ON WEIGHTED ESTIMATES OF NON-INCREASING
REARRANGEMENTS

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Dedicated to Professor P. L. Ul’yanov on the occasion of his 70th birthday

Abstract

Let \( \omega \) be a weight satisfying Muckenhoupt’s condition \( A_\infty \). In present paper the estimate of rearrangement \( f_\omega^*(t) \) was obtained

\[
f_\omega^*(t) \leq 2(M_\lambda^\# f)^*_\omega(2t) + f_\omega^*(2t) \quad (0 < t < \infty),
\]

where \( f \) is any measurable function, \( M_\lambda^\# f \) is local sharp maximal function due to John [12] and Strömberg [18]. Before (Bennett, DeVore and Sharpley [3], Bagby and Kurtz [1]) the similar estimates were expressed in terms of Fefferman-Stein sharp-function \( f^\# \) which is sufficiently larger then \( M_\lambda^\# f \).

In paper several applications of this estimate were pointed out.

1 Introduction

A non-negative locally integrable on \( \mathbb{R}^n \) function \( \omega(x) \) is called a weight. With any weight function we associate the measure \( \omega(E) = \int_E \omega(x) \, dx \). In case \( \omega(x) \equiv 1 \), \( \omega(E) = |E| \) is the Lebesgue measure. We shall assume that \( \omega(\mathbb{R}^n) = +\infty \).

Let \( f \) be a measurable function on \( \mathbb{R}^n \). The distribution function for \( f \) with respect to the measure \( \omega \) is defined by the equality

\[
\mu_f(\lambda) = \omega\{x \in \mathbb{R}^n : |f(x)| > \lambda\} \quad (0 < \lambda < \infty).
\]

Assume that \( \mu_f(\lambda) < \infty, \lambda > 0 \). We say that the function \( f_\omega^*(t) \) is a non-increasing rearrangement of \( f \) with respect to the measure \( \omega \) if it is non-increasing on \( (0, +\infty) \) and equimeasurable with \( |f(x)| \), that is,

\[
|\{t \in (0, +\infty) : f_\omega^*(t) > \lambda\}| = \mu_f(\lambda)
\]

for any \( \lambda > 0 \). We shall assume that the rearrangement is continuous from the left. Then it is uniquely determined and can be defined by the equality

\[
f_\omega^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) < t\}
\]
or by (see [5, p. 32], [14]):

\[ f^*_\omega(t) = \sup_{\omega(E) = t} \inf_{x \in E} |f(x)| \quad (0 < t < \infty) \]

(by virtue of the conditions on \( \omega \), for any \( t > 0 \), a set \( E \) exists with \( \omega(E) = t \). Denote \( f^{**}_\omega(t) = t^{-1} \int_0^t f^*_\omega(\tau) d\tau \). If \( \omega \) is the Lebesgue measure we use the notations \( f^*(t) \), \( f^{**}(t) \).

We say that the weight function \( \omega \) satisfies Muckenhoupt’s condition \( A_\infty \) if there exist constants \( c, \delta > 0 \) such that for any cube \( Q \), and for any measurable subset \( E \subset Q \), holds the inequality

\[ \omega(E) \leq c \left( \frac{|E|}{|Q|} \right)^\delta \omega(Q). \]

Many other equivalent definitions of \( A_\infty \) can be found in the paper [6] of Coifman and Fefferman.

In this paper an estimate for the rearrangement \( f^*_\omega(t) \) (\( \omega \in A_\infty \)) in terms of the maximal function \( M^\#_f \) is obtained.

For a given measurable function \( f \) the maximal function \( M^\# f \) is defined as

\[ M^\# f(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} (|f - c|\chi_Q)^*(\lambda|Q|), \quad 0 < \lambda \leq 1, \]

(the supremum is taken over all cubes \( Q \) containing the point \( x \); \( \chi_Q \) denotes the characteristic function of \( Q \)).

The definition (1.1) was given by John [12] in 1965. Later, in 1972, Fefferman and Stein [10] introduce the maximal function \( f^\# \) which measure the mean oscillations. It is defined as follows

\[ f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \quad (f_Q = |Q|^{-1} \int_Q f). \]

The space \( BMO \) consists of all locally integrable functions such that \( f^\# \in L^\infty \) and the \( BMO \)-semi-norm is defined as \( \|f\|_* = \|f^\#\|_\infty \).

