On some sharp weighted norm inequalities

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Abstract

Given a weight \( \omega \), we consider the space \( ML_p^p \) which coincides with \( L_p^p \) when \( \omega \in A_p \). Sharp weighted norm inequalities on \( ML_p^p \) for the Calderón–Zygmund and Littlewood–Paley operators are obtained in terms of the \( A_p \) characteristic of \( \omega \) for any \( 1 < p < \infty \).

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1. Introduction

The boundedness of many important operators in Harmonic Analysis on \( L_p^p \) for \( \omega \in A_p \), \( 1 < p < \infty \), has been known for a long time (see, for example, [21]). However, for a given operator \( T \), it can be a very difficult problem to find sharp bounds for the operator norms \( \| T \|_{L_p^p} \) in terms of the \( A_p \) characteristic of \( \omega \), \( \| \omega \|_{A_p} \). Here, \( L_p^p \) denotes, as usual, the space of all measurable functions \( f \) on \( \mathbb{R}^n \) with norm

\[
\| f \|_{L_p^p} \equiv \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p},
\]

where a weight \( \omega \) is supposed to be a non-negative locally integrable function. Given a Banach space \( X \) and a bounded operator \( T \) on \( X \), \( \| T \|_X \) is the standard operator

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norm of $T$ defined by $\sup_{\|f\|_X \leq 1} \|Tf\|_X$. A weight $\omega$ satisfies the $A_p$, $1 < p < \infty$, Muckenhoupt condition [20] if

$$\|\omega\|_{A_p} \equiv \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ with sides parallel to the axes.

In [3], Buckley proved that for the Hardy–Littlewood maximal operator,

$$\|\mathcal{M}\|_{L^p_{\omega}} \leq c \|\omega\|_{A_p}^{1/(p-1)} \quad (1 < p < \infty), \quad (1.1)$$

and this result is sharp in the sense that $\|\omega\|_{A_p}^{1/(p-1)}$ cannot be replaced by $\varphi(\|\omega\|_{A_p})$ for any positive non-decreasing function $\varphi$ growing slower than $t^{1/(p-1)}$. In the same paper, he showed that the $L^p_{\omega}$ operator norms of the convolution Calderón–Zygmund singular integral operators are at most a multiple of $\|\omega\|_{A_p}^{p/(p-1)}$, and the best power of $\|\omega\|_{A_p}$ must lie in the interval $[\max\{1, 1/(p - 1)\}, p/(p - 1)]$. For the Hilbert transform, in the case $p = 2$, Petermichl and Pott [24] improved the power of $\|\omega\|_{A_2}$ from 2 to $\frac{3}{2}$, which in turn was improved by Petermichl [23] to the best possible linear dependence on $\|\omega\|_{A_p}$ for any $p \geq 2$. We mention also a series of results due to Dragičević, Hukovic, Petermichl, Treil, Volberg, and Wittwer, where the sharp linear dependence on $\|\omega\|_{A_2}$ for the $L^2_{\omega}$ operator norms of the dyadic and continuous square functions [12,31], the martingale transform [30], and the Beurling transform [9,25] was obtained. In [26], the linear bound for the Hilbert transform on the disk in terms of the “Poisson-$A_2$” characteristic of $\omega$ (where mean values are replaced by Poisson averages) was proved.

All results establishing the linear dependence on $\|\omega\|_{A_2}$ for the above operators have been obtained by means of the Bellman function technique. This powerful method has certain limitations. For example, it is still an open question whether the above-mentioned result on the Hilbert transform [23] can be extended to other singular integrals. This is unknown even for the first-order Riesz transforms.

In a recent paper by Dragičević et al. [8], sharp $L^p_{\omega}$ estimates in the Rubio de Francia extrapolation theorem in terms of $\|\omega\|_{A_p}$ have been established. In particular, the main result of [8] shows that if a sublinear operator $T$ is bounded on $L^2_{\omega}$ with the linear bound for $\|T\|_{L^2_{\omega}}$ in terms of $\|\omega\|_{A_2}$, then $T$ is bounded on $L^p_{\omega}$, $1 < p < \infty$, and $\|T\|_{L^p_{\omega}}$ is at most a multiple of $\|\omega\|_{A_p}^{2p}$, where $\alpha_p = \max\{1, 1/(p - 1)\}$. Thus, by the above mentioned results, this yields that the $L^p_{\omega}$ operator norms of the square functions, of the Hilbert, Beurling, and martingale transforms are at most a multiple of $\|\omega\|_{A_p}^{2p}$. Moreover, it is shown in [8] that for the three latter transforms the exponent $\alpha_p$ is sharp for all $1 < p < \infty$, while for the dyadic square function it is sharp for $1 < p \leq 2$.

In this paper, we obtain sharp weighted norm inequalities for the Calderón–Zygmund and Littlewood–Paley operators (their precise definitions are given in Section 2) on the
space $ML^p_0$ which consists of all locally integrable functions on $\mathbb{R}^n$ such that

$$\|f\|_{ML^p_0} = \left( \int_{\mathbb{R}^n} (Mf(x))^p \omega(x) \, dx \right)^{1/p} < \infty \quad (1 < p < \infty),$$

where $M$ is the Hardy–Littlewood maximal operator. Clearly, by Muckenhoupt’s theorem [20], $ML^p_0 = L^p_0$ if and only if $\omega \in A_p$.

