ON THE SHARP UPPER BOUND RELATED TO THE WEAK MUCKENHOUPT-WHEEDEN CONJECTURE

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Abstract. We construct an example showing that the upper bound $|w|_{A_1} \log(e + |w|_{A_1})$ for the $L^1(w) \to L^{1,\infty}(w)$ norm of the Hilbert transform cannot be improved in general.

1. Introduction

Define the Hardy-Littlewood maximal operator on $\mathbb{R}$ by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|dy,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ containing the point $x$.

In [3], C. Fefferman and E. Stein established the following weighted weak type inequality for $M$: there exists an absolute constant $C > 0$ such that for every weight $w$,

$$\sup_{\alpha > 0} \alpha w\{x \in \mathbb{R} : Mf(x) > \alpha\} \leq C \int_{\mathbb{R}} |f(x)|Mw(x)dx$$

(1.1)

(here by a weight we mean any non-negative locally integrable function on $\mathbb{R}$, and we use the standard notation $w(E) = \int_E w$ for a measurable set $E \subset \mathbb{R}$).

Inequality (1.1) is important for several reasons. First, it is the key ingredient in extending the Hardy-Littlewood maximal theorem to a vector-valued case. Second, this result was a precursor of the weighted theory, which had started to develop rapidly from the beginning of the 70’s. In particular, if we define the $|w|_{A_1}$ constant of the weight $w$ by $|w|_{A_1} = \|Mw/w\|_{L^\infty}$, then, assuming $|w|_{A_1} < \infty$, (1.1) implies
immediately that
\begin{equation}
\|Mf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \|f\|_{L^1(w)}.
\end{equation}

Consider now the Hilbert transform,
\[ Hf(x) = \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy. \]

The inequality (1.1) with the maximal operator on the left-hand side replaced by the Hilbert transform has become known as the Muckenhoupt-Wheeden conjecture. Only recently this conjecture has been disproved by M. Reguera and C. Thiele [10] (see also [1, 2, 9] for some comple-
ments and extensions). Their result, however, left open the question whether a weaker form of the Muckenhoupt-Wheeden conjecture holds, with \( M \) replaced by \( H \) on the left-hand side of (1.2).

In [5], it was proved that
\begin{equation}
\|Hf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \log(e + [w]_{A_1}) \|f\|_{L^1(w)}.
\end{equation}

This improved a previous result in [4], where the right-hand side contained an additional factor double logarithmic in \([w]_{A_1}\). Notice also that actually (1.3) in [5] was proved for every Calderón-Zygmund operator on \( \mathbb{R}^n \) with sufficiently smooth kernel.

On the other hand, in [7], it was shown for the martingale transform (and explained how to transfer the result to the Hilbert transform case) that the dependence of \([w]_{A_1}\) in the weighted weak type \((1,1)\) inequality cannot in general be made better than \([w]_{A_1} \log^{1/5}(e + [w]_{A_1})\), thus disproving the weak Muckenhoupt-Wheeden conjecture. Later, in [8], the power of the logarithm was improved to \(1/3\) (this was again done for the martingale transform).

Summarizing the results in [5, 7, 8], if we denote by \( \alpha_H \) the best possible exponent for which the inequality
\[ \|Hf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \log^{\alpha_H}(e + [w]_{A_1}) \|f\|_{L^1(w)} \]
holds, then we have that \( \frac{1}{3} \leq \alpha_H \leq 1 \).

The main result of this paper shows, in particular, that \( \alpha_H = 1 \). For \( t \geq 1 \), define
\[ \varphi_H(t) = \sup_{[w]_{A_1} \leq t} \|H\|_{L^1(w) \to L^{1,\infty}(w)}. \]

Then (1.3) implies \( \varphi_H(t) \leq C t \log(e + t) \). We will show that actually \( \varphi_H(t) \approx t \log(e + t) \). Our main result is the following.

**Theorem 1.1.** There exists \( c' > 0 \) such that for all \( t \geq 1 \),
\[ \varphi_H(t) \geq c' t \log(e + t). \]
As a trivial corollary we obtain that the inequality
\[ \|Hf\|_{L^1(w)} \leq \psi([w]_{A_1})\|f\|_{L^1(w)} \]
fails in general for every increasing on \([1, \infty)\) function \(\psi\) satisfying \(\lim_{t \to \infty} \frac{\psi(t)}{t \log(e + t)} = 0\).

2. Proof of Theorem 1.1

2.1. An overview of the proof. At the first step we show that the definition of \(\varphi_H\) along with the standard extrapolation and dualization arguments yields
\[ \|Hf\|_{L^1(w)} \leq 4\varphi_H(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})(\int_0^1 w)\frac{1}{2}, \]
where \(\sigma = w^{-1}\). Notice that \(\|M\|_{L^2(\sigma) \to L^2(\sigma)} < \infty\) if and only if \(w \in A_2\), that is, if \(\sup_{I \in \mathbb{R}} \frac{w(I)\sigma(I)}{|I|^2} < \infty\). Therefore, we assume here that \(w \in A_2\).