Since \( tf^*(t) \leq tf^{**}(t) \leq \|f\|_1 \), it is clear that \( \lambda M^\#_f(x) \leq f^\#(x) \)

\((0 < \lambda \leq 1)\). Thus

\[ \lambda ||M^\#_f||_\infty \leq \|f\|_* \quad (0 < \lambda \leq 1). \]

John [12] and Strömberg [18] proved that, in case \( \lambda \leq 1/2 \), the converse statement is also true. The following theorem holds.

**Theorem** [12, 18]. If \( 0 < \lambda \leq 1/2 \) then

\[ \|f\|_* \leq c ||M^\#_f||_\infty \]
where $c$ depends only on the dimension.

Bennett, DeVore and Sharpley proved in [3] the inequality: For any function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and any cube $Q \subset \mathbb{R}^n$,
\begin{equation}
(f \cdot \chi_Q)^*(t) - (f \cdot \chi_Q)^*(t) \leq c(f^#)^*(t) \quad (0 < t < |Q|/6).
\end{equation}
This result implies, in particular, the theorem of Fefferman and Stein [10]:
\begin{equation}
\|Mf\|_p \asymp \|f^#\|_p \quad (1 < p < \infty)
\end{equation}
($Mf$ is the standard maximal operator of Hardy – Littlewood).

Afterwards Bagby and Kurtz [1] established the weighted inequality: If $\omega \in A_\infty$, then for any function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,
\begin{equation}
f^*_\omega(t) \leq c(f^#)^*_\omega(2t) + f^*_\omega(2t) \quad (t > 0).
\end{equation}
There and in the consequent paper [2] similar estimates connecting the rearrangements of the functions $Mf$ and $f^#$, as well as of $Tf$ and $Mf$, are obtained ($Tf$ is the singular integral operator of Calderón – Zygmund). These estimates are ”rearrangemental” analogies of earlier known ”good $\lambda$ inequalities” [10, 6].

The main result of the present paper is an estimate which is more precise than (1.3). Namely, we show that if $\omega \in A_\infty$, $f$ is a measurable function, and $Q$ is an arbitrary cube, then
\begin{equation}
(f \cdot \chi_Q)^*_\omega(t) \leq 2(M^# f)^*_\omega(2t) + (f \cdot \chi_Q)^*_\omega(2t) \quad (0 < t < \omega(Q)),
\end{equation}
\begin{equation}
f^*_\omega(t) \leq 2(M^# f)^*_\omega(2t) + f^*_\omega(2t) \quad (0 < t < \infty).
\end{equation}
Moreover, with the help of these inequalities certain known results concerning estimates of the $L^p_\omega$- and $BMO$-norms of maximal and singular integral operators can be obtained.

2 Auxiliary propositions

A measure $\omega$ is said to satisfy the doubling condition ($\omega \in D$), if for any cube $Q$ holds $\omega(2Q) \leq c\omega(Q)$. The condition $A_\infty$ yields $D$, but the inverse is not true (see [9]).

Lemma 2.1. Assume that the measure $\omega$ satisfies the doubling condition $D$ and let $0 < \lambda < 1$. 

3
(i) If $Q_0$ is a cube and $E \subset Q_0$ is an arbitrary measurable set of positive measure with $\omega(E) \leq \lambda \omega(Q_0)$, then there exist mutually disjoint cubes $\{Q_i\} \subset Q_0$ covering $E$ and such that

$$\lambda \omega(Q_i) < \omega(Q_i \cap E) \leq c_\omega \lambda \omega(Q_i).$$

(ii) If $E \subset \mathbb{R}^n$ is an arbitrary measurable set of finite positive $\omega$-measure, then there exist mutually disjoint cubes $\{Q_i\}$ covering $E$ and satisfying inequality (2.1).

Proof. Item (i) follows from the Calderón – Zygmund ”weight” lemma (see [15]): If $\frac{1}{\omega(Q_0)} \int_{Q_0} |f(y)| \omega(y) \, dy \leq \lambda$ then there exist mutually disjoint cubes $\{Q_i\} \subset Q_0$ such that $\lambda < \frac{1}{\omega(Q_i)} \int_{Q_i} |f(y)| \omega(y) \, dy \leq c_\omega \lambda$ and $|f(x)| \leq \lambda$ for almost all $x \in Q_0 \setminus \bigcup_i Q_i$ (the proof is the same as that of the ordinary lemma of Calderón – Zygmund [16, p. 27]). Letting $f(y) = \chi_E(y)$ we get (i). For the proof of item (ii), it is necessary to decompose $\mathbb{R}^n$ into sufficiently big non-overlapping cubes $Q_k$ in such a way that $\omega(E \cap Q_k) \leq \lambda \omega(Q_k)$. We can do this because the set $E$ has a finite $\omega$-measure. Then it remains to apply item (i) to each cube $Q_k$. The lemma is proved.

