq\mathrm{nacc}(D_\beta)$. Since $\beta\in Z\subseteq\mathrm{acc}^+(A_0)$, we have $\sup(X_\beta)=\sup(A_0\cap\beta)=\beta$, so that $\mathrm{nacc}(D_\beta)\subseteq X_\beta$. Altogether, we have $\mathrm{succ}_\chi(C^\bullet_\alpha\setminus\beta^-)\subseteq\mathrm{nacc}(D_\beta)\subseteq X_\beta\subseteq A_0$. Since $\sup(Z)=\alpha$, it is clear that also $\sup\{\beta^-\mid\beta\in Z\}=\alpha$.

(2) The definition of $\langle C^\bullet_\alpha\mid\alpha<\lambda^+\rangle$ that witnesses $\mathrm{P}^-(\lambda^+,2,{}_\chi{\sqsubseteq^*},\chi,\{E^{\lambda^+}_\lambda\},2,1)$ is as in the previous case, modulo the fact that the clubs D_β need to be chosen somewhat differently. Fix a bijection $\pi:\chi\times\lambda^+\leftrightarrow\lambda^+$. Denote $X^i_\beta := \{\alpha<\beta\mid\pi(i,\alpha)\in X_\beta\}$. Then, for all limit $\beta<\lambda^+$, let D_β be some club in β of order-type $\mathrm{cf}(\beta)$, with the additional constraint that if $\mathrm{cf}(\beta)=\chi$, then $\sup(\mathrm{nacc}(D_\beta)\cap X^i_\beta)=\beta$ for all $i<\chi$ such that $\sup(X^i_\beta)=\beta$. For any sequence $\langle A_i\mid i<\chi\rangle$ of cofinal subsets of λ^+ and any club $E\subseteq\lambda^+$, the verification includes applying Clause (5) of Lemma 6.1 with $X:=\bigcup_{i<\chi}\pi[\{i\}\times A_i]$ and $D := E\cap\bigcap_{i<\chi}\mathrm{acc}^+(A_i)\cap\{\beta<\lambda^+\mid\pi[\chi\times\beta]=\beta\}$. □

Theorem 6.3. *If CH_λ holds for a given regular uncountable cardinal λ, and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $\mathrm{P}(\lambda^+,2,{}_\lambda{\sqsubseteq^*},\theta,\{E^{\lambda^+}_\lambda\},2,\sigma)$ holds for all regular cardinals $\theta,\sigma<\lambda$.*

Proof. Suppose that $S\subseteq E^{\lambda^+}_{<\lambda}$ is stationary and nonreflecting. By Fact 8.2, CH_λ entails $\diamondsuit(S)$. Recalling that $E^{\lambda^+}_\lambda$ is always a nonreflecting stationary subset of λ^+ and $S\cap E^{\lambda^+}_\lambda=\emptyset$, we apply Theorem 3 of [KS93] with $S^* = E^{\lambda^+}_\lambda$ to obtain sequences $\langle C_\alpha\mid\alpha\in E^{\lambda^+}_\lambda\rangle$ and $\langle Z_\gamma\mid\gamma<\lambda^+\rangle$ that satisfy the following:

1. C_α is a club in α for all $\alpha\in E^{\lambda^+}_\lambda$;
2. if $\alpha\in E^{\lambda^+}_\lambda$ and $\bar\alpha\in\mathrm{acc}(C_\alpha)\cap E^{\lambda^+}_\lambda$, then $C_{\bar\alpha}\sqsubseteq^* C_\alpha$;
3. for every subset $Z\subseteq\lambda^+$ and every club $D\subseteq\lambda^+$, there exists some $\alpha\in E^{\lambda^+}_\lambda$ such that

$$\sup(\{\gamma\in\mathrm{nacc}(C_\alpha)\cap D\cap S\mid Z_\gamma = Z\cap\gamma\})=\alpha.$$

For every $\alpha \in E^{\lambda^+}_{\lambda}$, let

$$C^\bullet_\alpha := \{\min((Z_\gamma \cup \{\gamma\}) \setminus \sup(C_\alpha \cap \gamma)) \mid \gamma \in C_\alpha\}.$$