Our main results are the following.

**Theorem 1.1.** For any Calderón–Zygmund operator $T$,

$$\|T\|_{ML^p_0} \leq c\|\omega\|_{A_p}^p \quad (1 < p < \infty),$$

where $\alpha_p = \max\{1, 1/(p - 1)\}$, and a constant $c$ depends only on $p$ and on the underlying dimension $n$. The exponent $\alpha_p$ is sharp for all $1 < p < \infty$.

**Theorem 1.2.** Let $T$ be either the area integral $S(f)$ or the Littlewood–Paley function $g^\mu_p(f), \mu > 3$. Then

$$\|T\|_{ML^p_0} \leq c\|\omega\|_{A_p}^{\beta_p} \quad (1 < p < \infty),$$

where $\beta_p = \max\{1/2, 1/(p - 1)\}$, and a constant $c$ depends only on $p$ and $n$. The exponent $\beta_p$ is sharp for all $1 < p < \infty$.

Although, we do not know whether one can deduce from these theorems the same bounds for the $L^p_0$ operator norms, they are of interest in their own right. Theorem 1.1 strengthens a natural conjecture (which is contained implicitly in [8]) that the $L^p_0$ operator norms of any Calderón–Zygmund operator must be bounded by a multiple of $\|\omega\|_{A_p}^{\alpha_p}$. Observe that the same estimate as in Theorem 1.1 can be proved for many different operators (see Remark 5.1 in Section 5). Theorem 1.2 combined with Buckley’s result (1.1) implies easily that for the area integral $S$,

$$\|S\|_{L^p_0} \leq c\|\omega\|_{A_p}^{\beta_p + 1/(p - 1)} \quad (1 < p < \infty),$$

(1.2)

which gives a better bound than the linear one when $p > 3$. However, we conjecture that the best power of $\|\omega\|_{A_p}$ in (1.2) must be equal to $\beta_p$.

As we mentioned above, previously known sharp $L^p_0$ estimates were based on the Bellman function technique and on the extrapolation. Our approach to Theorems 1.1 and 1.2 is different. It mainly depends on the so-called local sharp maximal function, particularly on recent author’s paper [19], where various weighted inequalities for such a function were obtained. In Section 3, we establish the sharp dependence.
on \( \|\omega\|_{A_p} \) for some of results from \[19\]. The second ingredient of the proofs is pointwise inequalities for the Calderón–Zygmund and Littlewood–Paley operators by means of the sharp functions. These inequalities are given in Section 4. Observe that although a lot of pointwise estimates in terms of the sharp functions are well known, we have never seen in the literature the one for the Littlewood–Paley function \( g^\ast_{\mu}(f) \), \( \mu > 3 \), given in Proposition 4.2. The proofs of Theorems 1.1 and 1.2 are contained in Section 5.

### 2. Some basic definitions

#### 2.1. Weights

We recall first that the Hardy–Littlewood maximal operator is defined by

\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.
\]

A weight \( \omega \) satisfies the \( A_1 \) condition if there exists \( c > 0 \) such that

\[
M\omega(x) \leq c\omega(x) \text{ a.e.} \tag{2.1}
\]

The smallest possible \( c \) in (2.1) is denoted by \( \|\omega\|_{A_1} \).

Set \( A_\infty = \bigcup_{p \geq 1} A_p \). Observe that there are many equivalent characterizations of \( A_\infty \) (see [6]).

#### 2.2. Calderón–Zygmund operators

Let \( K(x,y) \) be a locally integrable function defined off the diagonal \( x = y \) in \( \mathbb{R}^n \times \mathbb{R}^n \), which satisfies the size estimate

\[
|K(x,y)| \leq \frac{c}{|x-y|^n} \tag{2.2}
\]

and, for some \( \varepsilon > 0 \), the regularity condition

\[
|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq c \frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}}, \tag{2.3}
\]

whenever \( 2|x-z| < |x-y| \).

A linear operator \( T : C_0^\infty(\mathbb{R}^n) \to L^1_{\text{loc}}(\mathbb{R}^n) \) is a Calderón–Zygmund operator if it extends to a bounded operator on \( L^2(\mathbb{R}^n) \), and there is a kernel \( K \) satisfying (2.2)
and (2.3) such that
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy
\]
for any \( f \in C_0^\infty(\mathbb{R}^n) \) and \( x \notin \text{supp}(f) \).

2.3. Littlewood–Paley operators

Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) with \( \int \varphi = 0 \). Write \( \varphi_t(y) = t^{-n} \varphi(y/t) \). The area integral \( S(f) \) and the Littlewood–Paley function \( g^{\ast}_\mu(f) \) are defined by
\[
S(f)(x) = \left( \int_0^\infty \int_{\{y:|y-x|<t\}} |f * \varphi_t(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}
\]
and
\[
g^{\ast}_\mu(f)(x) = \left( \int \int_{\mathbb{R}^n_{+1}} |f * \varphi_t(y)|^2 \left( \frac{t}{t + |x-y|} \right)^{\mu n} \frac{dy \, dt}{t^{n+1}} \right)^{1/2} \quad (\mu > 1),
\]
where \( \mathbb{R}^n_{+1} = \mathbb{R}^n \times \mathbb{R}_+ \).

