A NOTE ON THE MAXIMAL
GUROV–RESHETNYAK CONDITION

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Dedicated to Professor Bogdan Bojarski on the occasion of his 75th birthday.

Abstract. In a recent paper [17] we established an equivalence between the Gurov–Reshet-
nyak and $A_\infty$ conditions for arbitrary absolutely continuous measures. In the present paper we
study a weaker condition called the maximal Gurov–Reshetnyak condition. Although this condition
is not equivalent to $A_\infty$ even for Lebesgue measure, we show that for a large class of measures sat-
sifying Busemann–Feller type condition it will be self-improving as is the usual Gurov–Reshetnyak
condition. This answers a question raised independently by Iwaniec and Kolyada.

1. Introduction

Throughout the paper, $Q_0$ will be a bounded cube from $\mathbb{R}^n$, and $\mu$ will be a
non-negative Borel measure on $Q_0$ absolutely continuous with respect to Lebesgue
measure. For $f \in L_\mu(Q_0)$ and for any subcube $Q \subset Q_0$ set

$$f_{Q,\mu} = \frac{1}{\mu(Q)}\int_Q f(x) \, d\mu, \quad \Omega_\mu(f; Q) = \frac{1}{\mu(Q)}\int_Q |f(x) - f_{Q,\mu}| \, d\mu.$$ 

A function $f \in L_\mu(Q_0)$ is said to belong $BMO(\mu)$ if

$$\sup_{Q \subset Q_0} \Omega_\mu(f; Q) < \infty.$$ 

Also we recall that the classes $A_p(\mu)$ and $RH_r(\mu), 1 < p, r < \infty,$ consist of all
non-negative $f \in L_\mu(Q_0)$ for which there exists $c > 0$ such that for all $Q \subset Q_0,$

$$(f_{Q,\mu}) \left( (f^{-1/(p-1)})_{Q,\mu} \right)^{p-1} \leq c \quad \text{and} \quad (f^r)_{Q,\mu} \leq c(f_{Q,\mu})^r,$$

respectively. In the unweighted case (i.e., in the case when $\mu$ is Lebesgue measure) these objects were first considered in the classical works by John and Nirenberg [15],

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measures.
Muckenhoupt [20], Gehring [9], and Coifman and Fefferman [5]. It has been quickly realized that the theory developed in the unweighted case remains true for doubling measures (i.e., for those measures \( \mu \) for which there exists a constant \( c > 0 \) such that \( \mu(2Q) \leq c\mu(Q) \) for all cubes \( Q \)). Only recently, it was shown in the papers by Mateu et al. [18] and by Orobitg and Pérez [21] that most of important results concerning BMO(\( \mu \)), \( A_p(\mu) \) and RH\(_r\)(\( \mu \)) still hold for any absolutely continuous measure \( \mu \) (and even for a wider class of measures). First of all, we mention that all these objects are closely related. Namely (see [21]),

\[
A_\infty(\mu) \equiv \bigcup_{1 < p < \infty} A_p(\mu) = \bigcup_{1 < r < \infty} \text{BMO}_r(\mu),
\]

and \( \log f \in \text{BMO}(\mu) \) for any \( f \in A_\infty(\mu) \), and, conversely, given an \( f \in \text{BMO}(\mu) \), there is \( \lambda > 0 \) such that \( e^{\lambda f} \in A_\infty(\mu) \). Also, conditions expressed in the above definitions represent a kind of so-called self-improving properties. Indeed, if \( f \in L_\mu(Q_0) \) belongs to BMO(\( \mu \)), then \( f \) belongs to \( L^p_\mu(Q_0) \) for any \( 1 < p < \infty \) (see [18]). Moreover, \( A_\mu(\mu) \Rightarrow A_{p-\delta}(\mu) \) and \( RH_\mu(\mu) \Rightarrow RH_{r+\delta}(\mu) \) for some small \( \delta > 0 \) which may differ in each implication (see [21]).

