A dimer model for the Jones polynomial of pretzel knots

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Outline

1 Graph and knot polynomials
   - The balanced overlaid Tait graph $\Gamma$
   - Tutte’s activity
   - Main results

2 Constructing the activity matrix
   - The bipartite adjacency submatrix
   - Kauffman’s trick $\kappa(\varepsilon)$ giving a Kasteleyn weighting
   - Writhe weighting $w(\varepsilon)$ and activity weighting $\alpha(\varepsilon)$

3 Examples, more results, and questions
   - The trefoil, $8_{19}$, and the $(-2, 3, 7)$-pretzel knot
   - Extending this class; applications to Khovanov homology
   - Future work
A knot $K$ is $S^1$ embedded in $S^3$. We orient the knot.

A knot diagram $D$ is the projection of the knot onto $\mathbb{R}^2$ with under- and over-crossing information.

**Theorem (Reidemeister 1926):**

Two diagrams represent the same knot $\iff$

$\exists$ a sequence of Reidemeister moves taking one to the other.
A knot invariant is an evaluation on a knot diagram that is constant under each of the three Reidemeister moves.
Graphs from knots: the signed Tait graph $G$

A **signed graph** has edges weighted $+1$ or $-1$.

Checkerboard color the regions of a knot diagram $D$.

**Definition:**

The **signed Tait graph** $G$ associated with $D$ has $V(G) = \{\text{colored regions}\}$ and $E(G) = \{\text{crossings of } D\}$.

Positive crossings: $\times$

Negative crossings: $\times$

Note that the dual $G^*$ comes from the uncolored regions.
**Definition:**

The *overlaid Tait graph* $\hat{\Gamma}$ associated with $D$ is bipartite with

$$V(\hat{\Gamma}) = \left[ E(G) \cap E(G^*) \right] \sqcup \left[ V(G) \sqcup V(G^*) \right]$$

and $E(\hat{\Gamma})$ the half-edges of $G$ and $G^*$.

Each face in the overlaid Tait graph $\hat{\Gamma}$ is a square.
**Definition:**

The **balanced overlaid Tait graph** $\hat{\Gamma}$ associated with $D$ is obtained from $\hat{\Gamma}$ by removing two vertices from the larger set that lie on the same face:

```
   * *
  /   \
 /     \   
*     *
|     |
|     |
\       / \
   * *  *
```

“Balanced” means the two vertex sets are the same size.
Graphs from knots: the signed Tait graph $G$

The oriented knot $8_{19}$,
Graphs from knots: the signed Tait graph $G$

a checkerboard coloring,
Graphs from knots: the signed Tait graph $G$,

the corresponding signed Tait graph $G$, 
Graphs from knots: the signed Tait graph $G$.

the dual signed Tait graph $G^*$,
the overlaid Tait graph $\hat{\Gamma}$ (all faces are square),
Graphs from knots: the balanced overlaid Tait graph $\Gamma$

and the balanced overlaid Tait graph $\Gamma$. 

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A dimer model for the Jones polynomial of pretzel knots
**Definition (Tutte’s Activity words):**

For spanning tree $S$ of signed graph $G$ with ordered edges, assign an activity letter to each edge:

<table>
<thead>
<tr>
<th></th>
<th>live</th>
<th>dead</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>$L$</td>
<td>$D$</td>
</tr>
<tr>
<td></td>
<td>internal</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>$\bar{L}$</td>
<td>$\bar{D}$</td>
</tr>
<tr>
<td></td>
<td>external</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\ell$</td>
<td>$d$</td>
</tr>
<tr>
<td></td>
<td>external</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{\ell}$</td>
<td>$\bar{d}$</td>
</tr>
</tbody>
</table>

Activity ("live" or "dead") is determined by the ordering:
Tutte’s activity words: Definition

For external edge $e \notin S$, there is a unique cycle in $S \cup \{e\}$. $e \notin S$ is live if it is the lowest-ordered edge in the cycle.

For internal edge $e \in S$, the graph $S\setminus\{e\}$ is disconnected. $e \in S$ is live if it is the lowest-ordered edge that reconnects.