3. Estimates for local sharp maximal functions

Given a weight \( \omega \), set \( \omega(E) = \int_E \omega(x) \, dx \). The non-increasing rearrangement of a measurable function \( f \) with respect to a weight \( \omega \) is defined by (cf. [5, p. 32])
\[
f^{\ast}_\omega(t) = \sup_{\omega(E)=t} \inf_{x \in E} |f(x)| \quad (0 < t < \omega(\mathbb{R}^n)).
\]
If \( \omega \equiv 1 \), we use the notation \( f^{\ast}(t) \).

Given a measurable function \( f \), the local sharp maximal function \( M_{\hat{\lambda}}^{\#} f \) is defined by
\[
M_{\hat{\lambda}}^{\#} f(x) = \sup_{Q \ni x} \inf_c \left( (f - c) \chi_Q \right)^{\ast}(\hat{\lambda} \, |Q|) \quad (0 < \hat{\lambda} < 1).
\]
This function was introduced by Strömberg [27] motivated by an alternate characterization of the space \( BMO \) given by John [14]. Its different aspects were studied by Jawerth and Torchinsky [13], and by the author [16–19]. We mention here that \( M_{\hat{\lambda}}^{\#} f \) is essentially smaller than the usual Fefferman–Stein sharp function \( f^{\#} \) defined by
\[
f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.
\]
Roughly speaking, \( f^# \) is the Hardy–Littlewood maximal operator of \( M^# f \) (see [13,17]):

\[
c_1 MM^# f(x) \leq f^#(x) \leq c_2 MM^# f(x).
\]

In [13,16], the following version of the Fefferman–Stein inequality was established:

\[
\|f\|_{L^p} \leq c \|M^# f\|_{L^p},
\]

where \( 0 < p < \infty \), \( \omega \) is any weight satisfying the \( A_\infty \) condition, \( f \) is any measurable function with \( f^*(+\infty) = 0 \), and the constants \( c, \lambda \) depend on \( \omega \). Note that (3.1) can be used to deduce the usual Fefferman–Stein inequality (see [11,16]):

\[
\|Mf\|_{L^p} \leq c \|f^#\|_{L^p}.
\]

In recent papers [18,19], several weighted variants of (3.1) and (3.2) have been obtained for arbitrary (i.e., not necessarily \( A_\infty \)) weights. In particular, in [19] a self-improving principle was found which says that both inequalities like (3.1) and (3.2) can be improved by replacing \( f \) by \( Mf \) on the left-hand side of (3.1) without the changing its right-hand side. Here, we combine some ideas from [19] with the above-mentioned Buckley result to deduce the following sharp version of (3.1) and (3.2) for \( A_p \) weights in terms of \( \|\omega\|_{A_p} \).

**Theorem 3.1.** For any \( \omega \in A_p \) and for any locally integrable function \( f \) with \( f^*(+\infty) = 0 \) we have

\[
\|Mf\|_{L^p} \leq c \|\omega\|_{A_p}^{\gamma_{p,q}} \cdot \|M^#(|f|^q)\|_{L^p}^{1/q} (1 < p < \infty, 1 \leq q < \infty),
\]

where \( \gamma_{p,q} = \max\{1/q, 1/(p - 1)\} \), \( c \) depends only on \( p, q \) and on the underlying dimension \( n \), and \( \lambda_n \) depends only on \( n \).

To prove this theorem, we will need several definitions and auxiliary results.

Following Wilson [28], given any weight \( \omega \), define the maximal function \( P_\lambda \omega \) measuring a local un-\( A_\infty \) behavior of \( \omega \). For \( 0 < \lambda < 1 \) and any cube \( Q \) with \( \omega(Q) > 0 \), let \( E_\lambda \subset Q \) be any subset of minimal Lebesgue measure such that \( \omega(E_\lambda) = \lambda \omega(Q) \). Set

\[
P_\lambda \omega(x) = \sup_{Q \ni x} \log(1 + |Q|/|E_\lambda|),
\]

where the supremum is taken over all cubes \( Q \) with \( \omega(Q) > 0 \) containing \( x \). Note that \( P_\lambda \omega \in L^\infty \) if and only if \( \omega \in A_\infty \).
Given a measurable function $f$, define the local maximal function $m_\lambda f$ by

$$m_\lambda f(x) = \sup_{Q \ni x} (f^*_{Q})^*(\lambda |Q|) \quad (0 < \lambda < 1).$$

The following propositions are proved in [19] (see Propositions 2.1 and 4.2, and Theorem 5.4 there).

**Proposition 3.2.** Let $\omega$ be any weight such that $\omega(\mathbb{R}^n) = +\infty$. Then $f^*_{\omega}(+\infty) = 0$ iff the distribution function $\mu_{f,\omega}(z) = \omega(x : |f(x)| > z)$ is finite for any $z > 0$.

**Proposition 3.3.** For any locally integrable $f$ and for all $x \in \mathbb{R}^n$,

$$Mf(x) \leq 3 f^#(x) + m_{1/2} f(x), \quad (3.4)$$

$$f^#(x) \leq 8 M_{\lambda_n}^# f(x) \quad (3.5)$$

and

$$M_\lambda^# (m_{1/2} f)(x) \leq 4 M_{\lambda/2,\omega_n}^# f(x). \quad (3.6)$$

**Proposition 3.4.** For any weight $\omega$ such that $\omega(\mathbb{R}^n) = +\infty$ and for any measurable function $f$ with $f^*_{\omega}(+\infty) = 0$,

$$\|f\|_{L_{\omega^r}} \leq c_{r,n} \| (M_{\lambda_n}^# f)(P_{\omega_n} f)\|_{L_{\omega^r}} \quad (0 < r < \infty),$$

where $\lambda_n$ and $\lambda_n'$ depend only on $n$.

