AN ELEMENTARY APPROACH TO SEVERAL RESULTS
ON THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

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Abstract. We give new elementary proofs of theorems due to B. Muckenhoupt, B. Jawerth, and S. Buckley. By means of our approach we answer a question raised by J. Orobitg and C. Pérez.

1. Introduction

Let $w$ be a weight, i.e., $w \geq 0$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. Given a measurable set $E$, let $w(E) = \int_E w(x)dx$. The Hardy-Littlewood maximal operator with respect to $w$ is defined by

$$M_w f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)|w(y)dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$. If the supremum is taken only over cubes $Q$ centered at $x$, denote the corresponding operator by $M^c_w$.

We drop the subscript $w$ if $w \equiv 1$. Given a weight $w$ and $p > 1$, set $\sigma = w^{-1/(p-1)}$. We say that $w$ satisfies the $A_p$ condition if

$$\|w\|_{A_p} = \sup_Q \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p} < \infty.$$

In [8], B. Muckenhoupt proved the following fundamental result.

Theorem A. The Hardy-Littlewood maximal operator $M$ is bounded on $L^p_w$, $1 < p < \infty$, if and only if $w \in A_p$.

The original proof of this theorem was based on the deep property of $A_p$ weights saying that $A_p$ implies $A_{p-\varepsilon}$ for some $\varepsilon > 0$. Then R. Coifman and C. Fefferman [3] gave a simplified proof but still depended on this property. Later, E. Sawyer [10] solved the two weight problem for $M$. In [6], R. Hunt, D. Kurtz and C. Neugebauer established that in the case of equal weights Sawyer’s condition is equivalent to the $A_p$ condition, providing a new proof of Muckenhoupt’s theorem that completely avoids the implication $A_p \Rightarrow A_{p-\varepsilon}$. After that, a very simple proof was given by M. Christ and R. Fefferman [2]. However, the proof in [2] is based essentially on
the Calderón-Zygmund decomposition which makes it applicable only to classical
maximal operators.

In [7], B. Jawerth obtained an even more elementary proof with the advantage
that it can be applied to maximal operators with respect to a general basis $B$. By a
basis we mean a collection of open sets in $\mathbb{R}^n$. Assume that the supremum is taken
in (1) and (2) over $B \in B$ instead of cubes. Denote the corresponding objects by
$M_{w,B}$.

**Theorem B.** Let $1 < p < \infty$. Then $M_B$ is bounded on $L^p_w$ and on $L^p_{\sigma}$ if and
only if $w \in A_{p,B}$, $M_{\sigma,B}$ is bounded on $L^p_{\sigma}$, and $M_w$ is bounded on $L^p_w$, where
$1/p + 1/p' = 1$.

Observe that $w \in A_{p,B}$ if and only if $\sigma \in A_{p',B}$. Next, in the case where $B$
consists of cubes, the $A_p$ condition implies the doubling condition (there exists
c $> 0$ such that $w(2Q) \leq cw(Q)$ for any cube $Q$), and hence the boundedness
of $M_w$ on $L^p_w$, and $M_{w}$ on $L^p_w$ follows from $A_p$ in the standard way by the usual
covering argument. Thus, Theorem B contains Theorem A as a particular case.

In [1], S. Buckley found the sharp dependence of $\|M\|_{L^p_w}$ on $\|w\|_{A_p}$ in Mucken-
houpt’s theorem.

**Theorem C.** Let $1 < p < \infty$. Then $\|M\|_{L^p_w} \leq c_{p,n}\|w\|_{A_p}^{1/(p-1)}$, and the exponent
$1/(p-1)$ is best possible.

The original proof of this result is based on the property $A_p \Rightarrow A_{p-}$ and on
interpolation. It was mentioned in [1] that some proofs of the boundedness of $M$,
for instance, Jawerth’s proof [7], do not yield the sharp exponent $1/(p-1)$.

In this short note we give an extremely simple argument leading to the proofs of
both Theorems B and C. This yields a new proof of Theorem A as well. Also we
consider similar questions for $A_p$ weights with respect to non-doubling measures.