Remark 2.1. It is meant in the formulation of Lemma 2.1 that the cubes $Q_i$ cover the set $E$ almost everywhere, i.e., $|E \setminus \bigcup_i Q_i| = 0$. Note else that in the lemma cited in [15] the family $\{Q_i\}$ of cubes may be empty in case $|f(x)| \leq \lambda$ almost everywhere. But, in our case, this is impossible since we take $\lambda < 1$ and $f(x) = \chi_E(x)$.

Lemma 2.2. Let $\omega \in D$. Then, for any measurable function $f$ and for every cube $Q$, the following inequality holds

$$(f \cdot \chi_Q)\omega(\lambda \omega(Q)) \leq 2 \inf_{c \in \mathbb{R}} \left( (f - c) \chi_Q \right)\omega(\lambda \omega(Q)) + (f \cdot \chi_Q)\omega((1 - \lambda) \omega(Q))$$

$(0 < \lambda < 1)$.

Proof. If $1/2 \leq \lambda < 1$, then the lemma is trivial. Assume that $0 < \lambda < 1/2$. For any constant $c$ we have

$$|c| \leq \inf_{x \in \bar{Q}} (|c - f(x)| + |f(x)|) \leq \left( |c - f| + |f| \right)\omega(\lambda \omega(Q))$$

$$\leq \left( (f - c) \chi_Q \right)\omega(\lambda \omega(Q)) + (f \cdot \chi_Q)\omega((1 - \lambda) \omega(Q)).$$

Therefore

$$(f \cdot \chi_Q)\omega(\lambda \omega(Q)) \leq \left( (f - c) \chi_Q \right)\omega(\lambda \omega(Q)) + |c|$$
\[
\leq 2 \left( (f - c) \chi_Q \right)_\omega^* (\lambda \omega(Q)) + (f \cdot \chi_Q)_\omega^* ((1 - \lambda) \omega(Q)).
\]

The proof is completed.

**Remark 2.2.** It is clear that the last lemma is valid under even more general conditions on \( \omega \).

Consider now the weighted analogy of the function \( M^\#_\lambda f \):

\[
M^\#_\lambda, \omega f(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left( (f - c) \chi_Q \right)_\omega^* (\lambda \omega(Q)) \quad (0 < \lambda \leq 1).
\]

**Lemma 2.3.** If \( \omega \) satisfies the condition \( A_{\infty} \), then for every \( \lambda \leq 1 \) there exists \( \lambda', \lambda'' \leq 1 \) such that

\[
(2.2) \quad M^\#_{\lambda'} f(x) \leq M^\#_{\lambda, \omega} f(x) \leq M^\#_{\lambda''} f(x)
\]

for all \( x \in \mathbb{R}^n \).

**Proof.** Let \( E \subset Q \) and \( \omega(E) = \lambda \omega(Q) \). Then it follows from the definition of \( A_{\infty} \) that for certain \( c, \delta > 0 \) we have

\[
|E| \geq \frac{\lambda^{1/\delta}}{c} |Q|.
\]

Thus

\[
\inf_{x \in E} |f(x) - \xi| \leq \left( (f - \xi) \chi_Q \right)_\omega^* \left( \frac{\lambda^{1/\delta}}{c} |Q| \right), \quad \xi \in \mathbb{R}.
\]

Taking a supremum over all \( E \subset Q \) with \( \omega(E) = \lambda \omega(Q) \), we get

\[
\left( (f - \xi) \chi_Q \right)_\omega^* (\lambda \omega(Q)) \leq \left( (f - \xi) \chi_Q \right)_\omega^* \left( \frac{\lambda^{1/\delta}}{c} |Q| \right), \quad \xi \in \mathbb{R}.
\]

This inequality yields the right-hand side of (2.2) with \( \lambda' = \lambda^{1/\delta} / c \).