We have the following estimate for Wilson’s maximal function $P_\lambda \omega$.

**Lemma 3.5.** For any $1 \leq p < \infty$ and for any weight $\omega$,

$$\|P_\lambda \omega\|_{\infty} \leq c \|\omega\|_{A_p},$$

where $c$ depends only on $p$, $\lambda$ and $n$.

**Proof.** It is shown in [19] that

$$P_\lambda \omega(x) \leq c_{\lambda,n} \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q M_Q \omega(y) \, dy, \quad (3.7)$$
where $M_Q \omega$ is the Hardy–Littlewood maximal function relative to $Q$:

$$M_Q \omega(x) = \sup_{Q' \ni x, Q' \subset Q} \frac{1}{|Q'|} \int_{Q'} \omega (x \in Q).$$

In the case $p = 1$, the lemma follows immediately from (3.7) and from the definition of $A_1$ weights. Assume that $p > 1$. Using the fact that $\|\omega^{-1/(p-1)}\|_{A_p'} = \|\omega\|_{A_p}^{p' - 1}$, where $p' = p/(p - 1)$, the Hölder inequality, and Buckley estimate (1.1), we get

$$\int_Q M_Q \omega = \int_Q (M_Q \omega)\omega^{-1/p} \omega^{1/p} \leq \left( \int_Q (M_Q \omega)^{p'} \omega^{-1/(p-1)} \right)^{1/p'} \left( \int_Q \omega \right)^{1/p} \leq c \|\omega^{-1/(p-1)}\|_{A_p'}^{1/(p-1)} \omega(Q) \leq c \|\omega\|_{A_p} \omega(Q),$$

which along with (3.7) proves the lemma. □

**Proof of Theorem 3.1.** Without loss of generality, one can assume that $f \geq 0$. Using (3.4), (3.5), Minkowski’s inequality and (1.1), we obtain

$$\|Mf\|_{L^p} \leq c \|\omega\|_{A_p}^{1/(p-1)} \|M^\# f\|_{L^p} + \|m_{1/2} f\|_{L^p} \quad (1 < p < \infty). \quad (3.8)$$

Next, we observe that $\{x : m_{1/2} f(x) > \lambda\} \subset \{x : M_{\lambda} f(x) \geq \lambda\}$, and since $\omega \in A_p$, Muckenhoupt’s theorem [20] yields

$$\omega\{x : m_{1/2} f(x) > \lambda\} \leq c_{\omega, p, \lambda} \omega\{x : f(x) > \lambda\}. \quad (3.9)$$

Therefore, by Proposition 3.2, $f_{\omega}^{\#}(+\infty) = 0$ implies $(m_{1/2} f_{\omega})^{\#}(+\infty) = 0$. Thus, combining (3.6), Proposition 3.4, and Lemma 3.5, we get

$$\|m_{1/2} f\|_{L^r} \leq c \|\omega\|_{A_p} \|M^\#_{L^r} f\|_{L^r} \quad (1 \leq p < \infty, r > 0).$$

From this and from the fact that $m_{\lambda} (f^q) = (m_{\lambda} f)^q$, $q > 0$,

$$\|m_{1/2} f\|_{L^p} = \|m_{1/2} (f^q)^{1/q}\|_{L^{p/q}} \leq c \|\omega\|_{A_p} \|M^\#_{L^r} (f^q)\|_{L^{p/q}}^{1/q}.$$ 

Similarly, by a simple estimate $M^\#_{\lambda} (f^\delta) \leq M^\# (f^\delta), 0 < \delta \leq 1$,

$$\|M^\#_{\lambda} f\|_{L^p} \leq \|M^\# (f^q)\|_{L^{p/q}}^{1/q} \quad (1 \leq q < \infty).$$

Combining two latter estimates with (3.8) completes the proof. □
4. Pointwise estimates for Calderón–Zygmund and Littlewood–Paley operators

Given $0 < \delta < 1$, consider the maximal function $f^\#_\delta$ defined by

$$f^\#_\delta(x) = \sup_{Q \ni x} \inf_c \left( \frac{1}{|Q|} \int_Q |f(y) - c|^{\delta} \, dy \right)^{1/\delta}.$$ 

By Chebyshev’s inequality, it is easy to see that

$$M^\#_\lambda f(x) \leq (1/\lambda)^{1/\delta} f^\#_\delta(x). \quad (4.1)$$

In [2, Theorem 2.1], Alvarez and Pérez proved that for any Calderón–Zygmund operator $T$ and for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$(Tf)^\#_\delta(x) \leq c_{\delta,n}Mf(x) \quad (0 < \delta < 1).$$

From this and from (4.1) we have the following.

**Proposition 4.1.** For any Calderón–Zygmund operator $T$ and for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$M^\#_\lambda (Tf)(x) \leq c_{\lambda,n}Mf(x).$$

We note that for specific classes of Calderón–Zygmund operators this proposition is contained in [13,16].

