EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY 88537

Contents

1. Ceva, Menelaus

The material in this section is from the book by H. Perfect [9] (see the bibliography).

Collinearity and concurrency.

Ceva and Menelaus Theorems.

Explain "signed length" at great length.

Concurrency of altitudes, angle bisectors, and medians in a triangle.

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2. Desargues, Pappus, Pascal, Brianchon

There are two points of view on Desargues' theorem: the slick statement: "triangles in pespective from a point, are in perspective from a line", and a detailed statement in terms of specific intersections, etc.

On must insist on the explicit version, for otherwise students come away without a true understanding of Desargues' theorem.

Pappus's theorem.

Pascal's theorem (six points on a circle).

Brianchon's theorem.

Note that the usual statement of Brianchon's theorem in terms of the sides and diagonals of a hexagon, at least on the surface of it, is *less general* than the polar dual of Pascal.

Students need to be able to state this duality precisely in terms of *labeled points*. Here again the connection between the dual theorems needs to be explained in detail, otherwise the students don't learn to translate theorems to their duals/polars.

Internal and external bisectors.

Harmonic 4-tuple.

3. Axioms of affine planes and projective planes

The material is in Hartshorne [5].

3.1. **Axioms of affine plane.**

Axiomatisation of affine planes: 3 axioms.

Proofs of basic results derived from the axioms.

Note that a line by definition is parallel to itself (in previous years students protested, citing Margolis).

3.2. **Axiom of projective plane.**

The 4 axioms of a projective plane.

3.3. **Adding points at infinity.**

The model obtained by completing the affine line by adding points at infinity defined by pencils of parallel lines. This treatment follows Hartshorne [5].

Proof that this model satisfies the four axioms.

4. Isomorphic models

4.1. **Homogeneous coordinates.** Consider the example of projective plane, denoted S, defined as set of lines through the origin in $\mathbb{R}^3$.

Recall that a (projective) line of S is the set of 1-dimensional subspaces lying in a given 2-dimensional subspace of $\mathbb{R}^3$.

A point $P \in S$ is a line through $O = (0, 0, 0)$ (the origin). We will represent P by choosing any point (x_1, x_2, x_3) on the line, provided the point is different from the origin.

Definition 4.1. The numbers x_1, x_2, x_3 are the homogeneous coordinates of P.

Any other point on the line has the coordinates $(\lambda x_1, \lambda x_2, \lambda x_3)$, where $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Thus S is the collection of equivalence classes of triples (x_1, x_2, x_3) of real numbers, not all zero, where two triples (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) represent the same point if and only if there exists $\lambda \in \mathbb{R}$ such that

$$x'_i = \lambda x_i \text{ for each } i = 1, 2, 3.$$

Remark 4.2. If $x_3 = 0$ then the point $P = (x_1, x_2, 0)$ spans a line in the (x_1, x_2) plane of slope $m = \frac{x_2}{x_1}$. The slope is infinite, $m = \infty$, if and only if $x_1 = 0$.

Remark 4.3. Since the euqation of a plane in $\mathbb{R}^3$ passing through O is of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \quad \text{where not all } a_i \text{ are zero,} \tag{4.1}$$

we see that equation (4.1) is also the equation of a line of S in terms of the homogeneous coordinates.

4.2. **Isomorphism of two models.** We will prove that the completion at infinity and homogeneous coordinates give isomorphic models of the real projective plane.

Definition 4.4. Two projective planes S and S' are isomorphic if there exists a one-to-one correspondence $T\colon S \to S'$ which takes collinear points to collinear points.

Proposition 4.5. *The projective plane S defined by homogeneous coordinates which are real numbers, is isomorphic to the projective plane $\overline{E}$ obtained by completing the ordinary affine plane of Euclidean geometry.*

To prove the proposition, we will use the notation (x_1, x_2, x_3) for the homogeneous coordinates in S. We will use (x, y) for the coordinates in the Euclidean plane E, whose completion by points at infinity is denoted $\overline{E}$. Recall that $\overline{E} = E \cup \omega$. Thus our second model is $\overline{E}$. The points of $\overline{E}$ are

- points (x, y) of E, and
- ideal points at infinity of the form ℓ_∞ on the horizon ω, one ideal point for each pencil of parallel lines.