Let $a(e, S)$ be the activity letter for the edge $e$ and the tree $S$, and let $a(S)$ be the activity word associated to the tree $S$. 
Tutte’s activity words: Example

For the (all positive) graph $G$
Tutte’s activity words: Example

and the spanning tree $S_1$,
Tutte’s activity words: Example

the first edge is $L$,
Tutte’s activity words: Example

the second edge is $d$. 

$A$ dimer model for the Jones polynomial of pretzel knots
and the third edge is also $d$, 
Tutte’s activity words: Example

\[ a(S_1) = (Ldd). \]
For the spanning tree $S_2$, ...
the first edge is \( \ell \),
the second edge is $D$, 
and the third edge is $d$, 
Tutte’s activity words: Example

\[ a(S_2) = (\ell Dd). \]
And for the spanning tree $S_3$,
Tutte’s activity words: Example

the first edge is \( \ell \),
the second edge is $\ell$, 
Tutte's activity words: Example

and the third edge is $D$,
Tutte’s activity words: Example

giving the activity word \( a(S_3) = (\ell \ell D) \).
Thus the activity words are $(Ldd)$, $(\ell Dd)$, and $(\ell \ell D)$. 
Tutte polynomial $T(G; x, y)$

For (unsigned) graph $G$ and edge $e$, let $G \backslash e$ be the deletion of $e$ and $G / e$ the contraction.

**Definition (Tutte):**
The (unsigned) Tutte polynomial $T(G; x, y) =$

$$T(G \backslash e; x, y) + T(G / e; x, y) \text{ if } e \text{ is neither a bridge nor a loop,}$$

$$x \# \text{ bridges } y \# \text{ loops} \text{ if all edges are bridges and loops.}$$

**Theorem (Tutte):**

$$T(G; x, y) = \sum_S x^{\# L} y^{\# \ell} = \sum_S \prod_{e \in E(G)} a(e, S)|_T$$
### Tutte polynomial $T(G; x, y)$

The activity evaluations for the Tutte polynomial $T(G; x, y)$

<table>
<thead>
<tr>
<th>$a(e, S)$</th>
<th>$L$</th>
<th>$D$</th>
<th>$\ell$</th>
<th>$d$</th>
<th>$\bar{L}$</th>
<th>$\bar{D}$</th>
<th>$\bar{\ell}$</th>
<th>$\bar{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(e, S)</td>
<td>_T$</td>
<td>$x$</td>
<td>1</td>
<td>$y$</td>
<td>1</td>
<td>$___$</td>
<td>$___$</td>
<td>$___$</td>
</tr>
</tbody>
</table>
Signed Tutte polynomial $Q(G; A, B, \delta)$

**Definition (Kauffman):**
The signed Tutte polynomial $Q(G; A, B, \delta) =$

\[
\begin{cases}
AQ(G\setminus \overline{e}; A, B, \delta) + BQ(G/\overline{e}; A, B, \delta) & \text{non-bridge/loop } \overline{e}, \\
BQ(G\setminus e; A, B, \delta) + AQ(G/e; A, B, \delta) & \text{non-bridge/loop } e, \\
x \# \text{ bridges} + y \# \text{ loops} & \text{all bridges/loops},
\end{cases}
\]

setting $x = A + B\delta$ and $y = A\delta + B$.

**Theorem (Kauffman):**

\[
Q(G; A, B, \delta) = \sum_{S} \prod_{e \in E(G)} a(e, S)|_Q
\]
Signed Tutte polynomial $Q(G; A, B, \delta)$

<table>
<thead>
<tr>
<th>$a(e, S)$</th>
<th>$L$</th>
<th>$D$</th>
<th>$\ell$</th>
<th>$d$</th>
<th>$\bar{L}$</th>
<th>$\bar{D}$</th>
<th>$\bar{\ell}$</th>
<th>$\bar{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(e, S)</td>
<td>_Q$</td>
<td>$x$</td>
<td>$A$</td>
<td>$y$</td>
<td>$B$</td>
<td>$y$</td>
<td>$B$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

The activity evaluations for the signed Tutte polynomial $Q(G; A, B, \delta)$ with $x = A + B\delta$ and $y = A\delta + B$
**Kauffman bracket polynomial** $\langle K \rangle$

**Definition (Kauffman):**

The **Kauffman bracket polynomial** $\langle L \rangle$ of link $L$ satisfies

1. **Smoothing relation:** $\langle L \rangle = A \langle L_0 \rangle + A^{-1} \langle L_\infty \rangle$
2. **Stabilization:** $\langle U \sqcup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$
3. **Normalization:** $\langle U \rangle = 1$.

