A Theory of Stationary Trees, and the Balanced Baumgartner-Hajnal-Todorcevic Theorem for Trees

Ari Meir Brodsky

Partition Calculus — Milestones

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Finite Ramsey Theorem For all positive integers m, r, and k, there is some positive integer n such that

$$n \to (m)^r_k\,.$$

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- Erdős and Rado (1956): the first systematic study of partition relations for linear orders (especially cardinals and ordinals) of all cardinalities
- Todorcevic (early 1980s): systematically extending the partition calculus to trees and partial orders

Arrow Notation for Partial Orders

Suppose $\langle P, <_P \rangle$ is any partial order.

- If α is any ordinal, $[P]^\alpha$ denotes the set of all linearly ordered chains in P of order-type α.
- If μ is any cardinal and α is any ordinal, the statement

$$P \to (\alpha)^2_\mu$$

means: For any colouring (partition function) $c : [P]^2 \to \mu$, there is a chain $X \in [P]^\alpha$ that is c-homogeneous, that is, $c''[X]^2 = \{\chi\}$ for some colour $\chi < \mu$.

Trees

Definition

A partial order $\langle T, <_T \rangle$ is a tree if for every $t \in T$,

$$t{\downarrow} = \{s \in T : s <_T t\}$$

is a well-ordered set.

Nonspecial Trees

Definition (Todorcevic, 1981)

A tree T is nonspecial if it cannot be written as a union of countably many antichains:

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Examples

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- (Kurepa, 1956) $w\mathbb{Q}$ (and its variant $\sigma\mathbb{Q}$), the collection of all well-ordered subsets of $\mathbb{Q}$, ordered by end-extension

Taller Nonspecial Trees

Definition (Todorcevic, 1985)

For any infinite cardinal κ, a tree T is non-κ-special if it cannot be written as a union of $\leq \kappa$ many antichains:

$$T \to (2)^1_\kappa .$$

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For any infinite cardinal κ, a tree T is non-κ-special if it cannot be written as a union of $\leq \kappa$ many antichains:

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Fact

For any tree T,

$$T \text{ is non-}\kappa\text{-special} \iff T \to (\kappa)^1_\kappa.$$

From Trees to Partial Orders

Theorem (Todorcevic 1985)

Let r be any positive integer, let κ and θ be cardinals, and for each $\gamma < \theta$ let α_γ be an ordinal. If every non-κ-special tree T satisfies

$$T \to (\alpha_\gamma)^r_{\gamma<\theta}\,, \tag{$**$}$$

then every partial order P satisfying $P \to (\kappa)^1_\kappa$ also satisfies the above partition relation $(**)$.

Balanced Partition Relations for Pairs

Theorem (Erdős-Rado, 1956)

Let κ be any infinite cardinal. Then for any cardinal $\mu < \mathrm{cf}(\kappa)$,

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + 1)^{2}_{\mu} .$$

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Theorem (Todorcevic, 1985)

Let κ *be any infinite cardinal. Then for any cardinal* $\mu < \mathrm{cf}(\kappa)$*,*

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \to (\kappa+1)^2_\mu\,.$$

Balanced Partition Relations for Pairs

Theorem (Baumgartner-Hajnal-Todorcevic, 1991)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + \xi)^2_k \, .$$

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Theorem (Baumgartner-Hajnal-Todorcevic, 1991)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\left(2^{<\kappa}\right)^{+} \to (\kappa + \xi)^2_k \, .$$

Theorem (Brodsky — Main Theorem)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \to (\kappa + \xi)^2_k \, .$$

Optimality of the Result

Theorem (Brodsky)

Let κ be any infinite regular cardinal. If T is any tree, then the following are equivalent:

1. $T \to (2)^1_{2^{<\kappa}}$
2. $T \to (2^{<\kappa})^1_{2^{<\kappa}}$
3. $T \to (\kappa + 1, \kappa)^2$
4. *For any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have*
$$T \to (\kappa + \xi)^2_k \, .$$

Optimality of the Result

Theorem (Main Theorem)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree } \rightarrow (\kappa + \xi)^2_k \, .$$

Theorem (Rebholz, Donder)

If $V = L$, and if κ is any regular uncountable cardinal that is not weakly compact, then

$$\kappa^+ \not\rightarrow (\kappa + \tau)^2_2 \, ,$$

where τ is the first cardinal such that $2^\tau \geq \kappa$.