For the Littlewood–Paley function $g^*_\mu(f)$, it was proved by Cruz-Uribe and Pérez [7] that for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$(g^*_\mu(f))^\#_\delta(x) \leq c_{\delta,n}Mf(x) \quad (0 < \delta < 1, \mu > 2).$$

Therefore, a full analogue of Proposition 4.1 holds for $g^*_\mu(f)$. However, we will show that a more precise result holds, although our proof works in the case $\mu > 3$. Note also that our approach is different from the one of [7].

**Proposition 4.2.** Let $\mu > 3$. Then for any $f \in C_0^\infty$ and for all $x \in \mathbb{R}^n$,

$$M^\#_\lambda (g^*_\mu(f)^2)(x) \leq cMf(x)^2, \quad (4.2)$$

where $c$ depends on $\lambda, \mu$ and $n$. 
Proof. Given a cube $Q$, let $T(Q) = \{(y, t) : y \in Q, 0 < t < \ell_Q\}$, where $\ell_Q$ denotes the side length of $Q$. Next, without loss of generality, we can assume that $\text{supp} \varphi \subset \{x : |x| \leq 1\}$. Then for $(y, t) \in T(Q)$,

$$f * \varphi_t(y) = (f \chi_{3Q}) * \varphi_t(y).$$ \hfill (4.3)

Now, fix a cube $Q$ containing $x$. For any $z \in Q$ we decompose $g_\mu^*(f)^2$ into the sum of

$$I_1(z) = \int \int_{T(2Q)} |f * \varphi_t(y)|^2 \left( \frac{t}{t + |z - y|} \right)^{\mu n} dy dt \frac{t^{n+1}}{t^{n+1}}$$

and

$$I_2(z) = \int \int_{\mathbb{R}^{n+1} \setminus T(2Q)} |f * \varphi_t(y)|^2 \left( \frac{t}{t + |z - y|} \right)^{\mu n} dy dt \frac{t^{n+1}}{t^{n+1}}.$$

Since $g_\mu^*(f)$ is of weak type $(1, 1)$ for $\mu > 3$ (a direct proof of this fact is contained in [29, p. 689]), using (4.3), we get

$$(I_1^1)^*(\lambda|Q|) \leq \left( g_\mu^*(f \chi_{6Q}) \right)^* (\lambda|Q|)^2 \leq \left( \frac{c}{\lambda|Q|} \int_{6Q} |f| \right)^2 \leq cMf(x)^2. \hfill (4.4)$$

Further, for any $z_0 \in Q$ and $(y, t) \notin T(2Q)$, by the Mean Value Theorem,

$$(t + |z - y|)^{-\mu n} - (t + |z_0 - y|)^{-\mu n} \leq c\ell_Q(t + |z - y|)^{-\mu n - 1}.$$

From this and from (4.3), using again that $\mu > 3$, we get

$$|I_2(z) - I_2(z_0)|$$

$$\leq c\ell_Q \int \int_{\mathbb{R}^{n+1} \setminus T(2Q)} t^{\mu n} |f * \varphi_t(y)|^2 \left( \frac{1}{t + |z - y|} \right)^{\mu n + 1} dy dt \frac{t^{n+1}}{t^{n+1}}$$

$$\leq c \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{1}{2^k \ell_Q} \right)^{\mu n} \int \int_{T(2k+1Q) \setminus T(2kQ)} t^{\mu n} |f * \varphi_t(y)|^2 dy dt \frac{t^{n+1}}{t^{n+1}}$$

$$\leq c \sum_{k=1}^{\infty} \frac{2^{k+1} |Q|}{2^k (2^k \ell_Q)^{\mu n}} \left( \int_{2^k \ell_Q}^{2^{k+1} \ell_Q} t^{\mu n - 3n - 1} dt \right) \left( \int_{62^kQ} |f| \right)^2.$$
Combining this estimate with (4.4) yields

\[
\inf_c \left( (g^*_\mu(f)^2 - c)z_Q \right)^\ast (\lambda|Q|) \leq \left( (I_1 + I_2 - I_2(z_0))z_Q \right)^\ast (\lambda|Q|) \\
\leq (I_1)^\ast (\lambda|Q|) + cMf(x)^2 \leq cMf(x)^2,
\]

which proves the desired result. \(\square\)

5. Proof of main results

First, we show how to deduce the estimates contained in the statements of Theorems 1.1 and 1.2, and then we shall discuss the sharpness of exponents \(x_p\) and \(\beta_p\).

Observe that the \(L^p_{\omega}\) boundedness of the Calderón–Zygmund and Littlewood–Paley operators when \(\omega \in A_p\) is well known (see, e.g., [15,22]). Therefore, assuming that \(\|f\|_{ML^p_{\omega}}\) is finite, we clearly obtain that \((Tf)^\ast_\omega(+\infty) = 0\), where \(T\) is any one of the operators appearing in Theorems 1.1 and 1.2.

Suppose now that \(T\) is a Calderón–Zygmund operator. Letting \(Tf\) instead of \(f\) in (3.3) with \(q = 1\) and applying Proposition 4.1, we immediately obtain

\[
\|Tf\|_{ML^p_{\omega}} \leq c_{p,n} \|\omega\|_{A_p}^{\beta_p} \|f\|_{ML^p_{\omega}} \quad (1 < p < \infty),
\]

which proves the first part of Theorem 1.1.