In the mid 70’s, Gurov and Reshetnyak [10, 11] introduced in analogy with the definition of BMO(\( \mu \)) (in the unweighted case) the class \( GR_\varepsilon(\mu), 0 < \varepsilon < 2 \), which consists of all non-negative \( f \in L_\mu(Q_0) \) such that for any \( Q \subset Q_0 \),

\[
\Omega_\mu(f; Q) \leq \varepsilon f_{Q,\mu}.
\]

This class has found interesting applications in quasi-conformal mappings and PDE’s (see, e.g., [3, 14]). Observe that (1.2) trivially holds for \( \varepsilon = 2 \), and therefore only the case \( 0 < \varepsilon < 2 \) is of interest. It turned out that (1.2) also represents a kind of self-improving property. It was established in [3, 10, 11, 14, 19, 26] for Lebesgue measure and in [7, 8] for doubling measures that if \( \varepsilon \) is small enough, namely \( 0 < \varepsilon < c2^{-n} \), then the \( GR_\mu(\varepsilon) \) implies \( f \in L^p_\mu(Q_0) \) for some \( p > 1 \). In [16], it was shown that in the case of \( n = 1 \) and Lebesgue measure this self-improvement holds for the whole range \( 0 < \varepsilon < 2 \). In a recent paper [17], the authors have established a rather surprising analogue of (1.1), namely for any absolutely continuous \( \mu \),

\[
A_\infty(\mu) = \bigcup_{0 < \varepsilon < 2} GR_\varepsilon(\mu).
\]

First, this result shows a close relation between the classes \( GR_\varepsilon(\mu) \) and \( A_p(\mu) \). Second, it follows immediately that for any absolutely continuous \( \mu \) and \( n \geq 1 \) and for all \( 0 < \varepsilon < 2 \) the \( GR_\varepsilon(\mu) \) condition implies higher integrability properties of \( f \).

Iwaniec and Kolyada independently asked the authors whether a weaker variant of (1.2):

\[
f_{\mu,\#Q_0}(x) \leq \varepsilon M_{\mu,\#Q_0}f(x) \quad \mu\text{-a.e. in } Q_0
\]

has an analogous self-improving property for all \( \varepsilon < 2 \). Here, as usual,

\[
f_{\mu,\#Q_0}(x) = \sup_{Q \ni x, \mu > \mu(Q_0)} \Omega_\mu(f; Q) \quad \text{and} \quad M_{\mu,\#Q_0}f(x) = \sup_{Q \ni x, \mu > \mu(Q_0)} |f|_{Q,\mu}.
\]
The property expressed in (1.4) we call the maximal Gurov–Reshetnyak condition, and denote it by $MGR_\varepsilon(\mu)$.

Observe that the passing to maximal operators is a quite natural and well-known approach to many questions mentioned above. For example, Gehring’s approach to the reverse Hölder inequality [9] as well as Bojarski’s proof of the Gurov–Reshetnyak Lemma for small $\varepsilon$ [3] were based on maximal function estimates. Actually, many papers establishing the self-improving property of $GR_\varepsilon(\mu)$ for small $\varepsilon$ contain implicitly the same for $MGR_\varepsilon(\mu)$ (see, e.g., [3, 7, 19]). On the other hand, author’s proof of (1.3) cannot be directly generalized to the class $MGR_\varepsilon(\mu)$. Therefore, the question of Iwaniec and Kolyada is of interest for $0 < \varepsilon < 2$.

In this paper we show that for a large class of measures, including any doubling measures in $\mathbb{R}^n$ and any absolutely continuous measures in $\mathbb{R}^1$, the maximal Gurov–Reshetnyak condition $MGR_\varepsilon(\mu)$ is self-improving for any $0 < \varepsilon < 2$. The relevant class of measures will be given in the following definition.

**Definition 1.1.** We say that a measure $\mu$ satisfies the Busemann-Feller type condition (BF-condition) if

$$\varphi_\mu(\lambda) \equiv \sup_E \frac{\mu\{x : M_{\mu,Q_0}\chi_E(x) > \lambda\}}{\mu(E)} < \infty \quad (0 < \lambda < 1),$$

where the supremum is taken over all measurable sets $E \subset Q_0$ of positive $\mu$-measure.

In the case of Lebesgue measure and the maximal operator associated with the homothety-invariant differential basis, this condition coincides with the well-known Busemann–Feller density condition (see, e.g., [4] or [12, p. 122]).

Our main result is the following.