Assume that \( E \subset Q \) and \( |E| = \lambda'' |Q| \). Then \( |Q \setminus E| = (1 - \lambda'') |Q| \) and hence \( \omega(Q \setminus E) \leq c (1 - \lambda'')^\delta \omega(Q) \). Therefore

\[
\omega(E) \geq (1 - c (1 - \lambda'')^\delta) \omega(Q)
\]

and

\[
\inf_{x \in E} |f(x) - \xi| \leq \left( (f - \xi) \chi_Q \right)_\omega^* ((1 - c (1 - \lambda'')^\delta) \omega(Q)).
\]

Letting \( \lambda = 1 - c (1 - \lambda'')^\delta \) we get \( \lambda'' = 1 - (1 - \lambda)^{1/\delta} \). Inequality (2.2) is proved with \( \lambda'' = 1 - (1 - \lambda)^{1/\delta} / c \), \( \lambda' = \lambda^{1/\delta} / c \).
Denote by $L^p(\mathbb{R}^n)$ the space of all functions $f$ for which

$$
\|f\|_{p,\omega} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.
$$

By virtue of the equimeasurability of the rearrangement the following equality holds:

$$
\int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx = \int_0^\infty (f^\ast_\omega(t))^p \, dt \quad (p > 0).
$$

**Lemma 2.4.** Let the functions $f, g$ satisfy the inequality

$$
f^\ast_\omega(t) \leq cg^\ast_\omega(\gamma t) + f^\ast_\omega(2t) \quad (0 < t < \infty, \gamma > 0).
$$

Assume that $f^\ast_\omega(+\infty) = 0$. Then

$$
\|f\|_{p,\omega} \leq c \|g\|_{p,\omega} \quad (0 < p < \infty).
$$

**Proof.** Applying consecutively (2.4) we get

$$
f^\ast_\omega(t) \leq \sum_{k=0}^\infty g^\ast_\omega(2^k \gamma t) \leq \frac{c}{\log 2} \sum_{k=0}^\infty \int_{2^k \gamma t}^{2^{k+1} \gamma t} \frac{g^\ast(s)}{s} \, ds = \frac{c}{\log 2} \int_t^\infty \frac{g^\ast(s)}{s} \, ds.
$$

Hence, from Hardy inequation [16, p. 319] and (2.3), it follows that

$$
\|f\|_{p,\omega} = \|f^\ast_\omega\|_p \leq \frac{2c}{\log 2} \cdot \frac{1}{\gamma} \left\| \int_t^\infty \frac{g^\ast(s)}{s} \, ds \right\|_p \leq \frac{2c}{\log 2} \cdot \frac{1}{\gamma} \|g\|_{p,\omega} \quad (1 \leq p < \infty).
$$

If $0 < p < 1$ the arguments are similar since in this case (2.4) implies that

$$
(f^\ast_\omega(t))^p \leq (cg^\ast(\gamma t))^p + (f^\ast_\omega(2t))^p.
$$

The lemma is proved.

### 3 The basic inequality

**Theorem 3.1.** Let $\omega$ satisfy the doubling condition. Then, for any measurable function $f$ and for every cube $Q$, the following inequalities hold

$$
(f \cdot \chi_Q)^\ast_\omega(t) \leq 2(M^\#_{\lambda,\omega} f)^\ast_\omega(t) + (f \cdot \chi_Q)^\ast_\omega(2t) \quad (0 < t \leq \omega(Q)/5c_\omega),
$$

$$
f^\ast_\omega(t) \leq 2(M^\#_{\lambda,\omega} f)^\ast_\omega(t) + f^\ast_\omega(2t) \quad (0 < t < \infty),
$$

where $c_\omega$ is the constant from (2.1), $0 < \lambda \leq 1/5c_\omega$. 

6
Proof. Fix an arbitrary cube $Q \subset \mathbb{R}^n$. Let $\lambda \leq 1/5c_\omega$, $t \leq \omega(Q)/5c_\omega$ and let $E \subset Q$ be an arbitrary measurable set with $\omega(E) = t$. According to Lemma 2.1 (i) there exist mutually disjoint cubes $\{Q_i\} \subset Q$ covering $E$ and such that

\begin{equation}
\omega(Q_i \cap E) > \frac{1}{5c_\omega} \omega(Q_i), \tag{3.3}
\end{equation}

\begin{equation}
\sum_i \omega(Q_i) \geq 5 \sum_i \omega(Q_i \cap E) = 5\omega(E) = 5t. \tag{3.4}
\end{equation}

