Structure and colouring of almost immersed surfaces, their compliments and their
intersection graphs
(Bar-Ilan University, 2012)

STRUCTURE AND COLOURING OF STABLE "ALMOST
IMMERSED" SURFACES, THEIR COMPLIMENTS AND THEIR
INTERSECTION GRAPHS

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1. Summary

Say you you have a stable map from a closed (but not necessarily connected) surface $F$ to $S^3$. Stable meaning that only have singularities and intersections that cannot be solved by slight changes - I.O.W regular intersections of two or three planes and cross-caps.

Such a map will create a graph of intersections in $F$ (pre images of all intersections and singularities) and that will divide the surface into a collection of surfaces with boundaries - which I call 2-components. The map image will also cut $S^3$ into different 3-manifolds with a boundary (a single boundary component if $i(F)$ is connected) which I call 3-components. Three very elementary questions you can ask yourself are what intersection graphs and collections of 2 and 3 components may arise by such maps.

A result by J. J. Nuño Ballesteros and Osamu Saeki from 1999 indicates that that 3-components can be coloured by two colours, such that adjacent components will have opposed colours, (actually, for maps $i : F \to M$ between surfaces and 3 manifolds this is equivalent to $i_{\ast}[f] \in H_2(M)$ being 0, and this can be extended to manifolds with a boundary, non-compact ones or higher dimensions by taking better homology groups).

I found that there is a similar colouring for the 2-components iff $F$ is orientable. These two colourings can actually be used instead of orientations, as one can deduce these form these given some conventions.

One can also induce a colouring on the interaction graph from this, this is equivalent to deciding what orientations do the two planes intersecting in a graph line (actually the image of a graph line) have. Such a thing is called an ”oriented cross-surface” and in 1998 Gui-Song Li asked which of these can be realised by a map. I found an ”easy to check” solution for this.

On 3-components, Tahl Nowik (ask him when) proved that a collection of components, classified by color and genus of boundary, and numbers of triple points and cross caps has a connected closed surface $F$, a map $i : F \to S^3$ and a colouring of $S^3 \setminus i(F)$
that realise it iff uphold two “euler char induced equations”. The euler char’ of the surface is determined by the equations, but there is no indication to orientability of said surface.

A closer look at Nowik’s construction revels that his construction will almost always result in a non-orientable surface, and in face a slight modification shows that every collection that upholds the equation can be realised by a non-orientable surface or the sphere (if the needed euler char is 2).

I found additional conditions which are necessary and sufficient for such a collection to also be realizable by an orientable surface. The condition is somewhat harder to formalize and considerably harder to prove. More so, both mine and Nowik’s proves of sufficiency are constructive - we construct a map that realizes the collection, with me using colour as a tool to make sure the surfaces I construct are orientable. My construction is very larger, so I’ll only be able to give a general outline and show small glimpses of it.

Lastly, and still ongoing, is the study of 2-components. Nowik showed what collections of 2-components, classified by genus and number of boundary components, and number of triple points (they only worked with immersions, so no cross-caps) can be realised by a sphere and Tomonori Hirasa showed did the same for a torus. I initially extendend their result to a $\mathbb{T}^2$ but showed it failed for $\mathbb{T}^3$ (there is a boundary case).

I then solved the case for all maps with no triple points (allowing cross caps), this time the 2-components were classified by colour as well. Unfortunately the complexity of the possibilities increases with every additional triple point. I solved the graph question as the first step to solve this, but in order to fined patterns for the conditions I need to generate an exam many maps, and this require generating — missile attack brake — collections of graphs on surfaces.

Briefly, you can smooth the coloured, almost immersed surface by gluing the images of 2-components of the same colour to each other thus creating many smooth surfaces embedded in $S^3$, and the places where the gluings (can anyone tell me how to write gluings without pissing off the every auto correct ever)
happened are graphs on the surfaces. and they divide them into the 2-components. these are graphs of a very specific kind, and both their edges and their vertexes are coloured.

Finally, to entice the reader, here is a sample of my speciality - horrible illustrations of maps:
REFERENCES