**Theorem 1.2.** Let $\mu$ satisfy the BF-condition, and let $0 < \varepsilon < 2$. Assume that a non-negative $f \in L^p(\mu)$ satisfies the maximal Gurov–Reshetnyak condition $MGR_\varepsilon(\mu)$. Then there is $p_0 > 1$ depending on $\mu, \varepsilon$ and $n$ such that for all $1 \leq p < p_0$ one has

$$\left(f^p\right)_{Q_0,\mu} \leq c(f_{Q_0,\mu})^p,$$

where $c$ depends on $\mu, \varepsilon, p$ and $n$.

Some comments about this result are in order. By (1.1) and (1.3), the usual Gurov–Reshetnyak condition $GR_\varepsilon(\mu)$ implies (1.5) with any subcube $Q \subset Q_0$ instead of $Q_0$. Theorem 1.2 shows that although we cannot obtain from the $MGR_\varepsilon(\mu)$ condition such a nice conclusion, we still have a higher integrability result. In fact, $MGR_\varepsilon(\mu)$ is really much weaker than $GR_\varepsilon(\mu)$. Indeed, let $n = 1$, $Q_0 = (0, 1)$, $\mu$ is Lebesgue measure, and $f$, for example, is the characteristic function of the interval $(0, 7/8)$. Then it is easy to see that $f_{Q_0,\mu}(x) \leq 1/2$, while $M_{Q_0,\mu}f(x) > 7/8$ for all $x \in (0, 1)$. Thus, (1.4) holds for this function with $\varepsilon = 4/7$. However, $f \notin A_\infty(\mu)$, since it is zero on a set of positive measure. Hence, in view of (1.3), (1.2) cannot hold for this $f$ with any $\varepsilon < 2$. This example shows also that the class $GR_\varepsilon(\mu)$ in (1.3) cannot be replaced by $MGR_\varepsilon(\mu)$.
We make several remarks about the BF-condition. It is easy to see that this condition holds provided $M_\mu$ has a weak type $(p, p)$ property with respect to $\mu$ for some $p > 0$. A standard argument shows that if $\mu$ is doubling in $\mathbb{R}^n$, then $M_\mu$ is of weak type $(1, 1)$. It is well-known that in the case $n = 1$ the doubling condition can be completely removed; namely in this case, $M_\mu$ is of weak type $(1, 1)$. It is well-known that in the case $n \geq 2$ for a large class of radial (and non-doubling, in general) measures, including, for example, a Gaussian measure, $M_\mu$ will be of strong type $(p, p)$ for any $p > 1$. On the other hand, it was mentioned in [25] that there exists $\mu$ for which $M_\mu$ will not be of strong type $(p, p)$ for any $p > 1$. In Section 3 below we give an example of $\mu$ (in the case $n = 2$) for which the BF-condition does not hold.

We would like to emphasize that we still do not know whether the BF-condition in Theorem 1.2 is really necessary. In other words we do not know whether there exist an absolutely continuous measure $\mu$ on a cube $Q_0 \subset \mathbb{R}^n$, $n \geq 2$, and a function $f \in L_\mu(Q_0)$ such that $\mu$ does not satisfy the BF-condition, $f$ satisfies (1.4) for some $\varepsilon < 2$ and $f \not\in L_p^\mu(Q_0)$ for any $p > 1$.

Let us mention also that Theorem 1.2 gives yet another proof of the Gurov–Reshetnyak Lemma for the whole range of $\varepsilon$ if $\mu$ is a BF-measure.

The paper is organized as follows. In the next section we prove our main result. Section 3 contains a detailed analysis of the BF-condition.

2. Proof of main result

2.1. Some auxiliary propositions. We first recall that the non-increasing rearrangement of a measurable function $f$ on $Q_0$ with respect to $\mu$ is defined by

$$f_\mu^*(t) = \inf\{\alpha > 0 : \mu\{x \in Q_0 : |f(x)| > \alpha\} \leq t\} \quad (0 < t < \mu(Q_0)).$$

Set also $f_\mu^{**}(t) = t^{-1} \int_0^t f_\mu^*(\tau) d\tau$. Note that [2, pp. 43, 53]

$$\int_0^t f_\mu^*(\tau) d\tau = \sup_{E \subset Q_0 : \mu(E) = t} \int_E |f| d\mu \quad (2.1)$$

and

$$\int_{Q_0} |f|^p d\mu = \int_{\mu(Q_0)} f_\mu^*(\tau)^p d\tau \quad (p > 0). \quad (2.2)$$

We will need a local variant of the well-known Herz-type estimate

$$f_\mu^{**}(t) \leq c(M_\mu f)_\mu^*(t).$$

In the case of $Q_0 = \mathbb{R}^n$ and Lebesgue measure this result can be found in [2, p. 122]. It was extended to arbitrary absolutely continuous measures $\mu$ in [1]. The case of the bounded cube $Q_0$ requires a slightly modified argument based on the following covering lemma from [18].