Select from them the cubes $\{Q_i'\}$ which are contained in the set

$$E^* = \{ x \in \mathbb{R}^n : M_{\lambda,\omega}^f(x) > (M_{\lambda,\omega}^f)_{\omega}(2t) \},$$

In other words, for $Q_i'$ the following inequality holds

$$\inf_{c} ((f - c)\chi_{Q_i'})_{\omega}(\lambda\omega(Q_i')) \leq (M_{\lambda,\omega}^f)_{\omega}(2t).$$

This relation and Lemma 2.2 yield

\begin{equation}
(f \cdot \chi_{Q_i'})_{\omega}(\omega(Q_i')/5c_\omega) \leq 2 \inf_{c} ((f - c) \cdot \chi_{Q_i'})_{\omega}(\lambda\omega(Q_i')) + (f \cdot \chi_{Q_i'})_{\omega}((1 - 1/5c_\omega)\omega(Q_i')) \\
\leq 2(M_{\lambda,\omega}^f)_{\omega}(2t) + (f \cdot \chi_{Q_i'})_{\omega}((1 - 1/5c_\omega)\omega(Q_i')). \tag{3.5}
\end{equation}

Further, since $\omega(E^*) \leq 2t$ and the cubes $Q_i$ are mutually disjoint, taking into account (3.4), we get $\sum_i \omega(Q_i') \geq 3t$. Thus

$$\inf_i (f \cdot \chi_{Q_i'})_{\omega}((1 - 1/5c_\omega)\omega(Q_i')) \leq (f \cdot \chi_{Q_i'})_{\omega}((1 - 1/5c_\omega)2t) \leq (f \cdot \chi_{Q_i'})_{\omega}(2t).$$

Hence, by virtue of (3.3) and (3.5), we have

\begin{align*}
\inf_{x \in E} |f(x)| & \leq \inf_{x \in E \cap Q_i'} |f(x)| \leq \inf_{i} (f \cdot \chi_{Q_i'})_{\omega}(\omega(Q_i' \cap E)) \\
& \leq \inf_i (f \cdot \chi_{Q_i'})_{\omega}(\omega(Q_i')/5c_\omega) \leq 2(M_{\lambda,\omega}^f)_{\omega}(2t) + (f \cdot \chi_{Q_i'})_{\omega}(2t).
\end{align*}

Taking supremum in this inequality over all sets $E \subset Q$ with $\omega(E) = t$, we get (3.1). Inequality (3.2) may be proved either in the same manner using Lemma 3.1 (ii), or from (3.1) passing to the limit as $Q \to \mathbb{R}^n$. The theorem is proved.
extend the idea of transition from "good λ inequalities" to estimates of rearrangements.

Lemma 2.3 and inequality (3.2) yields the following statement.

**Corollary 3.1.** If ω ∈ A∞, then for every measurable function f we have

\[
(3.6) \quad f_\ast(t) \leq 2(M_\lambda \# f)_\ast(2t) + f_\ast(2t) \quad (0 < t < \infty, \; 0 < \lambda < \lambda_0(\omega)).
\]

Inequality (3.6) and Lemma 2.4 imply immediately

**Theorem 3.2.** Let ω ∈ A∞. Then, for any f satisfying \(f_\ast(+\infty) = 0\), holds the inequality

\[
(3.7) \quad \|f\|_{L^p}\omega \leq c_{p,\omega} \|M_\lambda \# f\|_{L^p}\omega \quad (0 < p < \infty, \; 0 < \lambda < \lambda_0(\omega)).
\]

In the non-weighted case this result was proved in [11].

### 4 Certain estimates for operators

The right-hand side of inequality (3.7) contains the \(L^p_\omega\)-norm of the function \(M_\lambda \# f\). As mentioned already in the introduction

\[
(4.1) \quad \|f\|_\ast \leq c_{n,\lambda} \|M_\lambda \# f\|_\infty
\]

for \(0 < \lambda \leq 1/2\).

It is shown in this section that application of our basic inequality (3.6) and John – Strömberg inequality (4.1), allow us to give short proofs of certain known results.

To this end consider first how the operator \(M_\lambda \#\) acts on the operators \(Mf\) and \(Tf\). Set

\[
Mf(x) = \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} |f(y)| \, dy \right).
\]

**Theorem 4.1.** Assume that \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\). If \(Mf(x) < \infty\) almost everywhere, then

\[
(4.2) \quad M_\lambda \#(Mf)(x) \leq c_{n,\lambda} f_\ast(x) \quad (0 < \lambda \leq 1)
\]

for all \(x \in \mathbb{R}^n\).