The key step is to show that there exist \(C_1, C_2, C_3 > 0\) such that for every \(t > C_3\), there is an \(A_2\) weight \(w\) satisfying
\[ \int_0^1 w = 1, \quad \|M\|_{L^2(\sigma) \to L^2(\sigma)} \leq C_1 t, \quad \|H(w\chi_{[0,1]})\|_{L^2(\sigma)} \geq C_2 t \log t. \]
Plugging these estimates into (2.1), we finish the proof.

2.2. Extrapolation and dualization. First, we apply the standard Rubio de Francia extrapolation trick. Given \(g \geq 0\) with \(\|g\|_{L^2(\sigma)} = 1\), define
\[ \mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})^k}. \]
Then \(g \leq \mathcal{R}g\), \(\|\mathcal{R}g\|_{L^2(\sigma)} \leq 2\), and \(\|\mathcal{R}g\|_{A_1} \leq 2\|M\|_{L^2(\sigma) \to L^2(\sigma)}\). These estimates along with the definition of \(\varphi_H\) and Hölder’s inequality imply
\[ \alpha \int_{\{x: |Hf(x)| > \alpha\}} g \leq \alpha \int_{\{x: |Hf(x)| > \alpha\}} \mathcal{R}g \leq \varphi_H(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})\|f\|_{L^1(\mathcal{R}g)} \]
\[ \leq \varphi_H(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})\|f\|_{L^2(w)} \|\mathcal{R}g\|_{L^2(w)} \]
\[ \leq 2\varphi_H(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})\|f\|_{L^2(w)}. \]
Taking here the supremum over all \(g \geq 0\) with \(\|g\|_{L^2(\sigma)} = 1\) yields
\[ \|Hf\|_{L^2(\sigma)} \leq 2\varphi_H(2\|M\|_{L^2(\sigma) \to L^2(\sigma)})\|f\|_{L^2(w)}. \]

We now use the following elementary estimate:
\[ \int_0^1 |Hf|w \leq 2\|Hf\|_{L^2(\sigma)}(\int_0^1 w)^{1/2}, \]
which along with (2.3) implies
\[ \left| \int_\mathbb{R} (H(w\chi_{[0,1]}))f \right| = \left| \int_0^1 (Hf)w \right| \leq 4\phi_H(2\|M\|_{L^2(\omega)\rightarrow L^2(\omega)}) \left( \int_0^1 w \right)^{1/2} \|f\|_{L^2(w)}. \]

Taking here the supremum over all \( f \) with \( \|f\|_{L^2(w)} = 1 \) proves (2.1).

To show (2.4), notice that for every \( \lambda > 0 \),
\[ \int_0^1 |Hf|w \leq \int_\lambda^\infty w\{x : |Hf(x)| > \alpha\}d\alpha + \lambda \int_0^1 w \leq \frac{1}{\lambda}\|Hf\|_{L^2,\infty(w)}^2 + \lambda \int_0^1 w. \]

Optimizing this estimate with respect to \( \lambda \), we obtain (2.4).

2.3. Construction of the weight. Fix \( t \gg 1 \). Take \( k \in \mathbb{N} \) such that \( t \leq 3^k \leq 3t \). Let \( \varepsilon = 3^{-k} \) and \( p = \frac{1}{6\varepsilon} \left( \frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon} \right) \). The reason for this definition of \( p \) will be clarified a bit later. Note that we will frequently use the obvious estimate \( \frac{1}{6\varepsilon} \leq p \leq \frac{2}{\varepsilon} \).

For every two positive numbers \( \omega \) and \( \sigma \) such that \( \omega\sigma = p \) and any interval \( I \subset \mathbb{R} \), we define inductively the sequence of weights \( w_\nu = w_\nu(\omega,\sigma,I) (\nu = 0,1,2,\ldots) \) supported on \( I \) as follows.

Let \( u = \sqrt{p} + \sqrt{p-1} \) be the larger root of \( u^2 - 2u = 2\sqrt{p} \). Define
\[ w_0(\omega,\sigma,I) = \frac{\omega}{\sqrt{p}} \left( u\chi_{I_0} + \frac{1}{u}\chi_{I_+} \right), \]
where \( I_- \) and \( I_+ \) are the left and the right halves of \( I \) respectively.