For knot $K$ with signed Tait graph $G$,

**Theorem (Thistlethwaite):**

$$\langle K \rangle = \sum_S \prod_{e \in E(G)} a(e, S) |_V$$
### Kauffman bracket polynomial $\langle K \rangle$

<table>
<thead>
<tr>
<th>$a(e, S)$</th>
<th>$L$</th>
<th>$D$</th>
<th>$\ell$</th>
<th>$d$</th>
<th>$\bar{L}$</th>
<th>$\bar{D}$</th>
<th>$\bar{\ell}$</th>
<th>$\bar{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(e, S)</td>
<td>_V$</td>
<td>$-A^{-3}$</td>
<td>$A$</td>
<td>$-A^3$</td>
<td>$A^{-1}$</td>
<td>$-A^3$</td>
<td>$A^{-1}$</td>
<td>$-A^{-3}$</td>
</tr>
</tbody>
</table>

The activity evaluations for the Kauffman bracket $\langle K \rangle$
The writhe $w(D)$ of an oriented diagram is the sum:

$w(D) = +1 - 1$

**Definition (Jones):**

The Jones polynomial $V_L(t)$ of link $L$ satisfies, for $A = t^{-1/4}$,

$$V_L(t) = (-A^{-3})^{w(D)} \langle L \rangle.$$

For a knot $K$ with signed Tait graph $G$,

**Theorem (Thistlethwaite):**

$$V_K(t) = (-A^{-3})^{w(D)} \sum_S \prod_{e \in E(G)} a(e, S)|_V$$
A \textit{dimer} in a (bipartite) graph is just an edge.

A \textit{perfect matching} $\mu$ is a collection of non-incident dimers that covers the graph.
## The correspondence between $G$ and $\Gamma$

<table>
<thead>
<tr>
<th>Signed Tait graph $G$</th>
<th>Balanced overlaid Tait graph $\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>edge $e \in E(G)$</td>
<td>edge $\epsilon \in E(\Gamma)$</td>
</tr>
<tr>
<td>Squared incidence matrix</td>
<td>Bipartite adjacency submatrix</td>
</tr>
<tr>
<td>Rooted spanning tree $S$ in $G$</td>
<td>Perfect matching $\mu$ in $\Gamma$</td>
</tr>
<tr>
<td>Activity $a(e, S)$</td>
<td>Activity weighting $\alpha(\epsilon)$</td>
</tr>
</tbody>
</table>
The $(n_1, n_2, \ldots, n_k)$-pretzel knot $P$ with an ordering on the crossings.
**Main Theorem:**

Summing over all perfect matchings \( \mu \) in \( \Gamma \) and taking the product over all edges \( \varepsilon \in \mu \),

\[
\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\varepsilon) = \sum_{S} a(S)
\]

gives the complete list of activity words \( a(S) \) associated with spanning trees \( S \) of \( G \) associated with the diagram of \( P \).
Main results: Jones polynomial

**Main Corollary:**

Summing over all perfect matchings \( \mu \) in \( \Gamma \) and taking the product over all edges \( \varepsilon \in \mu \),

\[
\sum_{\mu} \prod_{\varepsilon \in \mu} w(\varepsilon) \alpha(\varepsilon)|_V = V_P(t)
\]

gives the Jones polynomial \( V_P(t) \) of \( P \).
Main results: matrix determinant

**Computational Corollary:**

Let $\varepsilon_{ij}$ be the edge $\varepsilon \in E(\Gamma)$ between the $i$-th vertex coming from the crossings and the $j$-th vertex coming from the regions.

Let $A = (\kappa(\varepsilon_{ij}) w(\varepsilon_{ij}) \alpha(\varepsilon_{ij}) | V)$ be the activity weighting on the bipartite adjacency submatrix associated with $P$. Then

$$\det(A) = V_P(t)$$

gives the Jones polynomial $V_P(t)$ of $P$ up to sign.
The results above hold for pretzel knots $\forall k \in \mathbb{N}, |n_i| \in \mathbb{N}$.

One cannot hope to achieve this result for a general knot $K$.

