# A Theory of Stationary Trees, and the Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees

Ari Meir Brodsky

May 19, 2014 ©

# Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)

# Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

  Finite Ramsey Theorem For all positive integers $m, r$, and $k$, there is some positive integer $n$ such that
  $$n \to (m)^r_k \,.$$

  Infinite Ramsey Theorem For all positive integers $r$ and $k$, we have
  $$\omega \to (\omega)^r_k \,.$$

# Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

  Finite Ramsey Theorem For all positive integers $m$, $r$, and $k$, there is some positive integer $n$ such that

  $$n \to (m)^r_k \,.$$

  Infinite Ramsey Theorem For all positive integers $r$ and $k$, we have

  $$\omega \to (\omega)^r_k \,.$$

- Erdős and Rado (1956): the first systematic study of partition relations for linear orders (especially cardinals and ordinals) of all cardinalities

# Partition Calculus — Milestones

- Dirichlet's Pigeonhole principle (1834)
- Ramsey's Theorems (1930):

  Finite Ramsey Theorem For all positive integers $m, r$, and $k$, there is some positive integer $n$ such that
  $$n \to (m)^r_k .$$

  Infinite Ramsey Theorem For all positive integers $r$ and $k$, we have
  $$\omega \to (\omega)^r_k .$$

- Erdős and Rado (1956): the first systematic study of partition relations for linear orders (especially cardinals and ordinals) of all cardinalities
- Todorcevic (early 1980s): systematically extending the partition calculus to trees and partial orders

# Arrow Notation for Partial Orders

Suppose $\langle P, <_P \rangle$ is any partial order.

- If $\alpha$ is any ordinal, $[P]^\alpha$ denotes the set of all linearly ordered chains in $P$ of order-type $\alpha$.
- If $\mu$ is any cardinal and $\alpha$ is any ordinal, the statement

$$P \to (\alpha)^2_\mu$$

means: For any colouring (partition function) $c : [P]^2 \to \mu$, there is a chain $X \in [P]^\alpha$ that is $c$-homogeneous, that is, $c''[X]^2 = \{\chi\}$ for some colour $\chi < \mu$.

# Trees

**Definition**

A partial order $\langle T, <_T \rangle$ is a tree if for every $t \in T$,

$$t{\downarrow} = \{s \in T : s <_T t\}$$

is a well-ordered set.

# Nonspecial Trees

## Definition (Todorcevic, 1981)

A tree $T$ is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

# Nonspecial Trees

## Definition (Todorcevic, 1981)

A tree $T$ is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

## Fact

*For any tree $T$,*

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0} .$$

# Nonspecial Trees

## Definition (Todorcevic, 1981)

A tree $T$ is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

## Fact

*For any tree $T$,*

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0}.$$

## Examples

- $\omega_1$

# Nonspecial Trees

## Definition (Todorcevic, 1981)

A tree $T$ is nonspecial if it cannot be written as a union of countably many antichains:

$$T \to (2)^1_{\aleph_0}$$

## Fact

*For any tree $T$,*

$$T \to (2)^1_{\aleph_0} \iff T \to (\omega)^1_{\aleph_0}.$$

## Examples

- $\omega_1$
- (Kurepa, 1956) $w\mathbb{Q}$ (and its variant $\sigma\mathbb{Q}$), the collection of all well-ordered subsets of $\mathbb{Q}$, ordered by end-extension