**Lemma 2.1.** Let $E$ be a subset of $Q_0$ with $\mu(E) \leq \rho \mu(Q_0)$, $0 < \rho < 1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in $Q_0$ such that
(i) \( \mu(Q_i \cap E) = \rho \mu(Q_i) \);
(ii) \( \bigcup Q_i = \bigcup_{k=1}^{B_n} \bigcup_{i \in F_k} Q_i \), where each of the family \( \{Q_i\}_{i \in F_k} \) is formed by pairwise disjoint cubes and a constant \( B_n \) depends only on \( n \);
(iii) \( E' \subset \bigcup_i Q_i \), where \( E' \) is the set of \( \mu \)-density points of \( E \).

**Proposition 2.2.** For any \( f \in L_\mu(Q_0) \) we have

\[
(2.3) \quad f^\ast_\mu(t) \leq c_n(M_{\mu,Q_0}f)^\ast_\mu(t) \quad (0 < t < \mu(Q_0)),
\]

where \( c_n \) depends only on \( n \).

**Proof.** Let \( \Omega = \{ x \in Q_0 : M_{\mu,Q_0}f(x) > (M_{\mu,Q_0}f)^\ast_\mu(t) \} \). Then for some \( \delta_0 \) and for any \( \delta < \delta_0 \) we have \( \mu(\Omega) \leq t \leq (1 - \delta)\mu(Q_0) \). Fix such a \( \delta \) and apply Lemma 2.1 to the set \( \Omega \) and number \( \rho = 1 - \delta \). We get a sequence \( \{Q_i\} \) satisfying properties (i)–(iii) of the lemma. It follows easily from (i) that \( \mu(Q_i \cap \Omega^c) > 0 \), and hence, \( |f|_{Q_i,\mu} \leq (M_{\mu,Q_0}f)^\ast_\mu(t) \). From this and from properties (i)–(iii) we obtain

\[
\int_\Omega |f| \, d\mu \leq \sum_{k=1}^{B_n} \sum_{i \in F_k} \mu(Q_i)|f|_{Q_i,\mu} \leq \frac{B_n}{1 - \delta} t(M_{\mu,Q_0}f)^\ast_\mu(t).
\]

Letting \( \delta \to 0 \) yields \( \int_\Omega |f| \, d\mu \leq B_n t(M_{\mu,Q_0}f)^\ast_\mu(t) \). Therefore, for any measurable set \( E \subset Q_0 \) with \( \mu(E) = t \),

\[
\int_E |f| \, d\mu \leq \int_{E \setminus \Omega} |f| \, d\mu + \int_\Omega |f| \, d\mu \leq (\mu(E \setminus \Omega) + B_n t)(M_{\mu,Q_0}f)^\ast_\mu(t)
\]

\[
\leq (B_n + 1) t(M_{\mu,Q_0}f)^\ast_\mu(t).
\]

Taking the supremum over all \( E \subset Q_0 \) with \( \mu(E) = t \) and using (2.1) completes the proof. \( \Box \)

Given a measurable \( f \), define the local maximal function \( m_{\lambda,\mu}f \) (cf. [24]) by

\[
m_{\lambda,\mu}f(x) = \sup_{Q \ni x, Q \in Q_0} (f\chi_Q)^\ast_\mu(\lambda \mu(Q)) \quad (0 < \lambda < 1).
\]

**Proposition 2.3.** Suppose that \( \mu \) satisfies the BF-condition. Then for any measurable \( f \),

\[
(2.4) \quad (m_{\lambda,\mu}f)^\ast_\mu(t) \leq f^\ast_\mu(t/\varphi_\mu(\lambda)) \quad (0 < t < \mu(Q_0)).
\]