**Proof.** Fix a cube \(Q\). Let \(x \in Q\) and let \(Q'\) be an arbitrary cube containing \(x\). If \(Q' \subset 3Q\), then

\[
(4.3) \quad |f|_{Q'} \leq |f - f_{3Q}|_{Q'} + |f|_{3Q} \leq M((f - f_{3Q})\chi_{3Q})(x) + \inf_{\xi \in Q} Mf(\xi).
\]
Assume that $Q' \not\subset 3Q$. Then $Q \subset 3Q'$ and in this case
\[
|f|_{Q'} \leq |f - f_{3Q'}|_{Q'} + |f|_{3Q'} \leq 3^{n} \inf_{\xi \in Q} f^\#(\xi) + \inf_{\xi \in Q} Mf(\xi).
\]
Hence, using (4.3), we get
\[
(4.4) \quad Mf(x) \leq M((f - f_{3Q})\chi_{3Q})(x) + 3^{n} \inf_{\xi \in Q} f^\#(\xi) + \inf_{\xi \in Q} Mf(\xi)
\]
for all $x \in Q$. Making use of the fact that the operator $M$ is of weak type (1,1) [16, p. 15] and (4.4), we get
\[
\inf_{c}((Mf - c)\chi_{Q})^*(\lambda|Q|) \leq ((Mf - \inf_{Q} Mf)\chi_{Q})^*(\lambda|Q|) \leq (M((f - f_{3Q})\chi_{3Q}))^*(\lambda|Q|) + 3^{n} \inf_{\xi \in Q} f^\#(\xi) \leq \frac{c_{n}}{\lambda|Q|} \int_{3Q} |f(y) - f_{3Q}| \, dy + 3^{n} \inf_{\xi \in Q} f^\#(\xi).
\]
Taking supremum over all $Q$ containing $x$ we get (4.2). The theorem is proved.

Note that a weaker variant of inequality (4.2) for the local case was given without proof in [11].

**Corollary 4.1.**

(i) If $f \in BMO(\mathbb{R}^{n})$ and $Mf(x) < \infty$ a.e., then $Mf \in BMO(\mathbb{R}^{n})$ and
\[
\|Mf\|_{*} \leq c_{n}\|f\|_{*}.
\]

(ii) If $\omega \in A_{\infty}$, then
\[
(4.5) \quad (Mf)_{\infty}^{\omega}(t) \leq c_{n,\omega}(f^\#)_{\infty}^{\omega}(2t) + (Mf)_{\infty}^{\omega}(2t) \quad (t > 0)
\]
for any function $f \in L_{1}^{1}(\mathbb{R}^{n})$.

(iii) If $\omega \in A_{\infty}$, then
\[
(4.6) \quad \|Mf\|_{L_{p}^{\omega}} \leq c_{n,p,\omega}\|f^\#\|_{L_{p}^{\omega}} \quad (0 < p < \infty)
\]
for all functions $f$ with $(Mf)_{\omega}^{\infty}(+\infty) = 0$.

Item (i) follows from inequalities (4.1), (4.2). This result was proved in [3]. Inequality (4.5) was obtained in [1]. It follows immediately from (3.6) and (4.2). Inequality (4.6) is the weighted Fefferman – Stein theorem [10]. It follows either from (3.7) or (4.5).
Suppose that the kernel \( k(x) \) satisfies the standard conditions:

\[
|k(x)| \leq \frac{c}{|x|^n}, \quad \int_{R_1<|x|<R_2} k(x) \, dx = 0 \quad (0 < R_1 < R_2 < \infty),
\]

(4.7)

\[
|k(x) - k(x - y)| \leq \frac{c|y|^{\alpha}}{|x|^{n+\alpha}} \quad (|y| \leq |x|/2, \alpha > 0).
\]

Set, for \( f \in L^p(\mathbb{R}^n) \) \((1 \leq p < \infty)\),

\[
T_f(x) = \text{P.V.} \int_{\mathbb{R}^n} f(y)k(x - y) \, dy,
\]

\[
T^*_f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} f(y)k(x - y) \, dy \right|.
\]

**Theorem 4.2.** For all \( f \in L^p(\mathbb{R}^n) \) \((1 \leq p < \infty)\) we have

\[
M^\#_{\lambda} (Tf)(x) \leq c_{n,\lambda} Mf(x),
\]