Then:

1. $C^\bullet_\alpha$ is a club in α for every $\alpha \in E^{\lambda^+}_{\lambda}$, with $\mathrm{acc}(C^\bullet_\alpha) = \mathrm{acc}(C_\alpha)$;
2. if $\alpha \in E^{\lambda^+}_{\lambda}$ and $\bar\alpha \in \mathrm{acc}(C^\bullet_\alpha) \cap E^{\lambda^+}_{\lambda}$, then $C^\bullet_{\bar\alpha} \sqsubseteq^* C^\bullet_\alpha$;
3. for every cofinal subset $Z \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_{\lambda}$ such that

$$\sup(\mathrm{nacc}(C^\bullet_\alpha) \cap Z) = \alpha.$$

(To see (3), let $D := \mathrm{acc}^+(Z)$.)

By Fact 8.2, CH_λ entails $\diamondsuit(E^{\lambda^+}_{\theta})$ for every regular cardinal $\theta < \lambda$. Thus, we can fix a matrix $\langle X^i_\gamma \mid i < \lambda, \gamma < \lambda^+\rangle$ with the property that for every sequence $\langle A_i \mid i < \lambda\rangle$ of subsets of λ^+, and every regular $\theta < \lambda$, the following set is stationary:

$$\{\gamma \in E^{\lambda^+}_{\theta} \mid \forall i < \lambda(A_i \cap \gamma = X^i_\gamma)\}.$$

We shall construct $D_\alpha \subseteq \alpha$ for every $\alpha < \lambda^+$, as follows: First, let $D_{\alpha+1} = \{\alpha\}$ for all $\alpha < \lambda^+$.

Then, consider any limit $\alpha \in E^{\lambda^+}_{<\lambda}$.

▶ If there exists some club d in α such that $\mathrm{otp}(d) < \lambda$ and for all $i < \mathrm{cf}(\alpha)$

$$\sup\{\beta \in d \mid \mathrm{succ}_{\mathrm{cf}(\alpha)}(d \setminus \beta) \subseteq X^i_\alpha\} = \alpha,$$

then pick such a d, and call it D_α.

▶ Otherwise, let D_α be an arbitrary club subset of α of order-type $\mathrm{cf}(\alpha)$.

Finally, consider any $\alpha \in E^{\lambda^+}_{\lambda}$. Put

$$D_\alpha := C^\bullet_\alpha \cup \bigcup\{D_\gamma \setminus \sup(C^\bullet_\alpha \cap \gamma) \mid \gamma \in \mathrm{nacc}(C^\bullet_\alpha) \cap E^{\lambda^+}_{<\lambda}\}.$$

Clearly, D_α is a club subset of α for every limit $\alpha < \lambda^+$.

Fix $\alpha < \lambda^+$ and $\bar\alpha \in \mathrm{acc}(D_\alpha)$. If $\mathrm{cf}(\bar\alpha) < \lambda$, then automatically $D_{\bar\alpha} \ {}_{\lambda}{\sqsubseteq^*} D_\alpha$. If $\mathrm{cf}(\alpha) < \lambda$, then by construction of D_α we have $\mathrm{otp}(D_\alpha) < \lambda$, so that $\mathrm{cf}(\bar\alpha) \leq \mathrm{otp}(D_\alpha \cap \bar\alpha) < \mathrm{otp}(D_\alpha) < \lambda$, and again $D_{\bar\alpha} \ {}_{\lambda}{\sqsubseteq^*} D_\alpha$. Otherwise, we have $\bar\alpha, \alpha \in E^{\lambda^+}_{\lambda}$, so that $C^\bullet_{\bar\alpha} \sqsubseteq^* C^\bullet_\alpha$, and it follows from the definition of D_α that $D_{\bar\alpha} \sqsubseteq^* D_\alpha$ in this case.