That is, our Main Theorem is the best possible balanced generalization to trees of the Erdős-Rado Theorem for finitely many colours to ordinal goals.

Optimality of the Result

Theorem (Main Theorem)

Let κ be any infinite regular cardinal, let ξ be any ordinal such that $2^{|\xi|} < \kappa$, and let k be any natural number. Then

$$\textit{non-}\left(2^{<\kappa}\right)\textit{-special tree} \rightarrow (\kappa + \xi)^2_k \,.$$

Theorem

If $V = L$, and if κ is any successor cardinal, then

$$\kappa^+ \not\rightarrow (\kappa + 2)^2_{\kappa^-} \,.$$

Examples for Particular Cardinals

Example

Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers k and n,

$$\text{nonspecial tree } \to (\omega + n)^2_k .$$

Examples for Particular Cardinals

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Suppose $\kappa = \aleph_0$. Then $2^{<\kappa} = \aleph_0$, and we have, for any natural numbers k and n,

$$\text{nonspecial tree } \rightarrow (\omega + n)^2_k\,.$$

But this is subsumed by:

Theorem (Todorcevic, 1985)

For all $\alpha < \omega_1$ and $k < \omega$ we have

$$\textit{nonspecial tree } \rightarrow (\alpha)^2_k\,.$$

Examples for Particular Cardinals

Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have, for any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \rightarrow (\omega_1 + n)^2_k \,.$$

Examples for Particular Cardinals

Example

Let $\kappa = \aleph_1$. Then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have, for any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \rightarrow (\omega_1 + n)^2_k \,.$$

Since $\mathfrak{p}$ (the pseudo-intersection number) is regular and $2^{<\mathfrak{p}} = \mathfrak{c}$, we have:

Example

For any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \rightarrow (\mathfrak{p} + n)^2_k \,.$$

Ideals on Nonspecial Trees

Ideal of special subtrees corresponds to ideal of bounded subsets of an ordinal.

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Question

How do we generalize club, stationary, and nonstationary sets to trees?

Diagonal Unions

Definition

Let T be a tree. For a collection of subsets of T indexed by nodes of T, i.e.

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t{\uparrow}).$$

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$$\bigtriangledown_{t \in T} A_t = \bigcup_{t \in T} (A_t \cap t\!\uparrow) \, .$$

Definition

Let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal. We define

$$\bigtriangledown \mathcal{I} = \left\{ \bigtriangledown_{t \in T} A_t : \langle A_t \rangle_{t \in T} \subseteq \mathcal{I} \right\}.$$

Diagonal Unions

Lemma

Let T be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. Then

$$\nabla \mathcal{I} = \{X \subseteq T : \exists \text{ regressive } f : X \to T \text{ constant only on sets from } \mathcal{I}\}.$$

Diagonal Unions

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Lemma (Idempotence Lemma)

Let $\lambda = \mathrm{ht}(T)$, and suppose λ is any cardinal. If $\mathcal{I}$ is a λ-complete ideal on T, then $\nabla \nabla \mathcal{I} = \nabla \mathcal{I}$, that is, $\nabla \mathcal{I}$ is normal.

Proof of Idempotence Lemma I

Let $X \in \nabla\nabla\mathcal{I}$. We must show $X \in \nabla\mathcal{I}$.
As $X \in \nabla\nabla\mathcal{I}$, we can write

$$X = \nabla_{t \in T} A_t,$$

where each $A_t \in \nabla\mathcal{I}$. For each $t \in T$, we can write

$$A_t = \nabla_{s \in T} B_t^s,$$

where each $B_t^s \in \mathcal{I}$.