(4.8)

\[
M^\#_{\lambda} (T^*_f)(x) \leq c_{n,\lambda} Mf(x).
\]

(4.9)

This theorem was proved in [11]. To prove it, decompose the function into two parts. Let \( x \) be the center of the cube \( Q \). Set \( f_1 = f \cdot \chi_{3Q}, \ f_2 = f \cdot \chi_{\mathbb{R}^n \setminus 3Q} \). By virtue of the weak type \((1,1)\) of the operator \( T \) (see [16, p. 42]), we have

\[
(Tf_1)^*(\lambda|Q|) \leq \frac{c}{\lambda|Q|} \int_{3Q} |f(y)| \, dy \leq c' Mf(x).
\]

On the other hand, using the conditions (4.7) on the kernel, it is not difficult to show that for all \( z', z'' \in Q \) holds the estimate

\[
|Tf_2(z') - Tf_2(z'')| \leq c Mf(x).
\]

This yields (4.8). Inequality (4.9) can be proved in a similar way.

The above theorem and inequalities (3.6), (4.1) imply

**Corollary 4.2.**

(i) If \( f \in L^p \cap L^\infty(\mathbb{R}^n) \), then \( Tf, T^*_f \in BMO(\mathbb{R}^n) \). Moreover,

\[
\|Tf\| \leq c_n \|f\|_\infty, \quad \|T^*_f\| \leq c_n \|f\|_\infty.
\]

(ii) If \( \omega \in A_\infty \), then

\[
(Tf)_\omega^*(t) \leq c_{n,\omega} (Mf)_\omega^*(2t) + (Tf)^*_\omega(2t) \quad (t > 0),
\]

10
\[(T^* f)^*(t) \leq c_{n, \omega}(Mf)^*(2t) + (T^* f)^*(2t) \quad (t > 0)\]
for all \(f \in L^p(\mathbb{R}^n)\) \((1 \leq p < \infty)\).

(iii) If \(\omega \in A_\infty\), then
\[
\|T^* f\|_{L^p(\omega)} \leq c_{n,p,\omega}\|Mf\|_{L^p(\omega)} \quad (0 < p < \infty)
\]
for all functions \(f\) with \((T^* f)^*(+\infty) = 0\).

Item (i) goes back to Stein [17]. Item (ii) was obtained in [2]. More precisely, in the paper of Bagby and Kurtz [2], it was proved that for any \(\gamma \in (0, 1)\) there exists a constant \(c(\gamma)\) such that
\[
(T^* f)^*(t) \leq c(\gamma)(Mf)^*(\gamma t) + (T^* f)^*(2t) \quad (t > 0),
\]
and hence, their method does not allow to take \((Mf)^*(2t)\). Inequality (4.10) was proved in [6].

Cordoba and Fefferman proved in [8] the inequality
\[
(Tf)^\#(x) \leq c_{p}M_p f(x) \quad (1 < p < \infty)
\]
where \(M_p f(x) = (M|f|^p)^{1/p}(x)\). We shall show now that this inequality can be improved. To this end, we use the following estimate obtained in Jawerth and Torchinsky [11]: For all \(f \in L^1_{\omega}(\mathbb{R}^n)\) and all \(x \in \mathbb{R}^n\),
\[
(c_1MM^\# f(x) \leq f^\#(x) \leq c_2MM^\# f(x)
\]
where \(0 < \lambda < \lambda_0(n)\), \(c_1\) and \(c_2\) depends on \(n\) and \(\lambda\). The relations (4.8) and (4.12) imply the inequality:
\[
(Tf)^\#(x) \leq c_{M}M f(x).
\]

This inequality is more precise than (4.11) since, as shown in [7],
\[
MM_p f(x) \leq c_M p f(x) \quad (p > 1)
\]
and thus
\[
MM f(x) \leq MM_p f(x) \leq c_M p f(x)
\]
for \(p > 1\).

Let us note some consequences from inequality (3.1). Denote by \(BMO(\omega)\) the space of all functions \(f\) such that
\[
\|f\|_{\omega, \omega} \equiv \sup_{Q \subset \mathbb{R}^n} \frac{1}{\omega(Q)} \int_Q |f(y) - f_{Q, \omega}| \omega(y) dy < \infty
\]
Let us start with the John–Nirenberg theorem [13] in the weight case. The claim that this theorem can be extended to the case of weights satisfying the doubling condition was mentioned already in [15].