Claim 6.3.1. *For all regular cardinals $\sigma, \theta < \lambda$ and every sequence $\langle A_i \mid i < \theta\rangle$ of cofinal subsets of λ^+, there exist stationarily many $\alpha \in E^{\lambda^+}_{\lambda}$ such that for all $i < \theta$:*

$$\sup\{\beta \in D_\alpha \mid \mathrm{succ}_\sigma(D_\alpha \setminus \beta) \subseteq A_i\} = \alpha.$$

Proof. By increasing σ and θ if necessary, we may assume that $\sigma = \theta$ is an infinite regular cardinal.

Given $\langle A_i \mid i < \theta\rangle$ and a club $E \subseteq \lambda^+$, consider the stationary set

$$Z := \{\gamma \in E^{\lambda^+}_\theta \cap E \cap \bigcap_{i<\theta} \mathrm{acc}^+(\mathrm{acc}^+(A_i) \cap E^{\lambda^+}_\sigma) \mid \forall i < \theta(A_i \cap \gamma = X^i_\gamma)\}.$$

Fix $\alpha \in E^{\lambda^+}_\lambda$ such that $\sup(\mathrm{nacc}(C^\bullet_\alpha) \cap Z) = \alpha$. Then $\alpha \in \mathrm{acc}^+(Z) \subseteq \mathrm{acc}^+(E) \subseteq E$. So, it suffices to show that for every $\gamma \in \mathrm{nacc}(C^\bullet_\alpha) \cap Z$ and every $i < \theta$, there exists some $\beta > \sup(C^\bullet_\alpha \cap \gamma)$ for which $\mathrm{succ}_\sigma(D_\alpha \setminus \beta) \subseteq A_i$. Fix $\gamma \in \mathrm{nacc}(C^\bullet_\alpha) \cap Z$. As $\alpha \in E^{\lambda^+}_\lambda$ and $\gamma \in \mathrm{nacc}(C^\bullet_\alpha) \cap E^{\lambda^+}_{<\lambda}$, recalling the definition of D_α, it then suffices to prove that for all $i < \theta$

$$\sup\{\beta \in D_\gamma \mid \mathrm{succ}_\sigma(D_\gamma \setminus \beta) \subseteq A_i\} = \gamma.$$

Fix a surjection $f : \theta \to \theta$ such that the preimage of any singleton has size θ. Since $\gamma \in E^{\lambda^+}_\theta \cap \bigcap_{i<\theta} \mathrm{acc}^+(\mathrm{acc}^+(A_i) \cap E^{\lambda^+}_\sigma)$, let us pick a strictly increasing, continuous, and cofinal function $g : \theta \to \gamma$ with the property that $g(j + 1) \in \mathrm{acc}^+(A_{f(j)}) \cap E^{\lambda^+}_\sigma$ for all $j < \theta$. Next, for all $j < \theta$, fix $Y_j \subseteq A_{f(j)} \cap (g(j), g(j + 1))$ of order-type σ, and let d denote the closure of $\bigcup_{j<\theta} Y_j$. Then d is a club in γ of order-type $\sigma \cdot \theta < \lambda$, and for all $i < \theta$,

$$\sup\{\beta \in d \mid \mathrm{succ}_\sigma(d \setminus \beta) \subseteq X^i_\gamma\} = \gamma,$$

so D_γ was chosen to satisfy our needs. □

Thus, $\langle D_\alpha \mid \alpha < \lambda^+\rangle$ witnesses simultaneously $\mathrm{P}^-(\lambda^+, 2, {}_\lambda\sqsubseteq^*, \theta, \{E^{\lambda^+}_\lambda\}, 2, \sigma)$ for all regular cardinals $\theta, \sigma < \lambda$. □

Theorem 6.4. *Suppose that λ is a successor of a regular cardinal θ, $\mathrm{NS} \restriction E^\lambda_\theta$ is saturated, and CH_λ holds. Then $\mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq^*, \theta, \{E^{\lambda^+}_\lambda\}, 2, \theta)$ holds.*