# Taller Nonspecial Trees

## Definition (Todorcevic, 1985)

For any infinite cardinal $\kappa$, a tree $T$ is non-$\kappa$-special if it cannot be written as a union of $\leq \kappa$ many antichains:

$$T \to (2)^1_\kappa\,.$$

# Taller Nonspecial Trees

## Definition (Todorcevic, 1985)

For any infinite cardinal $\kappa$, a tree $T$ is non-$\kappa$-special if it cannot be written as a union of $\leq \kappa$ many antichains:

$$T \to (2)^1_\kappa \, .$$

## Fact

*For any tree $T$,*

$$T \text{ is non-}\kappa\text{-special} \iff T \to (\kappa)^1_\kappa .$$

# From Trees to Partial Orders

**Theorem (Todorcevic 1985)**

*Let $r$ be any positive integer, let $\kappa$ and $\theta$ be cardinals, and for each $\gamma < \theta$ let $\alpha_\gamma$ be an ordinal. If every non-$\kappa$-special tree $T$ satisfies*

$$T \to (\alpha_\gamma)^r_{\gamma<\theta}\,, \qquad (**)$$

*then every partial order $P$ satisfying $P \to (\kappa)^1_\kappa$ also satisfies the above partition relation $(**)$.*

# Balanced Partition Relations for Pairs

**Theorem (Erdős-Rado, 1956)**

*Let $\kappa$ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,*

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + 1)^{2}_{\mu}\,.$$

# Balanced Partition Relations for Pairs

Theorem (Erdős-Rado, 1956)

*Let $\kappa$ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,*

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + 1)^2_\mu .$$

Theorem (Todorcevic, 1985)

*Let $\kappa$ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,*

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \ \to (\kappa + 1)^2_\mu .$$

# Balanced Partition Relations for Pairs

### Theorem (Baumgartner-Hajnal-Todorcevic, 1991)

*Let $\kappa$ be any infinite regular cardinal, let $\xi$ be any ordinal such that $2^{|\xi|} < \kappa$, and let $k$ be any natural number. Then*

$$\left(2^{<\kappa}\right)^+ \rightarrow (\kappa + \xi)^2_k \, .$$

# Balanced Partition Relations for Pairs

**Theorem (Baumgartner-Hajnal-Todorcevic, 1991)**

*Let $\kappa$ be any infinite regular cardinal, let $\xi$ be any ordinal such that $2^{|\xi|} < \kappa$, and let $k$ be any natural number. Then*

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \xi)^2_k \,.$$

**Theorem (Brodsky — Main Theorem)**

*Let $\kappa$ be any infinite regular cardinal, let $\xi$ be any ordinal such that $2^{|\xi|} < \kappa$, and let $k$ be any natural number. Then*

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \to (\kappa + \xi)^2_k \,.$$

# Optimality of the Result

## Theorem (Brodsky)

*Let $\kappa$ be any infinite regular cardinal. If $T$ is any tree, then the following are equivalent:*

1. $T \to (2)^1_{2^{<\kappa}}$
2. $T \to (2^{<\kappa})^1_{2^{<\kappa}}$
3. $T \to (\kappa + 1, \kappa)^2$
4. *For any ordinal $\xi$ such that $2^{|\xi|} < \kappa$, and any natural number $k$, we have*
$$T \to (\kappa + \xi)^2_k \, .$$

# Optimality of the Result

**Theorem (Rebholz, Donder)**

*If $V = L$, and if $\kappa$ is any regular uncountable cardinal that is not weakly compact, then*

$$\kappa^+ \not\to (\kappa + \log \kappa)^2_2 \,,$$

*where $\log \kappa$ is the first cardinal $\tau$ such that $2^\tau \geq \kappa$.*

That is, our Main Theorem is the best possible balanced generalization to trees of the Erdős-Rado Theorem for finitely many colours to ordinal goals.

# Examples for Particular Cardinals

**Example**

Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers $k$ and $n$,

$$\text{nonspecial tree } \rightarrow (\omega + n)^2_k \, .$$

# Examples for Particular Cardinals

## Example

Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers $k$ and $n$,

$$\text{nonspecial tree } \to (\omega + n)^2_k\,.$$

But this is subsumed by:

## Theorem (Todorcevic, 1985)

*For all $\alpha < \omega_1$ and $k < \omega$ we have*

$$\textit{nonspecial tree } \to (\alpha)^2_k\,.$$

# Examples for Particular Cardinals

### Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but $\xi$ must still be finite, so we have, for any natural numbers $k$ and $n$,

$$\text{non-}\mathfrak{c}\text{-special tree} \rightarrow (\omega_1 + n)^2_k .$$

# Examples for Particular Cardinals

## Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but $\xi$ must still be finite, so we have, for any natural numbers $k$ and $n$,

$$\text{non-}\mathfrak{c}\text{-special tree} \ \rightarrow (\omega_1 + n)^2_k \, .$$

Since $\mathfrak{p}$ (the pseudo-intersection number) is regular and $2^{<\mathfrak{p}} = \mathfrak{c}$, we have:

## Example

For any natural numbers $k$ and $n$,

$$\text{non-}\mathfrak{c}\text{-special tree} \ \rightarrow (\mathfrak{p} + n)^2_k \, .$$

# Ideals on Nonspecial Trees

Ideal of special subtrees corresponds to ideal of bounded subsets of an ordinal.

# Ideals on Nonspecial Trees

Ideal of special subtrees corresponds to ideal of bounded subsets of an ordinal.

### Question

*How do we generalize club, stationary, and nonstationary sets to trees?*

# Diagonal Unions

### Definition

Let $T$ be a tree. For a collection of subsets of $T$ indexed by nodes of $T$, i.e.

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\uparrow) .$$

# Diagonal Unions

## Definition

Let $T$ be a tree. For a collection of subsets of $T$ indexed by nodes of $T$, i.e.

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\uparrow) .$$

## Definition

Let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal. We define

$$\bigtriangledown \mathcal{I} = \left\{ \bigtriangledown_{t \in T} A_t : \langle A_t \rangle_{t \in T} \subseteq \mathcal{I} \right\} .$$

# Diagonal Unions

**Lemma**

*Let $T$ be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on $T$. Then*

$$\nabla \mathcal{I} = \{X \subseteq T : \exists \text{ regressive } f : X \to T \text{ constant only on sets from } \mathcal{I}\}.$$

# Diagonal Unions

**Lemma**

*Let $T$ be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on $T$. Then*

$$\bigtriangledown \mathcal{I} = \{X \subseteq T : \exists \text{ regressive } f : X \to T \text{ constant only on sets from } \mathcal{I}\}.$$

**Lemma (Idempotence Lemma)**

*Let $\lambda = \mathrm{ht}(T)$, and suppose $\lambda$ is any cardinal. If $\mathcal{I}$ is a $\lambda$-complete ideal on $T$, then $\bigtriangledown \bigtriangledown \mathcal{I} = \bigtriangledown \mathcal{I}$, that is, $\bigtriangledown \mathcal{I}$ is normal.*

# Stationary Subtrees

## Definition

Let $B \subseteq T$, where $T$ is a tree of height $\kappa^+$.
$B$ is a nonstationary subtree of $T$ if we can write

$$B = \nabla_{t \in T} A_t,$$

where each $A_t$ is a $\kappa$-special subtree of $T$.
Otherwise, $B$ is a stationary subtree of $T$.

$$\begin{aligned} NS_\kappa^T &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \nabla \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

# Stationary Subtrees

### Definition

Let $B \subseteq T$, where $T$ is a tree of height $\kappa^+$.
$B$ is a nonstationary subtree of $T$ if we can write

$$B = \nabla_{t \in T} A_t,$$

where each $A_t$ is a $\kappa$-special subtree of $T$.
Otherwise, $B$ is a stationary subtree of $T$.

$$\begin{aligned} NS_\kappa^T &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \nabla \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

### Theorem

*For any tree $T$ of height $\kappa^+$, the ideal $NS_\kappa^T$ is a normal ideal on $T$.*

# Very Nice Collections of Elementary Submodels

### Definition

Let $\lambda$ be any regular uncountable cardinal, and let $T$ be a tree of height $\lambda$. Suppose $\theta$ is a large enough regular cardinal, and let $\kappa$ be an infinite cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a $\kappa$-very nice collection of elementary submodels of $H(\theta)$ indexed by $T$ if:

1. For each $t \in T$, $N_t \prec H(\theta)$;
2. For each $t \in T$, $|N_t| < \lambda$;
3. For each $t \in T$, $t\downarrow \subseteq N_t$;
4. For $s, t \in T$ with $s <_T t$, $N_s \in N_t$.
5. For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.
6. The collection is continuous (with respect to its indexing), meaning that for all $t \in T$ with height a limit ordinal,

$$N_t = \bigcup_{s <_T t} N_s.$$

# Very Nice Collections of Elementary Submodels

**Lemma**

*Suppose $\lambda$ is any regular uncountable cardinal, $T$ is a tree of height $\lambda$, and $\theta$ is a large enough regular cardinal. If $\kappa$ is an infinite cardinal such that*

$$(\forall \text{ cardinals } \nu < \lambda) \left[\nu^{<\kappa} < \lambda\right],$$

*then there is a $\kappa$-very nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$.*

# Very Nice Collections of Elementary Submodels

Lemma

*Suppose $\nu$ is any infinite cardinal, $T$ is a non-$\nu$-special tree (necessarily of height $\nu^+$), and $\theta$ is a large enough regular cardinal. Suppose $\kappa$ is an infinite cardinal, and $\langle N_t \rangle_{t \in T}$ is a $\kappa$-very nice collection of elementary submodels of $H(\theta)$. Then the set*

$$\left\{ t \in T : [N_t]^{<\kappa} \subseteq N_t \right\}$$

*is a stationary subtree of $T$.*

# Nonreflecting Ideals Determined by Elementary Submodels

### Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Define a collapsing function

$$\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t\downarrow)$$

by setting, for $B \subseteq T$ with $B \in N$,

$$\pi_{N,t}(B) = B \cap t\downarrow.$$

# Nonreflecting Ideals Determined by Elementary Submodels

## Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Define a collapsing function

$$\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t\downarrow)$$

by setting, for $B \subseteq T$ with $B \in N$,

$$\pi_{N,t}(B) = B \cap t\downarrow.$$

## Definition

$$I_{N,t} = \{X \subseteq t\downarrow : X \subseteq A \text{ for some } A \in N \text{ with } t \notin A\}$$

# Eligibility Condition

### Remark

*In the special case where* $T$ *is a cardinal* $\lambda$ *and* $t = \sup(N \cap \lambda)$*, we have* $N \cap T \subseteq t{\downarrow}$*, so that elementarity of* $N$ *implies that* $\pi_{N,t}$ *is one-to-one, and the ideal* $I_{N,t}$ *is proper.*
*Not necessarily true for trees in general!*

# Eligibility Condition

### Remark

*In the special case where $T$ is a cardinal $\lambda$ and $t = \sup(N \cap \lambda)$, we have $N \cap T \subseteq t{\downarrow}$, so that elementarity of $N$ implies that $\pi_{N,t}$ is one-to-one, and the ideal $I_{N,t}$ is proper.*
*Not necessarily true for trees in general!*

To ensure that our ideals are proper, we impose an eligibility condition:

### Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. We say that a node $t \in T$ is $N$-eligible if

$$\forall B \in N \, [t{\downarrow} \subseteq B \implies t \in B] \, .$$

# Eligibility Condition

**Lemma**

*Suppose $\nu$ is any infinite cardinal, $T$ is a non-$\nu$-special tree (necessarily of height $\nu^+$), and $\theta$ is a large enough regular cardinal. Suppose $\kappa$ is an infinite cardinal, and $\langle N_t \rangle_{t \in T}$ is a $\kappa$-very nice collection of elementary submodels of $H(\theta)$. Then the set*

$$\{t \in T : t \text{ is } N_t\text{-eligible and } [N_t]^{<\kappa} \subseteq N_t\}$$

*is a stationary subtree of $T$.*