**Theorem 4.3** [John–Nirenberg]. Let $\omega \in D$. Then, for every function $f \in BMO(\omega)$ and every cube $Q$ holds the inequality:

$$((f - f_Q,\omega)^{\ast}_{\omega}(t) \leq c \|f\|_{*\omega} \log^+ \frac{2\omega(Q)}{t} \quad (0 < t < \infty),$$

or, equivalently,

$$\omega\{x \in Q : |f(x) - f_Q,\omega| > \alpha\} \leq 2\omega(Q) \exp\left(-\frac{\alpha}{c \|f\|_{*\omega}}\right) \quad (0 < \alpha < \infty)$$

where $c$ depends only on $\omega$.

**Proof.** It follows from inequality (3.1) that

$$(f - f_Q,\omega)^{\ast}_{\omega}(t) \leq 10c\omega \|f\|_{*\omega} + ((f - f_Q,\omega)^{\ast}_{\omega}(2t)$$

for $t < \omega(Q)/5c_{\omega}$. But

$$((f - f_Q,\omega)^{\ast}_{\omega}(t) \leq \frac{5c\omega}{\omega(Q)} \int_Q |f(x) - f_Q,\omega| \omega(x) \, dx \leq 5c\omega \|f\|_{*\omega}$$

provided $t \geq \omega(Q)/5c_{\omega}$. Thus inequality (4.13) is valid for all $t > 0$.

Suppose now that $\omega(Q)/2^{k+1} < t \leq \omega(Q)/2^k$. Applying (4.13) $k$ times we get

$$((f - f_Q,\omega)^{\ast}_{\omega}(t) \leq 10c\omega \|f\|_{*\omega}(k + 1) \leq \frac{10c\omega}{\log 2} \|f\|_{*\omega} \log^+ \frac{2\omega(Q)}{t}.$$  

The theorem is proved.

**Remark 4.1.** Just in the same way the weighted analogy of John and Strömberg theorem follows from (3.1) : For any cube $Q$ there exists a constant $a_{Q,\omega}$ such that

$$((f - a_{Q,\omega})^\ast_{\omega}(t) \leq c \|M_{\lambda,\omega}^f\|_{\infty} \log^+ \frac{2\omega(Q)}{t} \quad (0 < t < \infty, \; 0 < \lambda \leq 1/5c_{\omega}).$$

Hence

$$\frac{1}{\omega(Q)} \int_Q |f(y) - f_{Q,\omega}| \omega(y) \, dy \leq 2\frac{1}{\omega(Q)} \int_Q |f(y) - a_{Q,\omega}| \omega(y) \, dy$$

12
\[
\leq \frac{c\|M_{\lambda,\omega}^# f\|_{\infty}}{\omega(Q)} \int_0^{\omega(Q)} \log^+ \frac{2\omega(Q)}{t} dt \leq c\|M_{\lambda,\omega}^# f\|_{\infty}.
\]

Thus
\[
(4.14) \quad \|f\|_{*,\omega} \asymp \|M_{\lambda,\omega}^# f\|_{\infty}.
\]

Since for \( \lambda \leq 1/2 \) we have \( \|f\| \asymp \|M_{\lambda} f\|_{\infty} \), the inequality (4.14) and Lemma 2.3 imply the result of Muckenhoupt and Wheeden [15], namely: If \( \omega \in A_\infty \), then \( BMO(\omega) = BMO \).

We shall show next that (3.1) implies the inequality of Bennett, DeVore and Sharpley (1.2). Indeed, integrating inequality (3.1) we get
\[
(f \cdot \chi_Q)_{\omega}^{**}(t) - (f \cdot \chi_Q)_{\omega}^{*}(t) \leq 2((f \cdot \chi_Q)_{\omega}^{*}(t) - (f \cdot \chi_Q)_{\omega}^{**}(2t)) \leq 4(M_{\lambda,\omega}^# f)_{\omega}^{**}(2t) \quad (0 < t < \omega(Q)/5c_\omega, 0 < \lambda \leq 1/5c_\omega).
\]

Let \( \omega \) be the Lebesgue measure. Then, according to Hardy – Littlewood – Hertz inequality (see [4]), we have
\[
c_1 f^{**}(t) \leq (Mf)^*(t) \leq c_2 f^{**}(t) \quad (0 < t < \infty).
\]

The left-hand side of this inequality, (4.12) and (4.15) yield (1.2).

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**References**