Proof. Recalling Theorem 6.3, we may assume that every stationary subset of $E^{\lambda^+}_\theta$ reflects. Recalling Theorem 5.6, it suffices to prove that $\diamondsuit(E^{\lambda^+}_\lambda)$ holds. Using Fact 8.2, let us fix a sequence, $\langle Z_\beta \mid \beta \in E^{\lambda^+}_\theta\rangle$ such that $G_Z := \{\beta \in E^{\lambda^+}_\theta \mid Z \cap \beta = Z_\beta\}$ is stationary for every $Z \subseteq \lambda^+$. As in [She84], for every $\delta \in E^{\lambda^+}_\lambda$, we let

$$\mathcal{S}_\delta := \{Z \subseteq \delta \mid G_Z \cap \delta \text{ is stationary in } \delta\}.$$

Of course, if Z, Z' are distinct elements of $\mathcal{S}_\delta$ then $G_Z \cap \delta$ and $G_{Z'} \cap \delta$ are stationary subsets of E^δ_θ with $\sup(G_Z \cap G_{Z'}) < \delta$. As $\mathrm{cf}(\delta) = \lambda$ and $\mathrm{NS} \restriction E^\lambda_\theta$ is saturated, it follows that $|\mathcal{S}_\delta| \leq \lambda$.

Claim 6.4.1. *For every $Z \subseteq \lambda^+$, there exist stationarily many $\delta \in E^{\lambda^+}_\lambda$ for which $Z \cap \delta \in \mathcal{S}_\delta$.*

Proof. Let $Z \subseteq \lambda^+$ be arbitrary. Let $D \subseteq \lambda^+$ be some club. Then $G := G_Z \cap D$ is a stationary subset of $E^{\lambda^+}_\theta$. As every stationary subset of $E^{\lambda^+}_\theta$ reflects, pick $\delta \in E^{\lambda^+}_\lambda$ such that $G \cap \delta$ is stationary in δ. Then $\delta \in D$ and $Z \cap \delta \in \mathcal{S}_\delta$. □

So $\langle \mathcal{S}_\delta \mid \delta \in E^{\lambda^+}_\lambda \rangle$ forms a $\diamondsuit^-(E^{\lambda^+}_\lambda)$-sequence, and hence $\diamondsuit(E^{\lambda^+}_\lambda)$ holds.[25] $\square$

Note that it is unknown whether the hypothesis of the preceding theorem is consistent (see [Rin11a, §4] for a list of open problems and known partial answers).

7. An open problem

Recall that by Corollary 3.10, for every singular cardinal λ, $\square_\lambda + \mathrm{CH}_\lambda$ entails $\boxtimes_\lambda(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$, and hence, by Proposition 2.5, the existence of a uniformly coherent λ^+-Souslin tree. Next, suppose that $\square_\lambda + \mathrm{CH}_\lambda$ holds for a given regular uncountable cardinal λ. We have that $\mathrm{P}(\lambda^+, 2, \mathcal{R}, \theta, \{S\})$ holds, in any of the following cases:

- $\mathcal{R} = \sqsubseteq$, $S = E^{\lambda^+}_{<\lambda}$ and all $\theta < \lambda$, by Corollary 3.9;
- $\mathcal{R} = \sqsubseteq^*$, $S = E^{\lambda^+}_\lambda$ and $\theta = \aleph_0$, by Corollary 6.2;
- $\mathcal{R} = {}_\lambda\sqsubseteq^*$, $S = E^{\lambda^+}_\lambda$ and all regular $\theta < \lambda$, by Theorem 6.3.

In particular, the case $\theta \in \{\lambda, \lambda^+\}$ remains open (when the second parameter is 2). A test question is the following:

Question. Suppose that λ is a regular uncountable cardinal.

Does $\square_\lambda + \mathsf{GCH}$ entail the existence of a uniformly coherent λ^+-Souslin tree?

Note that by [BR17a], for a singular cardinal λ, already $\square(\lambda^+) + \mathsf{GCH}$ entails the existence of a uniformly coherent λ^+-Souslin tree.