Proof of Idempotence Lemma II

Notice that for each $t \in T$, the only part of A_t that contributes to X is the part within $t\!\uparrow$. For each $s, t \in T$, the only part of B_t^s that contributes to A_t is the part within $s\!\uparrow$. We therefore have:

- If s and t are incomparable in T, we have $s\!\uparrow \cap\, t\!\uparrow = \emptyset$, so B_t^s does not contribute anything to X;
- If $t \leq_T s$ then $s\!\uparrow \cap\, t\!\uparrow = s\!\uparrow$, so the only part of B_t^s that contributes to X is within $s\!\uparrow$;
- If $s \leq_T t$ then $s\!\uparrow \cap\, t\!\uparrow = t\!\uparrow$, so the only part of B_t^s that contributes to X is within $t\!\uparrow$.

We collect the sets B_t^s whose contribution to X lies within any $r\!\uparrow$. We define, for each $r \in T$,

$$D_r = \bigcup_{t \leq_T r} B_t^r \cup \bigcup_{s \leq_T r} B_r^s.$$

Proof of Idempotence Lemma III

Since $\mathcal{I}$ is λ-complete and each r has height $< \lambda$, we have $D_r \in \mathcal{I}$.

Claim

We have

$$X = \bigtriangledown_{r \in T} D_r.$$

It follows that $X \in \bigtriangledown \mathcal{I}$, as required. □

Stationary Subtrees

Definition

Let $B \subseteq T$, where T is a tree of height κ^+.
B is a nonstationary subtree of T if we can write

$$B = \bigtriangledown_{t \in T} A_t,$$

where each A_t is a κ-special subtree of T.
Otherwise, B is a stationary subtree of T.

$$\begin{aligned} NS_\kappa^T &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \bigtriangledown \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

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$$\begin{aligned} NS_\kappa^T &= \{\text{nonstationary subtrees of } \kappa\} \\ &= \nabla \{\kappa\text{-special subtrees of } T\} \end{aligned}$$

Theorem
For any tree T of height κ^+, the ideal NS_κ^T is a normal ideal on T.

Stationary Subtrees

Theorem (Pressing-Down Lemma for Trees, Todorcevic)

Suppose T is a non-κ-special tree. Then NS_κ^T is a proper ideal on T, that is, $T \notin NS_\kappa^T$.

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Suppose T is a non-κ-special tree. Then NS_κ^T is a proper ideal on T, that is, $T \notin NS_\kappa^T$.

Theorem (Todorcevic, 1985)

If T is a non-κ-special tree, then the subtree

$$T \restriction S^{\kappa^+}_{\mathrm{cf}(\kappa)} = \{t \in T : \mathrm{cf}(\mathrm{ht}(t)) = \mathrm{cf}(\kappa)\}$$

is a stationary subtree of T.

Nonreflecting Ideals Determined by Elementary Submodels

Lemma

Suppose $N \prec H(\theta)$ *is an elementary submodel such that* $T \in N$. *Then the collection* $\mathcal{P}(T) \cap N$ *is a field of sets (set algebra) over the set* T.

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Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$. Then the collection $\mathcal{P}(T) \cap N$ is a field of sets (set algebra) over the set T.

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection

$$\{B \subseteq T : B \in N \text{ and } t \in B\}$$

is an ultrafilter in the set algebra $\mathcal{P}(T) \cap N$, and the collection

$$\{B \subseteq T : B \in N \text{ and } t \notin B\}$$

is the corresponding maximal (proper) ideal in the same set algebra.

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Define a collapsing function

$$\pi_{N,t} : \mathcal{P}(T) \cap N \to \mathcal{P}(t\downarrow)$$

by setting, for $B \subseteq T$ with $B \in N$,

$$\pi_{N,t}(B) = B \cap t\downarrow.$$

We then define the collection

$$\begin{aligned}\mathcal{A}_{N,t} = \mathrm{range}\left(\pi_{N,t}\right) &= \{B \cap t\downarrow : B \in \mathcal{P}(T) \cap N\} \\ &= \{B \cap t\downarrow : B \in N\} \subseteq \mathcal{P}\left(t\downarrow\right).\end{aligned}$$

