

15 July 2007. Projective Geometry, 88-524-01. moed Alef. Final Exam

ALL ANSWERS MUST BE JUSTIFIED

1. Let triangle ABC be inscribed in a circle M . Let $A'B'C'$ be a triangle such that the side $A'B'$ is tangent to M at the point C , while the side $B'C'$ is tangent to M at the point A , and the side $C'A'$ is tangent to M at the point B .
 - (a) make an appropriate drawing in the case when triangle ABC is an acute-angle triangle;
 - (b) make an appropriate drawing in the case when triangle ABC is an obtuse-angle triangle;
 - (c) Prove that the lines AA' , BB' , CC' are concurrent in case (a);
 - (d) Prove that the lines AA' , BB' , CC' are concurrent in case (b).

2. Consider the field $F = F_5$ with 5 elements. Let A be the affine plane over F , let FP^1 be the projective line over F , and let FP^2 be the projective plane over F .
 - (a) Find the number of points and the number of lines in FP^1 ;
 - (b) Find the number of points and the number of lines in A ;
 - (c) Find the number of points and the number of lines in FP^2 ;
 - (d) Calculate the number of points in the intersection between the pair of projective lines in FP^2 defined by the equations $2x + y + 3z = 0$ and $3x + 4y + 2z = 0$ in homogeneous coordinates;
 - (e) Calculate the number of points in the intersection between the pair of projective lines in FP^2 defined by the equations $x - y + 3z = 0$ and $2x + y - z = 0$ in homogeneous coordinates.

3. Let A, B, C be points on a line ℓ , and P point not on ℓ .
 - (a) Give a precise definition of a harmonic 4-tuple.
 - (b) Describe a geometric construction of a point D such that the 4-tuple A, B, C, D is harmonic.
 - (c) Draw a sequence of at least three careful and precise drawings illustrating each step of the construction.
 - (d) Describe the construction dual to the one in (a), starting with a triple of lines a, b, c concurrent in point L , and line p not through L .

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4. Let $R(A, B, C, D)$ be the cross-ratio (yachas hakaful) of points on the real line, when $A = \infty$, $B = 0$, and $C = 1$. Let $D = \frac{3k-8}{2}$, where $k = 1, 2, 3, 4, 5$.

- (a) What are the possible values of the cross ratio when $k = 1$?
- (a) Let $f(k)$ be the total number of distinct values of the cross-ratio of all the permutations of the 4-tuple (A, B, C, D_k) . Calculate $f(k)$ as an explicit function of the index $k = 1, \dots, 5$.

5. This problem concerns polarity.

- (a) Present a precise statement of Pascal's theorem on a conic \mathcal{C} .
- (b) Formulate the theorem polar to Pascal's theorem. Here polarity is with respect to the conic \mathcal{C} .
- (c) Draw a careful picture illustrating the theorem dual to Pascal's.
- (d) Formulate and prove the reciprocity theorem (mishpat hahada-diut).

GOOD LUCK!