Let $\mu$ be a non-negative Radon measure in $\mathbb{R}^n$ with the property that $\mu(\partial Q) = 0$
for any cube $Q$. Replacing in (1) $w$ by $\mu$ (and $w(y)dy$ by $d\mu(y)$), we get the maximal
operator $M_\mu$. As usual, by $M_\mu$ we denote the centered maximal operator. Given
a measurable set $E$ and a weight $w$, let $w_\mu(E) = \int_E w(x)d\mu(x)$. We say that $w$
satisfies the $A^r_p(\mu)$ ($0 < r \leq 1, p > 1$) condition if

$$\|w\|_{A^r_p(\mu)} \equiv \sup_Q A^r_p(Q) = \frac{w_\mu(rQ)}{\mu(rQ)} \left( \frac{\sigma_\mu(Q)}{\mu(Q)} \right)^{p-1} < \infty.$$  

If $r = 1$ we simply write $A_p(\mu)$. Denote by $L^p_w(\mu)$ the space of all measurable $f$
such that

$$\|f\|_{L^p_w(\mu)} \equiv \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)d\mu(x) \right)^{1/p} < \infty.$$  

In [9], J. Orobitg and C. Pérez proved the following variant of Muckenhoupt’s theorem.

**Theorem D.** If $w \in A_p(\mu), p > 1$, then $M_\mu^c$ is bounded on $L^p_w(\mu)$.

Using the same idea as was used in proving Theorems B and C, we get that $M_\mu^c$
is also bounded on $L^p_w(\mu)$ for $A^r_p(\mu)$ weights, $r < 1$.

**Theorem D’.** Let $0 < r < 1$. If $w \in A^r_p(\mu), p > 1$, then $M_\mu^c$ is bounded on $L^p_w(\mu)$.
It was asked in [9] whether the $A_p(\mu)$ condition is necessary for the $L^p_\mu(\mu)$-boundedness of $M^*_\mu$ as in the classical case. We show (see Example 1 in Section 2 below) that $A_p^{(r)}(\mu) \nsubseteq A_p(\mu), r < 1$. Therefore, Theorem D' provides a negative answer to this question.

In the next section we prove Theorems B, C and D'. To be more precise, we prove only the difficult parts of Theorems B and C, that is, the sufficiency part in Theorem B and the estimate for $\|M\|_{L^p_w}$ in Theorem C.

2. Proofs

**Proof of Theorem B.** By the symmetry between $w$ and $\sigma$, it suffices to prove that $Mg$ is bounded on $L^p_w$. Setting $A_p(B) = w(B)\sigma(B)^{p-1}/|B|^p$, we have

$$\frac{1}{|B|} \int_B |f| = A_p(B)^{\frac{1}{p-1}} \left\{ \frac{|B|}{w(B)} \left( \frac{1}{\sigma(B)} \int_B |f| \right)^{p-1} \right\}^{\frac{1}{p-1}} \leq \|w\|_{A_p,B}^{\frac{1}{p-1}} \left\{ \frac{1}{w(B)} \int_B M_{\sigma,B}(f(\sigma^{-1})^{p-1}w^{-1}) \, dx \right\}^{\frac{1}{p-1}},$$

and hence,

$$M_B f(x) \leq \|w\|_{A_p,B}^{\frac{1}{p-1}} M_{w,B}(M_{\sigma,B}(f(\sigma^{-1})^{p-1}w^{-1})) \left( x \right)^{\frac{1}{p-1}}.$$

Therefore,

$$\|M_B f\|_{L^p_w} \leq \|w\|_{A_p,B}^{\frac{1}{p-1}} \|M_{w,B}(M_{\sigma,B}(f(\sigma^{-1})^{p-1}w^{-1}))\|_{L^p_w} \leq \|w\|_{A_p,B}^{\frac{1}{p-1}} \|M_{w,B}\|_{L^p_w}^{\frac{1}{p-1}} \|M_{\sigma,B}\|_{L^p_w} \|f\|_{L^p_w}.$$

This completes the proof. \(\square\)

Observe that although the original proof of this theorem [7] (the same proof can be found in [9] p. 423) was also simple, it involved several additional ingredients such as a selection process for sets $B^k$ corresponding to the level set $\{2^k < M_B f \leq 2^{k+1}\}$ with the subsequent introduction of an auxiliary discrete measure.

**Proof of Theorem C.** Suppose that $B$ is formed by cubes. Note first that in the one-dimensional case $\|M_w\|_{L^p_w}$ and $\|M_{\sigma}\|_{L^p_w}$ are bounded uniformly in $w$ (see [11]). Hence, in this case (3) immediately yields the desired estimate for $\|M\|_{L^p_w}$.