8. Appendix: Combinatorial principles

Definition 8.1 (Jensen, [Jen72]). For a regular uncountable cardinal κ and a stationary subset $S \subseteq \kappa$, $\diamondsuit(S)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta \in S \rangle$ such that

- $Z_\beta \subseteq \beta$ for every $\beta \in S$;
- for every $Z \subseteq \kappa$, the set $\{\beta \in S \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

Fact 8.2 (Shelah, [She10]). *For every infinite cardinal λ and every stationary subset $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, the following are equivalent:*

1. CH_λ*;*
2. $\diamondsuit(S)$*.*

In particular, CH_λ *is equivalent to* $\diamondsuit(\lambda^+)$ *for every uncountable cardinal* λ*.*

Definition 8.3 (Ostaszewski, [Ost76]). For a stationary $S \subseteq \omega_1$, $\clubsuit(S)$ asserts the existence of a sequence $\langle X_\alpha \mid \alpha \in S \rangle$ such that:

[25] See [Kun80] for the definition of $\diamondsuit^-(S)$ and the proof that it implies $\diamondsuit(S)$.

- for all nonzero limit $\alpha \in S$, X_α is a cofinal subset of α of order-type ω;
- for every uncountable $X \subseteq \omega_1$, there exists a nonzero limit $\alpha \in S$ such that $X_\alpha \subseteq X$.

In [FSS97], Fuchino, Shelah and Soukup studied a weakening of the preceding principle, which they denoted by $\clubsuit_w(S)$. Their principle admits a natural generalization to arbitrary regular uncountable cardinals κ, as follows.

Definition 8.4. For a regular uncountable cardinal κ and a stationary $S \subseteq \kappa$, $\clubsuit_w(S)$ asserts the existence of a sequence $\langle X_\alpha \mid \alpha \in S\rangle$ such that:

- for all limit $\alpha \in S$, X_α is a cofinal subset of α of order-type $\mathrm{cf}(\alpha)$;
- for every cofinal $X \subseteq \kappa$, there exists a limit $\alpha \in S$ such that $\sup(X_\alpha \setminus X) < \alpha$.

Fact 8.5 (Devlin, [Ost76] and [Dev78])**.** *For every regular uncountable cardinal κ and stationary $S \subseteq \kappa$, the following are equivalent:*

1. $\diamondsuit(S)$;
2. $\diamondsuit(\kappa) + \clubsuit_w(S)$;
3. *There exists a partition $\langle S_i \mid i < \kappa\rangle$ of S such that $\diamondsuit(S_i)$ holds for all $i < \kappa$.*

Definition 8.6 (König-Larson-Yoshinobu, [KLY07])**.** Suppose that $\chi < \kappa$ are uncountable cardinals with κ regular, and $S \subseteq \kappa$ stationary.

$\curlywedge^*(\chi, S)$ asserts the existence of a sequence $\langle \mathcal{C}_\delta \mid \delta \in S\rangle$ such that:

1. for every $\delta \in S$, $\mathcal{C}_\delta$ is a club in $[\delta]^{<\chi}$;
2. for every club $\mathcal{D}$ in $[\kappa]^{<\chi}$, there exists a club $C \subseteq \kappa$ such that for all $\delta \in C \cap S$, there is $x_\delta \in [\delta]^{<\chi}$ with $\{y \in \mathcal{C}_\delta \mid x_\delta \subseteq y\} \subseteq \mathcal{D}$.

Definition 8.7 (Rinot, [Rin11b])**.** Suppose that $\chi < \kappa$ are uncountable cardinals with κ regular, and $S \subseteq \kappa$ is stationary.

$\curlywedge^-(\chi, S)$ asserts the existence of a matrix $\langle \mathcal{C}^i_\delta \mid \delta \in S, i < |\delta|\rangle$ such that:

1. for every $\delta \in S$ and $i < |\delta|$, $\mathcal{C}^i_\delta$ is cofinal in $[\delta]^{<\chi}$;
2. for every club $\mathcal{D}$ in $[\kappa]^{<\chi}$, the following set is stationary:

$$\{\delta \in S \mid \exists i < |\delta|\ \mathcal{C}^i_\delta \subseteq \mathcal{D}\}.$$

Definition 8.8 (Rinot, [Rin11b])**.** Suppose that λ is a regular uncountable cardinal, T is a stationary subset of λ, and S is a stationary subset of $E^{\lambda^+}_\lambda$.