**Proof.** It follows from the definitions that

\[
\{ x \in Q_0 : m_{\lambda,\mu}f(x) > \alpha \} = \{ x \in Q_0 : M_{\mu,Q_0}\chi_{\{|f| > \alpha\}}(x) > \lambda \}.
\]

Therefore,

\[
\mu\{ x \in Q_0 : m_{\lambda,\mu}f(x) > \alpha \} \leq \varphi_\mu(\lambda) \mu\{ x \in Q_0 : |f(x)| > \alpha \},
\]

which is equivalent to (2.4). \( \Box \)
Lemma 2.4. ([20, Lemma 4]) Let \( h \) be a non-negative and non-increasing function on the interval \([0, a]\), and assume that
\[
\frac{1}{s} \int_0^s h(\tau) \, d\tau \leq Dh(s) \quad (0 < s < a/r)
\]
for some \( D, r > 1 \). Then if \( 1 \leq p < D/(D - 1) \),
\[
\int_0^a h^p(\tau) \, d\tau \leq c \left( \int_0^a h(\tau) \, d\tau \right)^p,
\]
where \( c \) depends on \( r, p \) and \( D \).

Actually, this lemma was proved in [20] with \( r = 20 \) but exactly the same argument works for any \( r > 1 \).

2.2. Proof of Theorem 1.2. Choose some constants \( \alpha, \lambda \in (0, 1) \) such that \((1 - \alpha)\lambda > \varepsilon/2\). Take an arbitrary \( \delta \in (0, 1 - \lambda) \). Given a cube \( Q \subset Q_0 \), set
\[
E_Q = \{ x \in Q : f(x) > f_{Q, \mu} \},
\]
and let
\[
\mathcal{Q} = \{ Q \subset Q_0 : \mu(E_Q) \geq \alpha \mu(Q) \}.
\]
Observe that if \( Q \subset \mathcal{Q} \), then \( f_{Q, \mu} \leq (f_{\chi_Q})^\mu(\alpha \mu(Q)) \). Therefore,
\[
M_{\mu, Q_0} f(x) \leq \max \left( m_{\alpha, \mu} f(x), \sup_{Q \ni x, Q \notin \mathcal{Q}} f_{Q, \mu} \right).
\]

Assume that \( Q \notin \mathcal{Q} \). Set \( E_Q' = Q \setminus E_Q \),
\[
A_1(Q) = \{ x \in E_Q' : f_{Q, \mu} - f(x) > ((f_{Q, \mu} - f)\chi_{E_Q'})^\mu(\lambda \mu(E_Q')) \}
\]
and
\[
A_2(Q) = \{ x \in E_Q' : f(x) > (f_{\chi_{E_Q'}})^\mu(1 - \lambda - \delta)\mu(E_Q') \}.
\]
Then \( \mu(A_1(Q) \cup A_2(Q)) \leq (1 - \delta)\mu(E_Q') \) and \( \mu(E_Q') \geq (1 - \alpha)\mu(Q) \). Therefore we obtain
\[
f_{Q, \mu} \leq \inf_{y \in E_Q' \setminus \{ A_1(Q) \cup A_2(Q) \}} ((f_{Q, \mu} - f(y)) + f(y))
\]
\[
\leq ((f_{Q, \mu} - f)\chi_{E_Q'})^\mu(\lambda \mu(E_Q')) + (f_{\chi_{E_Q'}})^\mu((1 - \lambda - \delta)\mu(E_Q'))
\]
\[
\leq \mu(Q) \frac{1}{(Q, \mu(Q))} \int_{E_Q'} (f_{Q, \mu} - f) \, d\mu + (f_{\chi_{E_Q'}})^\mu((1 - \lambda - \delta)\mu(E_Q'))
\]
\[
\leq \frac{1}{2\lambda(1 - \alpha)} \Omega\mu(f; Q) + (f_{\chi_{E_Q'}})^\mu((1 - \lambda - \delta)(1 - \alpha)\mu(Q)).
\]