Lemma 4.6. *An ideal point ℓ_∞ is uniquely determined by its slope $m \in \mathbb{R} \cup \{\infty\}$.*

Proof. Indeed, a pencil of parallel lines $[\ell]$ is uniquely determined by its slope m. Here m may be either a real number or ∞ (when the lines of the pencil are vertical). $\square$

To prove Proposition 4.5, we will define a mapping $T\colon S \to \overline{E}$ which will exhibit an isomorphism between S and $\overline{E}$. Let $P = (x_1, x_2, x_3)$ be a point of S.

Remark 4.7. The idea is to cut all the lines by the plane $x_3 = 1$ in $\mathbb{R}^3$. Then non-horizontal lines will correspond to finite points of $\overline{E}$ whereas horizontal lines will correspond to ideal points at infinity of $\overline{E}$.

Thus, we consider the following two cases:

(1) If $x_3 \neq 0$, we define $T(P)$ to be the point of $E \subseteq \overline{E}$ with coordinates $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$. Note that this is uniquely determined, since if we replace (x_1, x_2, x_3) by $(\lambda x_1, \lambda x_2, \lambda x_3)$, then x and y do not change. Every point of $(x, y) \in E$ can be obtained this way by using the triple $(x, y, 1)$.
(2) If $x_3 = 0$, then we define $T(P)$ to be the ideal point of $\overline{E}$ with slope $m = \frac{x_2}{x_1}$. This makes sense since x_1 and x_2 cannot be both zero, replacing $(x_1, x_2, 0)$ by $(\lambda x_1, \lambda x_2, 0)$ does not change m, and each value of m occurs. Namely, if $m \neq 0$ then T sends $(1, m, 0)$ to the ideal point with slope m, and if $m = \infty$ then T sends $(0, 1, 0)$ to the ideal point with that slope.

The map $T\colon S \to \overline{E}$ thus defined is therefore one-to-one and onto. Therefore Proposition 4.5 results from the following theorem.

Theorem 4.8. *The map T defined above sends collinear points to collinear points.*

Recall that a line $\ell \subseteq S$ is given by equation (4.1), namely $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$. We now consider two cases.

Case 1. Suppose a_1 and a_2 are not both zero. Then the theorem follows from the following lemma.

Lemma 4.9. *Consider the images under T of the points of ℓ. Then*

(1) *points with nonzero x_3 remain collinear in $\overline{E}$, and form a line of slope $-\frac{a_1}{a_2}$;*
(2) *the point with $x_3 = 0$ is sent under T to the ideal point with slope $m = -\frac{a_1}{a_2}$.*

Proof. If $x_3 \neq 0$ we can choose a representative with homogeneous coordinates $(x_1, x_2, 1)$. Thus we have $a_1x_1 + a_2x_2 + a_1 = 0$. Hence the line ℓ gets mapped to the line $a_1x + a_2y = -a_3$ in $\overline{E}$. Equivalently,

$$y = -\frac{a_1}{a_2}x - \frac{a_3}{a_2},$$

which is a line of the required slope $m = -\frac{a_1}{a_2}$.

When $x_3 = 0$, the points satisfying (4.1) are given by $x_1 = \lambda a_2$ and $x_2 = -\lambda a_1$. The map T by definition sends this point to the ideal point with slope $m = -\frac{a_1}{a_2}$. □

Case 2. The remaining case is $a_1 = a_2 = 0$. In this case, the line ℓ in S is defined by the equation $x_3 = 0$. Any point of S with $x_3 = 0$ goes to an ideal point of $\overline{E}$, and these points form precisely the horizon line $\omega \subseteq \overline{E}$.

4.3. Affine neighborhoods.

Definition 4.10. The real projective plane $\mathbb{R}P^2$ is the model of the axioms $P1$ through $P4$ obtained either via homogeneous coordinates or via completion.

From now on when we mention the real projective plane we will refer to its isomorphism type, regardless of the construction chosen.

Definition 4.11. An *affine neighborhood* in the real projective plane $\mathbb{R}P^2$ is obtained by deleting a projective line.

Theorem 4.12. *Each affine plane of $\mathbb{R}P^2$ is isomorphic to the Euclidean plane.*

Proof. Deleting the projective line $\omega \subseteq \overline{E}$ produces the Euclidean plane E. The claim follows from the fact that any two planes in $\mathbb{R}^3$ differ by a linear transformation that preserves 1-dimensional subspaces and 2-dimensional subspaces. □

5. Cross-ratio

The material on cross-ratios is in Adler [1].

Definition 5.1. Cross-ratio (yachas hakaful).