The *reflected-diamond* principle, denoted $\langle T\rangle_S$, asserts the existence of a sequence $\langle C_\delta \mid \delta \in S\rangle$ and a matrix $\langle A^\delta_i \mid \delta \in S, i < \lambda\rangle$ such that:

1. for all $\delta \in S$, C_δ is a club subset of δ of order-type λ;
2. for every club $D \subseteq \lambda^+$ and every subset $A \subseteq \lambda^+$, there exist stationarily many $\delta \in S$ for which:

$$\{i \in T \mid C_\delta(i+1) \in D \ \&\ A \cap (C_\delta(i+1)) = A^\delta_{i+1}\} \text{ is stationary in } \lambda.$$

Fact 8.9 (Rinot, [Rin11b]). *Suppose that λ is a regular uncountable cardinal and $S \subseteq E^{\lambda^+}_{\lambda}$ is stationary.*

- *[Rin11b, Remark on p. 567] $\curlywedge^*(\lambda, S)$ entails $\curlywedge^-(\lambda, S)$;*
- *[Rin11b, Theorem 2.5] $\diamondsuit(S)$ entails $\curlywedge^-(\chi, S)$ for all uncountable cardinals $\chi \leq \lambda$;*
- *[Rin11b, Theorem 2.4] if $\chi < \lambda$ is an uncountable cardinal or $\chi = \lambda$ is a successor cardinal, then $\curlywedge^-(\chi, S) + \mathrm{CH}_\lambda$ entails $\langle T \rangle_S$ for every stationary $T \subseteq \lambda$.*

We now introduce a variation of $\langle T \rangle_S$ that makes sense also in the context of λ singular.[26]

Definition 8.10. Suppose that λ is an infinite cardinal, T is a cofinal subset of λ, and S is a stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

$\langle T \rangle^-_S$ asserts the existence of a sequence $\langle C_\delta \mid \delta \in S \rangle$ and a matrix $\langle A^\delta_i \mid \delta \in S, i < \lambda \rangle$ such that:

1. for all $\delta \in S$, C_δ is a club subset of δ of order-type λ;
2. for every club $D \subseteq \lambda^+$ and every subset $A \subseteq \lambda^+$, there exist stationarily many $\delta \in S$ for which:

$$\mathrm{otp}(\{i \in T \mid C_\delta(i+1) \in D \ \&\ A \cap (C_\delta(i+1)) = A^\delta_{i+1}\}) = \lambda.$$

Definition 8.11 (Jensen, [Jen72]). For an infinite cardinal λ, $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

- C_α is a club in α for all limit $\alpha < \lambda^+$;
- if $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- $\mathrm{otp}(C_\alpha) \leq \lambda$ for all $\alpha < \lambda^+$.

We now give three variations of the preceding. The first one is essentially due to Baumgartner (cf., [AC00, §2.2]).

Definition 8.12. For infinite cardinals $\chi \leq \lambda$, $\boxminus_{\lambda, \geq\chi}$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in E^{\lambda^+}_{\geq\chi} \rangle$ such that:

- C_α is a club in α for all $\alpha \in E^{\lambda^+}_{\geq\chi}$;
- $C_\alpha \cap \gamma = C_\beta \cap \gamma$ for all $\alpha, \beta \in E^{\lambda^+}_{\geq\chi}$ and $\gamma \in \mathrm{acc}(C_\alpha) \cap \mathrm{acc}(C_\beta)$;
- $\mathrm{otp}(C_\alpha) \leq \lambda$ for all $\alpha \in E^{\lambda^+}_{\geq\chi}$.

Definition 8.13 (Todorcevic, [Tod87]). For a regular uncountable cardinal κ, $\square(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

[26] Note that $\langle \lambda \rangle^-_S$ is equivalent to $\langle \lambda \rangle_S$ for every successor cardinal λ.