Nonreflecting Ideals Determined by Elementary Submodels

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$.
Then the collection $\mathcal{A}_{N,t}$ is a set algebra over the set $t{\downarrow}$, and the collapsing function $\pi_{N,t}$ defines a surjective homomorphism of set algebras

$$\pi_{N,t} : \langle \mathcal{P}(T) \cap N, \cup, \cap, \setminus, \emptyset, T \rangle \to \langle \mathcal{A}_{N,t}, \cup, \cap, \setminus, \emptyset, t{\downarrow} \rangle \, .$$

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define the collections

$$\begin{aligned}\mathcal{G}_{N,t} &= \{\pi_{N,t}(A) : A \in \mathcal{P}(T) \cap N \text{ and } t \notin A\} \\ &= \{A \cap t{\downarrow} : A \in N \text{ and } t \notin A\} \subseteq \mathcal{A}_{N,t}, \text{ and} \\ \mathcal{G}^*_{N,t} &= \{\pi_{N,t}(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \in B\} \\ &= \{B \cap t{\downarrow} : B \in N \text{ and } t \in B\} \subseteq \mathcal{A}_{N,t}.\end{aligned}$$

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\mathcal{G}^*_{N,t} &= \{\pi_{N,t}(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \in B\} \\
&= \{B \cap t{\downarrow} : B \in N \text{ and } t \in B\} \subseteq \mathcal{A}_{N,t}.
\end{aligned}$$

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $\mathcal{G}_{N,t}$ is a (not necessarily proper) ideal in the set algebra $\mathcal{A}_{N,t}$, and $\mathcal{G}^*_{N,t}$ is the dual filter corresponding to $\mathcal{G}_{N,t}$.

Nonreflecting Ideals Determined by Elementary Submodels

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. We define

$$I_{N,t} = \{X \subseteq t\downarrow : X \subseteq Y \text{ for some } Y \in \mathcal{G}_{N,t}\}.$$

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Lemma

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq A \textit{ for some } A \in N \textit{ with } t \notin A\}$$

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Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the collection $I_{N,t}$ is a (not necessarily proper) ideal on $t{\downarrow}$, that is, an ideal in the whole power set $\mathcal{P}(t{\downarrow})$.

Eligibility Condition

Remark

In the special case where T is a cardinal λ and $t = \sup(N \cap \lambda)$, we have $N \cap T \subseteq t{\downarrow}$, so that elementarity of N implies that $\pi_{N,t}$ is one-to-one, and the ideal $I_{N,t}$ is proper.
Not necessarily true for trees in general!

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Not necessarily true for trees in general!

To ensure that our ideals are proper, we impose an eligibility condition:

Definition

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$.
We say that a node $t \in T$ is N-eligible if

$$\forall B \in N \, [t{\downarrow} \subseteq B \implies t \in B] \, .$$

Eligibility Condition

Lemma

Suppose $N \prec H(\theta)$ is an elementary submodel such that $T \in N$, and let $t \in T$. Then the following are all equivalent:

1. *t is N-eligible;*
2. *$t{\downarrow} \notin \mathcal{G}_{N,t}$, that is, $\mathcal{G}_{N,t}$ is a proper ideal in $\mathcal{A}_{N,t}$;*
3. *$t{\downarrow} \notin I_{N,t}$, that is, $I_{N,t}$ is a proper ideal on $t{\downarrow}$;*
4. *For all $A, B \in N$ with $A \cap t{\downarrow} = B \cap t{\downarrow}$, we have $t \in A \iff t \in B$ (even if $\pi_{N,t}$ is not injective);*
5. *For all $B \in N$, we have*

$$t \in B \iff B \cap t{\downarrow} \in \mathcal{G}^{+}_{N,t}.$$

Recursive Construction of Homogeneous Chain

Definition

If $c : [T]^2 \to \mu$ is a colouring, where μ is some cardinal, and $\chi < \mu$ is some ordinal (colour), and $t \in T$, define

$$c_\chi(t) = \{s <_T t : c\{s, t\} = \chi\} .$$

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$$c_\chi(t) = \{s <_T t : c\{s, t\} = \chi\}.$$

Lemma

Suppose we have cardinals μ and κ, a colouring $c : [T]^2 \to \mu$, and some colour $\chi < \mu$. Suppose also that $N \prec H(\theta)$ is an elementary submodel such that $T, c, \chi \in N$, and also $[N]^{<\kappa} \subseteq N$. Let $t \in T$ be a node such that $t\downarrow \subseteq N$.