**Theorem (Jaeger-Vertigan-Welsh):**

Determining the Jones polynomial is $\#P$-hard.
The **incidence matrix** has rows labelled by edges and columns labelled by vertices.

\[ m_{ij} = 0 \text{ if the } i\text{-th edge is not incident with the } j\text{-th vertex.} \]

This \(|E| \times |V|\) matrix is in general not square.

The **squared incidence matrix** is the incidence matrix of the graph together with the incidence matrix for the dual graph with a column of each deleted.

This \(|E| \times [(|V| - 1) + (|F| - 1)]\) matrix is square.
Matrices from graphs: the adjacency matrix

The **adjacency matrix** rows and columns labelled by vertices. 

$m_{ij} = 0$ if the $i$-th vertex is not adjacent to the $j$-th vertex. 

For a bipartite graph, present this square matrix in block form

$$
\begin{pmatrix}
0 & M \\
M^T & 0
\end{pmatrix}
$$

The **bipartite adjacency submatrix** is the block $M$. 

**Proposition:**

The squared incidence matrix of the Tait graph $G$ is the bipartite adjacency submatrix of the balanced overlaid Tait graph $\Gamma$. 

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A dimer model for the Jones polynomial of pretzel knots
Recall the determinant of a matrix $M = (m_{ij})$

$$det(M) = \sum_{\sigma \in S} \prod_{i} (-1)^{\text{sign}(\sigma)} m_{i\sigma(i)}$$

The **permanent** or **unsigned determinant** is

$$\text{perm}(M) = \sum_{\sigma \in S} \prod_{i} m_{i\sigma(i)}$$
Proposition:

The terms in the permanent expansion of a bipartite adjacency submatrix associated with a(n unsigned) balanced bipartite graph give the complete list of perfect matchings of the graph.

Proof:

Each term in the permanent expansion is a permutation \( \sigma \) matching each vertex \( i \) in the first vertex set to a vertex \( \sigma(i) \) in the second vertex set. \( \square \)
Kauffman’s trick $\kappa(\varepsilon)$: signing the entries

This will be used to sign the corresponding entries in the matrix.

A **Kasteleyn weighting** of a plane bipartite graph is a signing of the edges such that the number of negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

**Lemma:**

Suppose $G$ has a Kasteleyn weighting. Then so does $G\setminus e$. 
Proof:

Let $e$ be incident with two faces of length $f_1$ and $f_2$. Delete $e$ to replace these with a face of length $f_1 + f_2 - 2$.

<table>
<thead>
<tr>
<th>$f_1$</th>
<th># negs</th>
<th>$f_2$</th>
<th># negs</th>
<th>$f_1 + f_2 - 2 \mod 4$</th>
<th># negs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>odd</td>
<td>0</td>
<td>odd</td>
<td>2</td>
<td>even</td>
</tr>
<tr>
<td>0</td>
<td>odd</td>
<td>2</td>
<td>even</td>
<td>0</td>
<td>odd</td>
</tr>
<tr>
<td>2</td>
<td>even</td>
<td>0</td>
<td>odd</td>
<td>0</td>
<td>odd</td>
</tr>
<tr>
<td>2</td>
<td>even</td>
<td>2</td>
<td>even</td>
<td>2</td>
<td>even</td>
</tr>
</tbody>
</table>

Then $\#$ negs changes by 0 or 2 (an even number) compared with the sum of $\#$ negs in $f_1$ and $f_2$. □
Kauffman’s trick $\kappa(\varepsilon)$: signing the entries

**Kauffman’s trick** $\kappa(\varepsilon)$ to distribute signs to the edges of the balanced overlaid Tait graph $\Gamma$ coming from a knot diagram:

![Graph Diagram]

**Proposition:**

Kauffman’s trick $\kappa(\varepsilon)$ provides a Kasteleyn weighting.
Proof:

Each face in the overlaid Tait graph $\hat{\Gamma}$ is a square. The balanced overlaid Tait graph $\Gamma$ is obtained by edge deletions.

The assigning of a negative edge affects exactly one of the NW and SW sides of the square. □
**Proposition:**

The determinant expansion of a bipartite adjacency submatrix associated with a Kasteleyn-weighted balanced bipartite graph gives the complete list of perfect matchings up to sign.