Suppose that \( w_{\nu-1}(\omega,\sigma,I) \) is already defined for all \( \omega,\sigma \) with \( \omega\sigma = p \) and all \( I \subset \mathbb{R} \). To construct \( w_\nu(\omega,\sigma,I) \), first denote by \( I_m, m = 0,1,\ldots,k-1 \) the interval with the same right endpoint as \( I \) of length \( 3^{-m}|I| \), so
\[ I_{k-1} \subset I_{k-2} \subset \ldots \subset I_0 = I \]
and \( |I_{k-1}| = 3\varepsilon|I| \).

Given an interval \( J \), denote by \( J^{(i)}, i = 1,2,3 \), the \( i \)-th from the left subinterval of \( J \) in the partition of \( J \) into 3 equal intervals.

Define \( w_\nu(\omega,\sigma,I) \) by
\[ w_\nu(\omega,\sigma,I) = \frac{\omega}{p} \left( \sum_{m=0}^{k-2} \chi_{I_m^{(1)}} + \chi_{I_{k-1}^{(1)}} + \chi_{I_{k-1}^{(2)}} + \frac{4\varepsilon}{1+\varepsilon} \chi_{I_{k-1}^{(3)}} \right) \]
\[ + \sum_{m=0}^{k-2} w_{\nu-1} \left( \frac{2\omega}{\sigma}, \frac{\sigma}{2}, I_m^{(2)} \right). \]
Figure 1. $w_\nu(\omega, \sigma, I)$ on intervals $I^{(i)}_m$ for $i = 1, 2$ and $0 \leq m \leq k - 2$ and on $I^{(i)}_{k-1}$ for $i = 1, 2, 3$.

Note that the interval $I^{(3)}_{k-1}$ plays a rather special role in the final step of this recursive construction. We shall call any interval of this type arising at any step in the construction of the weight $w_\nu(\omega, \sigma, I)$ a “tail interval”, so within $I$ we shall have one tail interval $I^{(3)}_{k-1}$ arising at the final stage of the construction, $k - 1$ tail intervals $(I^{(2)}_m)^{(3)}_{k-1}$ arising in the construction of the weights $w_{\nu-1}(2\omega, \sigma/2, I^{(2)}_m)$, and so on.

Finally, we define $w$ as the 1-periodization of $w_n(1, p, [0, 1))$ with $n = 3^{k-1}$. For $l = 0, 1, \ldots, n$, we shall say that an interval $I$ “carries $w_{n-l}$” if $w = w_{n-l}(2^l, 2^{-l}p, I)$ on $I$. Denote by supp $w_{n-l}$ the union of all intervals carrying $w_{n-l}$. For example, supp $w_n = \bigcup_{k \in \mathbb{Z}} [k, k+1) = \mathbb{R}$ as $[k, k+1)$ carries $w_n$ for every $k \in \mathbb{Z}$.

Let us now establish several useful properties of $w_\nu(\omega, \sigma, I)$.

**Proposition 2.1.** For every $\nu \geq 0$,

\[(2.6) \quad \frac{1}{|I|} \int_I w_\nu(\omega, \sigma, I) dx = \omega \quad \text{and} \quad \frac{1}{|I|} \int_I w^{-1}_\nu(\omega, \sigma, I) dx = \sigma.\]

**Proof.** The proof is by induction on $\nu$. For $\nu = 0$,

\[\frac{1}{|I|} \int_I w_0(\omega, \sigma, I) dx = \frac{\omega}{\sqrt{p}} \frac{1}{2} (u + 1/u) = \omega,\]

and

\[\frac{1}{|I|} \int_I w^{-1}_0(\omega, \sigma, I) dx = \frac{\sqrt{p}}{\omega} \frac{1}{2} (1/u + u) = \frac{p}{\omega} = \sigma.\]

Assume that the statement holds for $\nu - 1$ and let us prove it for $\nu$. Observe that $w_\nu(\omega, \sigma, I)$ equals $\frac{\nu}{p}$ on a subset of $I$ of total measure

\[\frac{1 - 3\varepsilon}{2} |I| + 2\varepsilon |I| = \frac{1 + \varepsilon}{2} |I|,\]
it equals \( \frac{4\varepsilon}{1+\varepsilon} \) on a set of measure \( \varepsilon |I| \), and the average of \( w_{\nu-1}(2\omega, \cdot, \cdot) \) over the remaining set of measure \( \frac{1-3\varepsilon}{2} |I| \) is \( 2\omega \) by the inductive assumption. Thus

\[
\frac{1}{|I|} \int_I w_\nu(\omega, \sigma, I) \, dx = \frac{\omega}{p} \left( \frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon} \right) + \omega(1-3\varepsilon) = \omega + \left( \frac{1}{p} \left( \frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon} \right) - 3\varepsilon \right) \omega = \omega
\]

(it is this equation that was used to determine \( p \)).