Definition 5.2. Perspectivity from a point.

Prove invariance under perspectivity, using areas.

Definition 5.3. Cross ratio of a pencil of lines.

The 6 cross-ratios: $\lambda, 1-\lambda, \frac{1}{\lambda}$, etc.
Role of the symmetric group on 4 letters and of the Klein 4-group.
Exceptional case: the 3 cross-ratios.

Remark 5.4. Over the complex numbers: an additional exceptional case of only 2 distinct values, when

$$\lambda = e^{\pm i\pi/3}.$$

The cross-ratio of 4 points on a circle.
Relation to polarity (which has not been treated formally yet): work with tangent lines to a circle instead of points on a circle.
Then a variable tangent line t meets a 4-tuple of fixed tangents in a 4-tuple of points whose cross-ratio is independent of t.

6. Geometric constr, projective transf, transitivity on triples

Exceptional values $0, 1, \infty$ of the cross-ratio when some of the points collide.

Theorem 6.1. $R(\infty, 0, 1, \lambda) = \lambda$.

Constructions in projective geometry.
An explicit geometric construction of the 4th harmonic point, using Ceva and Menelaus.
Recall the notion of a perspectivity.

Definition 6.2. A projectivity is a transformation preserving the cross-ratio.

The notation: a wedge under the equality sign.

Theorem 6.3. *Thansitivity of projective transformations on triples of collinear points.*

Proof by composition of suitable perspectivities.

Corollary 6.4. *On every line in projective plane, given a triple of points, a fourth point is uniquely determined by the cross-ratio.*

More axiomatics: prove from the 4 axioms the following:

Theorem 6.5. *There is a 1-1 correspondence between points on a line ℓ and lines through a point A not on ℓ.*

7. Projective plane over an arbitrary field

Construction of the projective plane over an arbitrary field.
Formulas for numbers of points in finite planes.

8. Duality, self-dual axiom systems

Duality.

The four axioms of projective geometry give rise to a self-dual system, i.e. the dual of each axiom can be proved from the original list of four.

Discussion the construction in homogeneous coordinates over any field, using a generalisation of the vector product.

Counting points in a projective plane, discuss in a bit more detail the notion of an affine neighborhood (to break the idea that the affine plane is "special").

Detailed discussion of the case over the field F_2, writing out the homogeneous coordinates of all the points, and explicit equations of some of the lines.

9. Cross-ratio in homogeneous coordinates

The definition of cross-ratio in homogeneous coordinates follows the book by Kaplansky [8].

Here if $A, B, C, D \in \mathbb{RP}^1$ we view A, B, C, D as 1-dimensional subspaces in $\mathbb{R}^2$. We choose representative nonzero vectors $\alpha \in A$, $\beta \in B$, $\gamma \in C$, and $\delta \in D$. We show that the vectors can be picked in such a way as to satisfy the relations

$$\gamma = \alpha + \beta$$

and

$$\delta = k\alpha + \beta, \tag{9.1}$$

where $k \in \mathbb{R}$ is suitably chosen. Then the coefficent k in (9.1) is the cross-ratio of A, B, C, D:

Theorem 9.1. *The coefficient k is independent of choices made and satisfies $R(A, B, C, D) = k$.*

10. Conic sections

Conic sections: intersection of cone in $\mathbb{R}^3$ and plane.

Ellipse, parabola, hyperbola and number of points at infinity: 0, 1, 2.

Theorem 10.1. *Every nondegenerate nonempty real conic section is projectively equivalent to the circle.*

Example 10.2. To transform a circle into a parabola by a projective transformation, consider the equation of the circle

$$+x_1^2 + x_2^2 - x_3^2 = 0. \tag{10.1}$$

Here in the affine neighborhood $x_3 \neq 0$ we obtain the usual circle equation

$$x^2 + y^2 = 1 \tag{10.2}$$

where $x = \frac{x_1}{x_3}$ and $y = \frac{x_2}{x_3}$. We would like to tranform this into a parabola

$$X_2 X_3 = X_1^2. \tag{10.3}$$

Here in the affine neighborhood $X_3 \neq 0$ this becomes the usual equation of a parabola $Y = X^2$, where $X = \frac{X_1}{X_3}$ and $Y = \frac{X_2}{X_3}$. We rewrite (10.3) as

$$(X_2 + X_3)^2 - (X_2 - X_3)^2 = (2X_1)^2,$$

or

$$+(2X_1)^2 + (X_2 - X_3)^2 - (X_2 + X_3)^2 = 0. \tag{10.4}$$

Note that the signs $+, +, -$ in equations (10.1) and (10.4) are compatible. Therefore we exploit the transformation

$$x_1 = 2X_1,\ x_2 = X_2 - X_3,\ x_3 = X_2 + X_3.$$

This is a linear transformation in homogeneous coordinates and therefore defines a projective transformation on the projective planes.