**Proof:**

Two permutations differ by a transposition $\leftrightarrow$

$\exists$ four non-zero terms in a rectangle in the matrix $\leftrightarrow$

$\exists$ a square face in the graph.

$\exists!$ negative sign in each square, so these have opposite signs in both the matrix and the perfect matching. $\square$
**Proposition:**

Given a knot diagram, there is a bijection between perfect matchings of the balanced overlaid Tait graph $\Gamma$ and rooted spanning trees of the Tait graph $G$.

**Proof:**

\[
\{\text{perfect matchings of } \Gamma\} \cong \\
\{\text{permanent expansion of the bipartite adjacency submatrix}\} \cong \\
\{\text{permanent expansion of the squared incidence matrix}\} \cong \\
\{\text{partition of edges } T \subset G \text{ and } T^c \subset G^*\}
\]

$T$ spans; if $\exists$ cycle $C$, then $*$ must be on one side of $C$.

$T^c$ spans; $\exists$ cycle in the dual on the same side of $C$.

Repeat this process, yielding an infinite graph. \(\square\)
Correspondence between edges $\varepsilon$ of the overlaid Tait graph $\hat{\Gamma}$ and directed edges $e$ of the (directed) Tait graph $G$. 
The **writhe weighting** $w(\varepsilon)$ on $\varepsilon \in E(\Gamma)$ is $(-A)^{-3}$ or $(-A)^3$:
Let $\varepsilon_{ij}$ be the edge $\varepsilon \in E(\Gamma)$ btwn the $i$-th vertex coming from the crossings and the $j$-th vertex coming from the regions.

The **writhe weighting** $w(\varepsilon_{ij})$ is determined by the sign of the $i$-th vertex coming from the crossings.

At the level of the bipartite adjacency submatrix, this means multiplying all entries in each row by $(-A)^{-3}$ or $(-A)^3$. 
Activity weighting $\alpha(\varepsilon)$: edges $\varepsilon \in E(\Gamma)$

The bipartition of the vertices in $\Gamma$ is really the tripartition

$V(\Gamma) = [E(G) \cap E(G^*)] \sqcup [V(G)] \sqcup [V(G^*)] = V_E \sqcup V_V \sqcup V_F$

**Definition**

The *activity weighting* $\alpha(\varepsilon)$ on $\varepsilon = v_i v_j \in E(\Gamma)$ is given by:

- an edge incident with $v_i \in V_E$ is $+$ or $-$ if $e \in E(G)$ is $+$ or $-$;
- an edge incident with $v_j \in V_V$ is internal, and
- an edge incident with $v_j \in V_F$ is external; and
- an edge is live if it connects the lowest-ordered $v_i \in V_E$ to the vertex $v_j \in V_V \sqcup V_F$ and dead otherwise.
Activity weighting $\alpha(\varepsilon)$: bipartite adjacency submatrix

The entries of the bipartite adjacency submatrix associated to the balanced overlaid Tait graph $\Gamma$ obey the following rules:

- ordered rows associated with $V_E$ are all positive or all negative;
- columns associated with $V_V$ are internal and $V_F$ are external;
- the first non-zero entry in a column is live, the rest are dead.
Graph and knot polynomials

Constructing the activity matrix

Examples, more results, and questions

The bipartite adjacency submatrix

Kauffman’s trick \( \kappa(\varepsilon) \) giving a Kasteleyn weighting

Writhe weighting \( w(\varepsilon) \) and activity weighting \( \alpha(\varepsilon) \)

\[
\begin{bmatrix}
L & \ell & L \\
D & : & D \\
& \ddots & \ddots \\
& \ddots & L \\
& & D \\
\end{bmatrix}
\]
A note on the proof

The proof that the terms in the determinant expansion give the

**exact activity words** for the pretzel knots comes from

a technical lemma (C.) on the activity of paths.