This along with the maximal Gurov–Reshetnyak condition (1.4) yields
\[
\sup_{Q \ni x, Q \notin \mathcal{Q}} f_{Q, \mu} \leq \frac{\varepsilon}{2\lambda(1 - \alpha)} M_{\mu, Q_0} f(x) + m_{\gamma, \mu} f(x),
\]
where \( \gamma = (1 - \lambda - \delta)(1 - \alpha) \). From this and from (2.5) we obtain
\[
M_{\mu, Q_0} f(x) \leq cm_{\alpha', \mu} f(x),
\]
where \( c = \frac{2\lambda(1-\alpha)}{2\lambda(1-\alpha)-2} \) and \( \alpha' = \min(\alpha, \gamma) \). Taking the rearrangements of both parts and using Propositions 2.2 and 2.3, we have
\[
f_{\mu}^{**}(t) \leq cf_{\mu}^{*}\left(\frac{t}{\varphi_{\mu}(\alpha')}\right) \quad (0 < t < \mu(Q_0)).
\]
This implies easily
\[
f_{\mu}^{**}(t) \leq c\varphi_{\mu}(\alpha')f_{\mu}^{*}(t) \quad (0 < t < \mu(Q_0)/\varphi_{\mu}(\alpha')),
\]
which along with (2.2) and Lemma 2.4 completes the proof. \( \square \)

3. On the BF-condition

First of all, we observe that we do not know whether for some absolutely continuous \( \mu \) the function \( \varphi_{\mu} \) can take both finite and infinite values. The following proposition represents only a partial answer to this question.

**Proposition 3.1.** There exists a constant \( \gamma < 1 \) depending only on \( n \) such that if \( \varphi_{\mu}(\gamma) < \infty \), then \( \varphi_{\mu}(\lambda) < \infty \) for all \( 0 < \lambda < 1 \).

**Proof.** Given a set \( E \subset Q_0 \) and \( 0 < \lambda < 1 \), let \( E_\lambda = \{ x \in Q_0 : M_{\mu,Q_0} \chi_E(x) > \lambda \} \).

By Proposition 2.2, for any \( E \subset Q \subset Q_0 \),
\[
\min(1, \mu(E)/t) \leq c_n(M_{\mu,Q} \chi_E)^*_{\mu}(t)
\]
or, equivalently,
\[
\frac{\mu(E)}{c_n \lambda} \leq \mu\{ x \in Q : M_{\mu,Q} \chi_E(x) > \lambda \} \leq \mu(Q \cap E_\lambda).
\]
Hence,
\[
M_{\mu,Q_0} \chi_E(x) \leq c_n \lambda M_{\mu,Q_0} \chi_{E_\lambda}(x) \quad (x \in Q_0),
\]
which yields
\[
(3.1) \quad \mu\{ x \in Q_0 : M_{\mu,Q_0} \chi_E > c_n \xi \lambda \} \leq \mu\{ x \in Q_0 : M_{\mu,Q_0} \chi_{E_\lambda} > \xi \}.
\]
Therefore,
\[
\varphi_{\mu}(c_n \lambda \xi) \leq \varphi_{\mu}(\lambda) \varphi_{\mu}(\xi) \quad (\lambda, \xi \in (0, 1) : \lambda \xi < 1/c_n).
\]
This clearly implies the desired result if \( 0 < \lambda < 1/c_n \). The case \( 1/c_n < \lambda < 1 \) follows from the monotonicity of \( \varphi_{\mu}(\lambda) \). \( \square \)

**Remark 3.2.** The last proposition means that Theorem 1.2 still holds if one relaxes the BF-condition to
\[
\mu\{ x \in Q_0 : M_{\mu,Q_0} \chi_E > \gamma \} \leq c\mu(E) \quad \forall E \subset Q_0
\]
with some \( 0 < \gamma < 1 \). Note that a similar condition with \( \gamma = 1/2 \) for the directional maximal operator appeared in [6] (see also [12, p. 372]), where it was called a Tauberian condition.

**Remark 3.3.** Inequality (3.1) is a full analogue of the Lemma from [13], which was proved there in a different context.
We give now an example of the absolutely continuous measure \( \mu \) on the unit cube \((0,1)^2\) which does not satisfy the BF-condition.