Next, this can be expressed in an affine neighborhood by noting that

$$\frac{x_1}{x_3} = \frac{2X_1}{X_2 + X_3} = \frac{2\frac{X_1}{X_3}}{\frac{X_2}{X_3} + 1}$$

and

$$\frac{x_2}{x_3} = \frac{X_2 - X_3}{X_2 + X_3} = \frac{\frac{X_2}{X_3} - 1}{\frac{X_2}{X_3} + 1}.$$

In affine coordinates, we obtain

$$x = \frac{X}{Y+1}, \quad y = \frac{Y-1}{Y+1}. \tag{10.5}$$

Substituting (10.5) into the circle equation (10.2) we obtain the equation of parabola $Y = X^2$.

Example 10.3. Transform parabola into hyperbola.

Example 10.4. Transform ellipse $x^2 + xy + y^2 = 1$ into parabola $Y = X^2$.

11. Polarity, reciprocity

Definition of polar line.
Metric characterisation of polar lines.
Axioms of Fano, Desargues, and Pappus.
Discussion of relation between algebraic properties and geometric axioms:

Theorem 11.1. *Suppose a projective plane π satisfies the axioms P1, ... P4 as well as Desargues' axiom. Then there exists a division ring D such that $\pi = DP^2$.*

Theorem 11.2. *Suppose in addition to the hypotheses above, π satisfies Fano's axiom (the diagonal points of a complete quadrilateral are not collinear). Then char $D \neq 2$.*

Theorem 11.3. *Suppose in addition to the hypotheses above, π satisfies Pappus' axiom. Then $\pi = DP^2$ where D is a field.*

This point of view may be found in the book by Kadison and Kromann [7, chapter 8]. It originates with Hilbert's book [6], see chapter 5 there, particularly paragraph 24: "Introduction of an algebra of segments based upon Desargues's theorem and independence of the axioms of congruence", starting on page 79. Hilbert mentions that this was also discussed by Moore.

1. proof of the reciprocity theorem: if Q is on p, then P is on q.
2. proof of the fact that polarity is a projective transformation, in two stages. First one proves it for 4 points lying on a tangent to the conic. Then one proves it for an arbitrary collinear 4-tuple.
3. A nice application is the theorem that every conic defines a projective transformation from points on a tangent, to points on another tangent. Namely, a point B on a tangent t is sent to a point B' on tangent t' if and only if the line BB' is tangent to the conic.
4. Present another example of a construction in projective geometry. So far the only construction we had is the construction of the fourth harmonic point, using Menelaus theorem.
5. Using the result that polarity is a projective transformation, construct a conic from 5 pieces of data. The 5 pieces are points L and L', the corresponding tangent lines l and l' through them, and an additional tangent line a". One constructs the map as in item 3 above, as the composition of two perspectivities.

Geometric constructions using projective theorems is an important topic in projective geometry that we have barely touched upon.

12. Constructing generic point on conic through 5 points

Construction of a generic point on a conic passing through 5 given points, using Pascal's theorem.

Translating it to a polar statement, so as to construct the polar pencil of parallel lines to the conic.

Finding a projective map between a pair of pencils of lines through a pair of points on a conic.

13. Mobius transformations

Every projective map from P^1 to itself is of the form

$$x \to \frac{ax+b}{cx+d}.$$

I already mentioned the fact that projective transformations correspond to linear maps when you write them in homogeneous coordinates. The fractional-linear presentation is a consequence of this.

More material on axioms of Fano, Desargues, Pappus, related material on the polar line, perhaps a proof of Desargues assuming existence of imbedding in projective 3-space.

14. Hyperbolic geometry

We introduce the Poincaré disk model following Greenberg [3].

Here a point is represented by the interior of a Euclidean circle γ.

A line is represented by one of the following:

(1) an open diameter of the disk bounded by γ, or
(2) an arc of circle δ contained in the disk and perpendicular to γ.

The notion of hyperbolic distance is defined via the cross-ratio.

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