One difficulty to extending this class is producing

a complete list of activity words for more general knots.
Example 1: the Jones polynomial for the trefoil

Example 1: the $(1, 1, 1)$-pretzel knot
Example 1: the Jones polynomial for the trefoil

Tait graph $G$

![Tait Graph](image)
Example 1: the Jones polynomial for the trefoil

balanced overlaid Tait graph $\Gamma$
Example 1: the Jones polynomial for the trefoil

The spanning trees give activity words \((Ldd), (ℓDd),\) and \((ℓℓD)\):

\[
\left(\begin{array}{c|cc}
L & ℓ & \\
D & -d & ℓ \\
D & -d & \\
\end{array}\right)
\]

With writhe \((-A^{-3})^{-3}\), the determinant is \(A^4 + A^{12} - A^{16} = t^{-1} + t^{-3} - t^{-4}\), the Jones polynomial of the trefoil.
Example 2: the Jones polynomial for $8_{19}$

\textit{Example 2:} the $(-2, 3, 3)$-pretzel knot
Example 2: the Jones polynomial for $8_{19}$

Tait graph $G$

![Tait graph G](image-url)
Example 2: the Jones polynomial for $8_{19}$

balanced overlaid Tait graph $\Gamma$
Example 2: the Jones polynomial for $8_{19}$

With writhe $(-A^{-3})^8$, the determinant is $-A^{-32} + A^{-20} + A^{-12}$
$= -t^8 + t^5 + t^3$, the Jones polynomial of $8_{19}$. 

\[
\begin{pmatrix}
\bar{L} & \bar{\ell} \\
\bar{D} & -d \\

\begin{array}{ccc}
-L & D & D \\
D & -L & -L \\
D & -L & -L \\
\end{array}
\end{pmatrix}
\]
**Example 3:** the Jones polynomial for the \((-2, 3, 7)\)-pretzel knot

\[
\begin{pmatrix}
\bar{L} \\
\bar{D}
\end{pmatrix}
\begin{pmatrix}
-L \\
D \\
-D \\
L
\end{pmatrix}

= \begin{pmatrix}
\bar{L} \\
\bar{D} \\
\bar{L} \\
\bar{D}
\end{pmatrix}
\begin{pmatrix}
\ell \\
-d \\
\ell \\
d \\
\ell \\
d \\
d \\
d
\end{pmatrix}

With writhe \((-A^{-3})^{12}\), the determinant is

\[-A^{-40} + A^{-36} - A^{-32} + A^{-16} + A^{-8} = -t^{10} + t^9 - t^8 + t^2.\]
**Property: (Subdivision/Doubling)**

Let $e_n \in E(G)$ be incident with the omitted vertex and face. Then if the activity weighting on $\Gamma$ provides a dimer model for $G$, this can be extended to one for $G \cup \{e_{n+1}\}$ that subdivides or doubles $e_n$. 

![Graph Diagram](image-url)
Leaving the class of pretzel knots

**Proof:**

Row $e_n$ in squared incidence matrix only has $D$ and $d$.

Subdivide to get a new row $e_{n+1}$ and a new vertex column.

Entries in this column are 0 except for $L$ and $D$.

Determinant expansion terms give $DD$ and $dD$, preserving
the first $n$ pivots, or $Ld$, preserving the first $n-1$ pivots.

These cases are exactly the possibilities for activity words.

The dual case of doubling works similarly. □
Another corollary to the Main Theorem

**Reduced Khovanov homology chain complex** $\widetilde{\mathcal{C}Kh}$:

<table>
<thead>
<tr>
<th>$a(e, S)$</th>
<th>$L$</th>
<th>$D$</th>
<th>$\ell$</th>
<th>$d$</th>
<th>$\overline{L}$</th>
<th>$\overline{D}$</th>
<th>$\overline{\ell}$</th>
<th>$\overline{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(e, S)</td>
<td>_K$</td>
<td>$uv$</td>
<td>$v$</td>
<td>$u^{-1}$</td>
<td>$1$</td>
<td>$u^{-1}$</td>
<td>$1$</td>
<td>$u$</td>
</tr>
</tbody>
</table>
**Corollary:**

Summing over all perfect matchings $\mu$ in $\Gamma$ and taking the product over all edges $\varepsilon \in \mu$,

$$\sum\prod_{\mu \varepsilon \in \mu} \alpha(\varepsilon)|\kappa$$

gives the two-variable polynomial $\widetilde{CKh}_P(q, t)$ for the reduced Khovanov chain complex of $P$ up to sign.
Questions

What can these easy computations teach us about the Jones polynomial of the class of pretzel knots?

The activity weighting can be extended to a larger class of knots, but how far can it go?

The first-order differential of reduced Khovanov homology can be found in the activity matrix, but the higher-order ones?