Similarly, letting \(g^*_\mu(f)\) instead of \(f\) in (3.3) with \(q = 2\) and applying Proposition 4.2, we get

\[
\|g^*_\mu(f)\|_{ML^p_{\omega}} \leq c_{p,n} \|\omega\|_{A_p}^{\beta_p} \|f\|_{ML^p_{\omega}} \quad (1 < p < \infty).
\]

Since \(S(f)(x) \leq c_{\mu,n} g^*_\mu(f)(x)\), we have the same estimate for \(S(f)\), and hence the proof of the first part of Theorem 1.2 is complete.

Let us show now that the exponents \(x_p\) and \(\beta_p\) are best possible. It is worth noting that even weaker variants of Theorems 1.1 and 1.2 are sharp in the cases \(p \geq 2\) and \(p \geq 3\), respectively, namely, in the inequalities

\[
\|Tf\|_{L^p_{\omega}} \leq c_{p,n} \|\omega\|_{A_p} \|f\|_{ML^p_{\omega}} \quad (p \geq 2)
\]

and

\[
\|S(f)\|_{L^p_{\omega}} \leq c \|\omega\|_{A_p}^{1/2} \|f\|_{ML^p_{\omega}} \quad (p \geq 3)
\]
the exponents 1 and $\frac{1}{2}$ are best possible. This follows easily from general observations by an argument of Fefferman and Pipher [10].

Suppose, for example, that for some $p_0 \geq 2$ we have (5.1) but with $\|\omega\|_{A_{p_0}}$ replaced by $\varphi(\|\omega\|_{A_{p_0}})$, where $\varphi$ is a non-decreasing function such that $\varphi(t)/t \to 0$ as $t \to +\infty$. Then we clearly obtain

$$\|Tf\|_{L_{p_0}^p} \leq c \varphi(\|\omega\|_{A_1}) \|f\|_{ML_{p_0}^p}. \quad (5.3)$$

From this, arguing exactly as in [10, pp. 356–357], i.e., using the Rubio de Francia algorithm and the duality, we get

$$\|T\|_{L_p} \leq c_1 \varphi(c_2 p) \quad \text{as } p \to \infty. \quad (5.4)$$

But it is well known that $\|T\|_{L_p} = O(p)$, and this is sharp, in general. Thus, we have obtained a contradiction which shows that the exponent 1 in (5.1) is sharp.

For the sake of completeness, we outline briefly how (5.4) follows from (5.3). Let $p > p_0$, and $\psi \geq 0$, $\|\psi\|_{L_{(p/p_0)'}^1} = 1$. Form the operator

$$R\psi = \psi + \sum_{k=1}^{\infty} \frac{M^k \psi}{(2\|M\|_{L_{(p/p_0)'}})^k}.$$  

Then $\|R\psi\|_{L_{(p/p_0)'}^1} \leq 2$ and $\|R\psi\|_{A_1} \leq 2\|M\|_{L_{(p/p_0)'}^1} = O(p)$ as $p \to \infty$. Therefore, by (5.3) and Hölder’s inequality,

$$\int_{\mathbb{R}^n} |Tf|^{p_0} \psi \leq \int_{\mathbb{R}^n} |Tf|^{p_0} R\psi \leq c \varphi(\|R\psi\|_{A_1})^{p_0} \int_{\mathbb{R}^n} (Mf)^{p_0} R\psi$$  

$$\leq 2c \varphi(c' p) \|Mf\|_{L_{p_0}^p} \leq c'' \varphi(c' p) Mf \|f\|_{L_{p_0}^p}.$$  

Taking the supremum over all $\psi$ with $\|\psi\|_{L_{(p/p_0)'}^1} = 1$ yields (5.4).

Exactly the same observations show that in (5.2), $\|\omega\|_{A_{p_0}^{1/2}}$, $p_0 \geq 3$, cannot be replaced by $\varphi(\|\omega\|_{A_{p_0}})$ with $\varphi(t)/\sqrt{t} \to 0$ as $t \to +\infty$, because it is well known that $\|S\|_{L_p} = O(\sqrt{p})$, and this is sharp, in general (see, e.g., [4]).

It remains to show that the exponents $\alpha_p$ and $\beta_p$ are sharp in the cases $1 < p < 2$ and $1 < p < 3$, respectively. We will use the same examples as in [3,8]. Let $n = 1$ and $T = H$ is the Hilbert transform (analogous examples can be found for $n > 1$ and for any one of the Riesz transforms). Let

$$\omega(x) = |x|^{(p-1)(1-\delta)}, \quad f(x) = x^{\delta-1} I_{(0,1)} \quad (0 < \delta < 1). \quad (5.5)$$
Then \( \|\omega\|_{A_p} \sim \delta^{1-p} \),

\[
Mf(x) \sim \frac{1}{\delta} \left(|x|^{\delta-1} \chi_{\{|x|<1\}} + |x|^{-1} \chi_{\{|x|\geq 1\}}\right)
\]

and

\[
Hf(x) \geq \frac{c}{\delta x} \quad (x > 2).
\]

Next, simple calculations show that

\[
\|Mf\|_{L^p} \leq \frac{c}{\delta^{1+1/p}}
\]

and

\[
\|M(Hf)\|_{L^p} \geq \frac{c}{\delta} \left( \int_1^\infty (\log x/x)^p x^{(p-1)(1-\delta)} \, dx \right)^{1/p}
\]

\[
= c \left( \int_1^\infty (\log x/x)^p \, dx \right)^{1/p} \frac{1}{\delta^{2+1/p}}.
\]

Therefore,

\[
\frac{\|Hf\|_{ML^p}}{\|f\|_{ML^p}} \geq \frac{c}{\delta} \geq c \|\omega\|^{1/(p-1)}_{A_p}.
\]

This shows that the exponent \( \beta_p \) is sharp for \( 1 < p < 2 \).