In the multi-dimensional case we have only to slightly modify the above argument. Setting $A_p(Q) = w(Q)\sigma(3Q)^{p-1}/|Q|^p$, we have

$$\frac{1}{|Q|} \int_Q |f| = A_p(Q)^{\frac{1}{p-1}} \left\{ \frac{|Q|}{w(Q)} \left( \frac{1}{\sigma(3Q)} \int_Q |f| \right)^{p-1} \right\}^{\frac{1}{p-1}} \leq 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} \left\{ \frac{1}{w(Q)} \int_Q M_{\sigma}^*(f(\sigma^{-1})^{p-1}w^{-1}) \, dx \right\}^{\frac{1}{p-1}}.$$

From this and from the fact that $Mf(x) \leq 2^{n}M^* f(x)$ we get

$$Mf(x) \leq 2^{n}3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} M_{w}^*(M_{\sigma}^*(f(\sigma^{-1})^{p-1}w^{-1})) \left( x \right)^{\frac{1}{p-1}}.$$
Using this inequality and the well-known fact (based on the Besicovitch covering theorem) that \( \|M_w^c\|_{L^p_w} \) and \( \|M_{\sigma}^c\|_{L^p_{\sigma}} \) are bounded uniformly in \( w \), exactly as above, we get Buckley’s theorem. \( \square \)

**Proof of Theorem D’**. Define the maximal operator \( M^{(r)}_{\mu} \) by

\[
M^{(r)}_{\mu}f(x) = \sup_{rQ \ni x} \frac{1}{\mu(rQ)} \int_Q |f(y)|d\mu(y),
\]

where the supremum is taken over all cubes \( Q \) such that \( x \in rQ \). By the Besicovitch covering theorem (see [5, pp. 6-7]) we have that the operator \( M^{(r)}_{\mu} \) is bounded on \( L^p_\mu \) for all \( p > 1 \), as the usual centered maximal operator (here we essentially use that \( r < 1 \)). We have

\[
\frac{1}{\mu(Q)} \int_Q |f|d\mu = A^{(r)}_\mu(Q)^{\frac{1}{p-1}} \left\{ \frac{\mu(rQ)}{w_\mu(rQ)} \left( \frac{1}{\sigma_\mu(Q)} \int_Q |f|d\mu \right)^{p-1} \right\}^{\frac{1}{p-1}} \leq \|w\| A^{(r)}_\mu(\mu) \left\{ \frac{1}{w_\mu(rQ)} \int_{rQ} M^{(r)}_{\sigma(\mu)}(f\sigma^{-1})^{p-1}d\mu \right\}^{\frac{1}{p-1}},
\]

and hence,

\[
M^{c}_{\mu}f(x) \leq \|w\| A^{(r)}_\mu(\mu) M^{c}_{w(\mu)}(M^{(r)}_{\sigma(\mu)}(f\sigma^{-1})^{p-1}w^{-1})(x)^{\frac{1}{p-1}}.
\]

From this, using the fact that \( M^{c}_{w(\mu)} \) is bounded on \( L^p_w(\mu) \) and \( M^{(r)}_{\sigma(\mu)} \) is bounded on \( L^p_\mu(\mu) \), we get the \( L^p_\mu(\mu) \)-boundedness of \( M^{c}_{\mu} \).

**Remark.** It is well-known (see [12] and [13]) that for \( n \geq 2 \) the maximal operator \( M_\mu \) does not bounded on \( L^p_\mu \), in general. Therefore, the above proof does not work when \( n \geq 2 \) and \( r = 1 \). Note that the proof of Theorem D [9] was based on the property \( A_p(\mu) \Rightarrow A_{p-1}(\mu) \). We do not know how to avoid this property in the multi-dimensional case. On the other hand, in the case \( n = 1 \) and \( r = 1 \) a full analogue of inequality [4] for non-centered maximal functions holds which shows that the \( A_p(\mu) \) condition is sufficient (and trivially necessary) for the boundedness of \( M_\mu \) on \( L^p_w(\mu) \).

In order to show that \( A^{(r)}_p(\mu) \not\subset A_p(\mu) \), \( r < 1 \), we give the following example.

**Example 1.** Let \( n = 1 \). Set \( d\mu = e^{x}|dx| \) and \( w(x) = e^{(p-1)|x|} \). Let us show that \( w \in A^{(r)}_p(\mu) \) for any \( r < 1 \) but \( w \not\in A_p(\mu) \). Indeed, it suffices to estimate \( A^{(r)}_p(I) \) for any \( I \subset (0, \infty) \). Let \( I = (a, a + h) \). A straightforward calculation shows that

\[
A^{(r)}_p(I) = \frac{h^{p-1} e^{(p-1)(r-1)h^2} (e^{ph} - 1)}{p (e^{ph} - 1)(e^h - 1)^{p-1}}.
\]

Hence, sup\( I A_p(I) = \infty \), while sup\( I A^{(r)}_p(I) < \infty \) for any \( r < 1 \).

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References


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