If $X \subseteq c_\chi(t)$ is such that $X \in I^+_{N,t}$, then there is a χ-homogeneous chain $Y \in [X]^\kappa$.

Proof I

We shall recursively construct a χ-homogeneous chain

$$Y = \langle y_\eta \rangle_{\eta < \kappa} \subseteq X,$$

of order type κ, as follows:
Fix some ordinal $\eta < \kappa$, and suppose we have constructed χ-homogeneous

$$Y_\eta = \langle y_\iota \rangle_{\iota < \eta} \subseteq X$$

of order type η. We need to choose $y_\eta \in X$ such that $Y_\eta <_T \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous.
Since $Y_\eta \subseteq X \subseteq t\downarrow \subseteq N$ and $|Y_\eta| < \kappa$, the hypothesis that $[N]^{<\kappa} \subseteq N$ gives us $Y_\eta \in N$. Define

$$Z = \{s \in T : (\forall y_\iota \in Y_\eta)\,[y_\iota <_T s \text{ and } c\,\{y_\iota, s\} = \chi]\}\,.$$

Proof II

Since Z is defined from parameters T, Y_η, c, and χ that are all in N, it follows by elementarity of N that $Z \in N$, so that $Z \cap t{\downarrow} \in \mathcal{A}_{N,t}$.
Since $Y_\eta \subseteq X \subseteq c_\chi(t)$, it follows from the definition of Z that $t \in Z$. But then we have $Z \cap t{\downarrow} \in \mathcal{G}^*_{N,t} \subseteq I^*_{N,t}$. By assumption we have $X \in I^+_{N,t}$. The intersection of a filter set and a co-ideal set must be in the co-ideal, so we have $X \cap Z \in I^+_{N,t}$. In particular, this set is not empty, so we choose $y_\eta \in X \cap Z$. Because $y_\eta \in Z$, we have $Y_\eta <_T \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous, as required.

Very Nice Collections of Elementary Submodels

Definition

Let λ be any regular uncountable cardinal, and let T be a tree of height λ. Suppose θ is a large enough regular cardinal, and let κ be an infinite cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a κ-very nice collection of elementary submodels of $H(\theta)$ indexed by T if:

1. For each $t \in T$, $N_t \prec H(\theta)$;
2. For each $t \in T$, $|N_t| < \lambda$;
3. For each $t \in T$, $t{\downarrow} \subseteq N_t$;
4. For $s, t \in T$ with $s <_T t$, $N_s \in N_t$.
5. For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.
6. The collection is continuous (with respect to its indexing), meaning that for all $t \in T$ with height a limit ordinal,

$$N_t = \bigcup_{s <_T t} N_s.$$

Very Nice Collections of Elementary Submodels

Lemma

Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and θ is a large enough regular cardinal. If κ is an infinite cardinal such that

$$(\forall \text{ cardinals } \nu < \lambda) \left[\nu^{<\kappa} < \lambda\right],$$

then there is a κ-very nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$.

Very Nice Collections of Elementary Submodels

Lemma

Suppose ν is any infinite cardinal, T is a non-ν-special tree (necessarily of height ν^+), and θ is a large enough regular cardinal. Suppose κ is an infinite cardinal, and $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then the set

$$\left\{ t \in T : [N_t]^{<\kappa} \subseteq N_t \right\}$$

is a stationary subtree of T.

κ-Completeness and Eligibility Condition

Lemma

Suppose ν is any infinite cardinal, T is a non-ν-special tree (necessarily of height ν^+), and θ is a large enough regular cardinal. Suppose κ is an infinite cardinal, and $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then the set

$$\left\{ t \in T : t \text{ is } N_t\text{-eligible and } [N_t]^{<\kappa} \subseteq N_t \right\}$$

is a stationary subtree of T.