The same example works for a large class of square functions. Assume, for instance, that \( \varphi(x) \sim \text{const} \) for \( x \in \left[\frac{1}{4}, 4\right] \). Let \( \omega, f \) be as in (5.5). Let \( x > 4 \). For any \( t \in (x/2, x) \) and for all \( y \in (x-t, x+t) \) we have \( f \ast \varphi_t(y) \sim 1/t\delta \). Therefore,

\[
S(f)(x) \geq \left( \int_{x/2}^x \int_{\{|y|:|y-x|<t\}} |f \ast \varphi_t(y)|^2 \frac{dy \, dt}{t^2} \right)^{1/2} \geq c/\delta x,
\]

and, exactly as for the Hilbert transform, we have

\[
\frac{\|S(f)\|_{ML^p}}{\|f\|_{ML^p}} \geq c \|\omega\|^{1/(p-1)}_{A_p},
\]

which proves that the exponent \( \beta_p \) is sharp for \( 1 < p < 3 \).

The proof of Theorems 1.1 and 1.2 is complete.
Remark 5.1. In [1], pointwise estimates by means of \( f_\emptyset^\# \) for a large class of operators have been obtained. As a result, a full analogue of Proposition 4.1 holds for weakly strongly singular integral operators by Fefferman, for pseudo-differential operators in the Hörmander class, for oscillatory integral operators introduced by Phong and Stein. The proof of Theorem 1.1 shows that for all these operators, their \( ML_{p,0}^p \) operator norms are bounded by a multiple of \( \| \omega \|_{A_p}^{2p} \), \( 1 < p < \infty \).

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Appendix A.

In order to make this paper slightly more self-contained, we outline briefly main steps and ideas used in the proofs of Propositions 3.3 and 3.4. For complete proofs of these results we refer to [19].

Proof of Proposition 3.3 (Sketch). To prove the first estimate of Proposition 3.3, it is convenient to use the notion of median values. A median value of \( f \) over \( Q \) is a, possibly non-unique, real number \( m_f(Q) \) such that

\[
\max \left\{ |\{ x \in Q : f(x) > m_f(Q) \}|, |\{ x \in Q : f(x) < m_f(Q) \}| \right\} \leq \frac{|Q|}{2}.
\]

It follows easily that \( |m_f(Q)| \leq (f \chi_Q)^*(|Q|/2) \) and

\[
\int_Q |f(x) - m_f(Q)| \, dx \leq 3 \inf_c \int_Q |f(x) - c| \, dx,
\]

which yields readily (3.4).

The second estimate of Proposition 3.3 is deeper. It was first deduced (with different constants) by Jawerth and Torchinsky [13] from their good-\( \lambda \) inequality related \( f \) and \( M_{\lambda}^f \). Note that this approach was rather complicated. Afterwards, using a simple covering lemma of Calderón–Zygmund type, the author [16] proved the following rearrangement inequality:

\[
(f \chi_Q)^*(t) \leq 2 \left( (M_{\lambda_n}^f) \chi_Q \right)^*(2t) + (f \chi_Q)^*(2t) \quad (0 < t \leq \lambda_n |Q|). \tag{A.1}
\]

Integrating both parts of (A.1) along with some standard transformations implies

\[
\inf_c \int_Q |f(x) - c| \, dx \leq 8 \int_Q M_{\lambda_n}^f(x) \, dx
\]

(a complete proof of this can be found in [18]), which gives (3.5).
The third estimate of Proposition 3.3 is based on a quite standard technique. Namely, using the fact that for two intersecting cubes $Q_1$ and $Q_2$ we have either $Q_1 \subset 3Q_2$ or $Q_2 \subset 3Q_1$, one can show that for all $x, y \in Q$,

$$m_{1/2}f(y) \leq m_{1/2}\left( ((f - mf(3Q)) \chi_{3Q}) (y) + 2M_{1/2, 3^n} f(x) + \inf_Q m_{1/2} f.\right)$$