Let \( \delta, L > 0 \) and
\[
d\mu = (\delta \chi_{(-L,0)^2}(x,y) + \chi_{(0,L)^2}(x,y)) \, dx \, dy.
\]

Given a point \((x,y) \in (0,L)^2\), let \( Q \) be a minimal cube contained in \((-2L,2L)^2\) and containing \((x,y)\) and the cube \((-L,0)^2\). Then \( \mu(Q) = xy + \delta L^2 \). Hence, setting \( E = (-L,0)^2 \), we get
\[
M_{\mu((-2L,2L)^2)} \chi_E(x,y) \geq \frac{\mu(Q \cap E)}{\mu(Q)} = \frac{\delta L^2}{xy + \delta L^2}.
\]

Denote \( c_\lambda = \frac{1}{\lambda} - 1 \). Assuming \( \delta c_\lambda < 1 \), we get, by (3.2),
\[
\frac{\mu\{M_{\mu((-2L,2L)^2)} \chi_E(x,y) > \lambda\}}{\mu(E)} \geq \frac{\mu\{(x,y) \in (0,L)^2 : xy < \delta L^2 c_\lambda\}}{\delta L^2} \geq c_\lambda \int_{\delta c_\lambda L}^L \frac{dx}{x} = c_\lambda \log \frac{1}{\delta c_\lambda}.
\]

Choose now a sequence of cubes \( \{Q_i\} \) such that the cubes \(2Q_i\) are pairwise disjoint and \( \bigcup_{i=1}^{\infty} 2Q_i \subset (0,1)^2 \). We divide each cube \( Q_i \) into four equal quadrants, and let \( Q_i^1 \) and \( Q_i^2 \) be the first and the third quadrants respectively. Let \( \{\delta_i\} \) be a sequence of positive numbers such that \( \delta_i \to 0 \) as \( i \to \infty \). Set
\[
d\mu = \sum_{i=1}^{\infty} \left( \delta_i \chi_{Q_i^1}(x,y) + \chi_{Q_i^2}(x,y) \right) \, dx \, dy.
\]

Given a \( \lambda \in (0,1) \), there is an \( N \) such that \( \delta_i c_\lambda < 1 \) for all \( i \geq N \). Hence, by (3.3),
\[
\frac{\mu\{M_{\mu((0,1)^2)} \chi_{Q_i^2} > \lambda\}}{\mu(Q_i^2)} \geq c_\lambda \log \frac{1}{c_\lambda \delta_i} \quad (i \geq N).
\]

This shows that \( \mu \) does not satisfy the BF-condition, since the right-hand side of (3.4) tends to \( \infty \) as \( i \to \infty \).

In conclusion we give one more proposition concerning the function \( \varphi_\mu \) which is probably of some independent interest. We recall that the operator \( M_\mu \) is said to be of restricted weak type \((p,p)\) if there exists \( c > 0 \) such that
\[
\varphi_\mu(\lambda) \leq c \lambda^{-p} \quad (0 < \lambda < 1).
\]

By the well-known interpolation theorem of Stein and Weiss [2, p. 233], (3.5) implies the strong type \((q,q)\) of \( M_\mu \) for \( q > p \). The following proposition shows first that a slightly better estimate than (3.5) allows us to get the strong type \((p,p)\), and, second, it yields a very simple proof of the Stein–Weiss theorem for \( M_\mu \).

**Proposition 3.4.** If \( \int_0^1 \varphi_\mu(\lambda)^{1/q} \, d\lambda < \infty \), then \( M_\mu \) is bounded on \( L^q_\mu \) and
\[
\|M_\mu f\|_{L^q_\mu} \leq \left( \int_0^1 \varphi_\mu(\lambda)^{1/q} \, d\lambda \right) \|f\|_{L^q_\mu} \quad (1 \leq q < \infty).
\]
Proof. By (2.2),
\[
\frac{1}{\mu(Q)} \int_Q |f| \, d\mu = \int_0^1 (f \chi_Q)^* (\lambda \mu(Q)) \, d\lambda,
\]
and thus,
\[
M_\mu f(x) \leq \int_0^1 m_{\lambda, \mu} f(x) \, d\lambda.
\]
Applying Minkowski’s inequality along with Proposition 2.3 yields
\[
\|M_\mu f\|_{L_p^\mu} \leq \int_0^1 \|m_{\lambda, \mu} f\|_{L_p^\mu} \, d\lambda \leq \left( \int_0^1 \varphi_\mu(\lambda)^{1/q} \, d\lambda \right) \|f\|_{L_q^\mu},
\]
as required. □

References


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