- C_α is a club in α for all limit $\alpha < \kappa$;
- if $\bar{\alpha} \in \operatorname{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- there exists no club $C \subseteq \kappa$ such that $C_\alpha = C \cap \alpha$ for all $\alpha \in \operatorname{acc}(C)$.

Definition 8.14 (Rinot, [Rin14b]). For an uncountable cardinal λ, $\boxed{\clubsuit}_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ having the following additional feature. For every sequence $\langle A_i \mid \iota < \lambda\rangle$ of cofinal subsets of λ^+, every limit ordinal $\theta < \lambda$, and every club $D \subseteq \lambda^+$, there exists some limit ordinal $\alpha < \lambda^+$ such that:

(i) $\operatorname{otp}(C_\alpha) = \theta$;
(ii) $C_\alpha(i+1) \in A_i$ for all $i < \theta$;
(iii) for every $i < \theta$ there is some $\beta \in D$ such that $C_\alpha(i) < \beta < C_\alpha(i+1)$.

Fact 8.15 (Rinot, [Rin14b]). *For every uncountable cardinal λ, $\square_\lambda + \mathrm{CH}_\lambda$ entails $\boxed{\clubsuit}_\lambda$.*

Definition 8.16 (Gray, [Gra80]). For a regular uncountable cardinal λ, $\boxed{\diamondsuit}_\lambda$ asserts the existence of a sequence $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

1. for every limit $\alpha < \lambda^+$, D_α is a club in α of order-type $\leq \lambda$, and $X_\alpha \subseteq \alpha$;
2. if $\bar{\alpha} \in \operatorname{acc}(D_\alpha)$, then $D_\alpha \cap \bar{\alpha} = D_{\bar{\alpha}}$ and $X_\alpha \cap \bar{\alpha} = X_{\bar{\alpha}}$;
3. for every subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists a limit $\alpha < \lambda^+$ with $\operatorname{cf}(\alpha) = \lambda$ such that $X \cap \alpha = X_\alpha$ and $\operatorname{acc}(D_\alpha) \subseteq E$.

We take the liberty of generalizing the preceding, as follows.[27]

Definition 8.17. For an infinite cardinal λ, $\boxed{\diamondsuit}_\lambda$ asserts the existence of a sequence $\langle (D_\alpha, X_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

1. for every limit $\alpha < \lambda^+$, D_α is a club in α of order-type $\leq \omega \cdot \lambda$, and $X_\alpha \subseteq \alpha$;
2. if $\bar{\alpha} \in \operatorname{acc}(D_\alpha)$, then $D_\alpha \cap \bar{\alpha} = D_{\bar{\alpha}}$ and $X_\alpha \cap \bar{\alpha} = X_{\bar{\alpha}}$;
3. for every subset $X \subseteq \lambda^+$ and club $E \subseteq \lambda^+$, there exists a limit $\alpha < \lambda^+$ with $\operatorname{otp}(\operatorname{acc}(D_\alpha)) = \lambda$ such that $X \cap \alpha = X_\alpha$ and $\operatorname{acc}(D_\alpha) \subseteq E$.

Fact 8.18 (Rinot, [Rin14b] and [Rin15b]). *For every singular cardinal λ, the following are equivalent:*

- $\square_\lambda + \mathrm{CH}_\lambda$;
- $\boxed{\clubsuit}_\lambda$;
- $\boxed{\diamondsuit}_\lambda$.

[27] The generalization from λ regular uncountable to λ singular was done in [ASS87, §2], by replacing $\operatorname{cf}(\alpha) = \lambda$ with $\operatorname{otp}(D_\alpha) = \lambda$ in Item (3) of Definition 8.16. Here, we also address the overlooked case $\lambda = \omega$, by taking into account Item (2)(d) of [Rin14b, Definiton 1.4]. Recalling the proof of [Rin14b, Lemma 2.8], this extra requirement is not entirely trivial. Nevertheless, one would expect that $\boxed{\diamondsuit}_{\aleph_0}$ should follow from $\diamondsuit(\aleph_1)$. This is indeed the case, as can be seen by combining Lemma 3.5 with [Rin15b, Claim 4].

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