Next, (3.9) in the unweighted case gives

$$(m_{1/2} f)^*(t) \leq f^*(t/2 \cdot 3^n). \quad (A.2)$$

Combining the last two inequalities yields easily (3.6). □

**Proof of Proposition 3.4 (Sketch).** The proof of Proposition 3.4 is more complicated. We define the weighted centered versions of $M_{\vec{\lambda}} f$ and $m_{\vec{\lambda}} f$ by

$$\tilde{M}_{\vec{\lambda}, \omega} f(x) = \sup_{Q \ni x} \inf_c \left( (f - c) \chi_Q \right)^*_\omega \left( \omega(Q) \right)$$

and

$$\tilde{m}_{\vec{\lambda}, \omega} f(x) = \sup_{Q \ni x} \left( f \chi_Q \right)^*_\omega \left( \omega(Q) \right) \quad (0 < \vec{\lambda} < 1),$$

where the supremum is taken over all cubes centered at $x$. The main tool used in Proposition 3.4 is the following rearrangement inequality which holds for arbitrary weights $\omega$ (see [19, Theorem 1.2]):

$$f^*_\omega(t) \leq 2\left( \tilde{M}_{\vec{\lambda}, \omega} f \right)^* (t/2) + f^*_\omega(2t) \quad \left(0 < \vec{\lambda} \omega(R^n)\right). \quad (A.3)$$

This inequality resembles (A.1). Observe that actually a weighted variant of (A.1) was proved in [16] but only for doubling weights with corresponding constants depending on $\omega$. Then, a refined argument was used in [17] to remove the doubling condition. However, the main estimate from [17] gives only a variant of (A.3) with the usual uncentered local sharp maximal function instead of the centered one. Thus, (A.3) has two features: arbitrary weights and the centered maximal function. The proof of (A.3) combines some ideas from [16,17] along with the Besicovitch covering theorem.

If $\omega(R^n) = +\infty$ and $f^*_\omega(+\infty) = 0$, then the iteration of (A.3) along with the integral Hardy inequality implies

$$\| f \|_{L_r^\omega} \leq c_{r,n} \| \tilde{M}_{\vec{\lambda}, \omega} f \|_{L_r^\omega} \quad (0 < r < \infty). \quad (A.4)$$
Next, the most technical part of the proof is the following pointwise estimate:

$$\tilde{M}_{\lambda,\omega}^# f(x) \leq c_n \tilde{m}_{\lambda,\omega} \left( M_{\lambda_n}^# f P_{\lambda/2} \right)(x). \quad (A.5)$$

Observe that (A.5) along with (A.4) implies almost immediately the desired result. Indeed, exactly as in proving (A.2), one can show (using the weighted weak type \((1,1)\) property of the weighted centered Hardy–Littlewood maximal operator) that

$$\left( \tilde{m}_{\lambda,\omega} f \right)^*_\omega (t) \leq f^*_\omega (\lambda t / c_n).$$

Therefore, the operator \(\tilde{m}_{\lambda,\omega}\) is bounded on \(L^r_{\omega,\omega} \) for any \(r > 0\) with the operator norms not depending on \(\omega\). Combining this fact with (A.5) and (A.4) gives the statement of Proposition 3.4.

The proof of (A.5) is based on several ingredients. The first one [19, Lemma 5.2] says that for any closed set \(F \subset Q\) with \(|F| > 0\) there is a function \(g\) such that \(f = g\) on \(F\) and

$$\|M^#_{1/2;Q} g\|_{\infty} \leq \sup_{x \in F} M^#_{\lambda_n;Q} f(x) \quad (A.6)$$

(one can take \(\lambda_n = 1/100^n\)). Here, \(M^#_{\lambda,\omega} f\) denotes the local sharp maximal function restricted to a cube \(Q\). Note that this result is close in spirit to characterizing the \(E\)-functional for \((L^0, BMO)\) [13, Theorem 3.2]. The second ingredient is the John–Strömberg theorem [14,27] saying that for any cube \(Q \subset Q_0\),

$$\left( \left( f - m^f (Q) \chi_Q \right)^* \right)^*_\omega (t) \leq c_n \|M^#_{1/2;Q_0} f\|_{\infty} \log \frac{2|Q|}{t} \quad (0 < t < |Q|). \quad (A.7)$$

Fix now a cube \(Q\) centered at \(x\). Set

$$F = \left\{ \xi \in Q : M^#_{\lambda_n;Q} f(\xi) \leq \left( M^#_{\lambda_n;Q} f \right)^*_\omega (\lambda_\omega(Q)/4) \right\}$$

and choose a function \(g\) satisfying (A.6) such that \(f = g\) on \(F\). Observe that

$$\omega(\text{supp}(f - g)) \leq \omega(Q \setminus F) \leq \lambda_\omega(Q)/4.$$

Hence,

$$\inf_c \left( \left( f - c \right) \chi_Q \right)^*_\omega (\lambda_\omega(Q)) \leq \left( \left( g - m^g (Q) \chi_Q \right)^* \right)^*_\omega (\lambda_\omega(Q)/2). \quad (A.8)$$

By the definition of the rearrangement, for any subset \(E_\lambda \subset Q\) of minimal Lebesgue measure with \(\omega(E_\lambda) = \lambda_\omega(Q)/2\) the right-hand side of (A.8) is bounded by
\((g - m_g(Q)1_Q)^* (|E|)\). Applying to this term (A.7) and then (A.6), we get
\[
\inf_c \left( \frac{1}{(f - c)1_Q} \right)^* (\omega(Q)) \leq c_n \left( \sup_f M^\#_Q f \right) \log \frac{2|Q|}{|E|} \leq c_n \left( M^\#_Q f \right)^* (\omega(Q)/4) \inf_Q P_{1/2} \omega \\
\leq c_n \left( (M^\#_{1/2} P_{1/2} \omega)1_Q \right)^* (\omega(Q)/4).
\]
This implies (A.5), and therefore the proof is complete. 

References