On the other hand, \( w_{\nu-1}(\omega, \sigma, I) \) equals \( \frac{\omega}{\nu} = \sigma \) on a subset of \( I \) of measure \( \varepsilon |I| \) (the same set on which \( w_\nu \) is defined as \( \omega/p \)), it equals \( 1 + \frac{\varepsilon}{4} \sigma \) on a set of measure \( \varepsilon |I| \), and its average over the remaining set of measure \( 1-3\varepsilon |I| \) equals \( 2\sigma \). Thus

\[
\frac{1}{|I|} \int_I w_{\nu-1}(\omega, \sigma, I) \, dx = \frac{1+\varepsilon}{2} + \frac{1+\varepsilon}{4} \sigma + \frac{(1-3\varepsilon)\sigma}{2} = \sigma,
\]

which completes the proof. \( \square \)

In particular, it follows from Proposition 2.1 that for the constructed weight \( w \),

\[
\int_0^1 w \, dx = \int_0^1 w_n(1, p, [0, 1)) \, dx = 1.
\]

**Proposition 2.2.** Let \( I = [a, a+h) \). Then, for every \( \nu \geq 0 \) and for all \( 0 < \tau < h \),

\[
\left(2.7\right) \quad \frac{1}{\tau} \int_a^{a+\tau} w_\nu(\omega, \sigma, I) \, dx \leq 3\omega \quad \text{and} \quad \frac{1}{\tau} \int_{a+\tau}^{a+h} w_\nu(\omega, \sigma, I) \, dx \leq \frac{9}{2}\omega.
\]

**Proof.** For \( \nu = 0 \) the statement is obvious since \( w_0(\omega, \sigma, I) \leq 2\omega \) on \( I \).

Assume that \( \nu \geq 1 \).

Since \( w_\nu(\omega, \sigma, I) = \frac{\omega}{\nu} \) on \( I^{(1)} \), we have that \( \frac{1}{\tau} \int_a^{a+\tau} w_\nu(\omega, \sigma, I) = \frac{\omega}{\nu} \) for \( 0 < \tau < h/3 \). But if \( \tau \geq h/3 \), then, by Proposition 2.1,

\[
\frac{1}{\tau} \int_a^{a+\tau} w_\nu(\omega, \sigma, I) \, dx \leq \frac{3}{|I|} \int_I w_\nu(\omega, \sigma, I) \, dx = 3\omega.
\]

We now turn to the proof of the second estimate in (2.7). Let \( I_m, m = 0, 1, \ldots, k-1 \), be the intervals appearing in the definition of \( w_\nu(\omega, \sigma, I) \).

Since \( w_\nu(\omega, \sigma, I) \leq \frac{\omega}{\tau} \) on \( I_{k-1} \), the estimate is trivial if \( a+h-\tau \in I_{k-1} \).

Assume that \( a+h-\tau \in I_m \setminus I_{m+1}, m = 0, \ldots, k-2 \). Then

\[
\left(2.8\right) \quad \frac{1}{\tau} \int_{a+h-\tau}^{a+h} w_\nu(\omega, \sigma, I) \, dx \leq \frac{1}{|I_{m+1}|} \int_{I_m} w_\nu(\omega, \sigma, I) \, dx.
\]
Next, by Proposition 2.1,
\[
\int_{I_m} w_\nu(\omega, \sigma, I) \, dx = \sum_{j=m}^{k-2} \left( \frac{\omega}{p} |I_j^{(1)}| + \int_{I_j^{(2)}} w_{\nu-1}(2\omega, \sigma/2, I_j^{(2)}) \, dx \right) + \int_{I_{k-1}} w_\nu(\omega, \sigma, I) \, dx
\]
\[
\leq \omega \sum_{j=m}^{k-1} |I_j| \leq \frac{3\omega}{2} \frac{|I|}{3^m} = \frac{9\omega}{2} |I_{m+1}|,
\]
which along with (2.8) completes the proof. \( \square \)

Assume that \( I \) carries \( w_{n-l} \). Consider the corresponding tail intervals contained in \( I \) (that is, the intervals on which \( w = \frac{4\varepsilon}{1+\varepsilon} \frac{2^j}{p} \), \( j = l, \ldots, n-1 \).)

These intervals will play the central role in the estimate of the Hilbert transform of \( w\chi_{[0,1)} \). There is only one tail interval in \( I \setminus \text{supp } w_{n-(l+1)} \), and its measure equals \( \frac{1}{3^k} |I| \). Next, there are \( k-1 \) tail intervals in
\[
I \cap \left( \text{supp } w_{n-(l+1)} \setminus \text{supp } w_{n-(l+2)} \right)
\]
of total measure \( \frac{1}{2} \left(1 - \frac{1}{3^{k-1}}\right) \frac{1}{3^k} |I| \). Similarly, the measure of the union of tail intervals in
\[
I \cap \left( \text{supp } w_{n-(l+j)} \setminus \text{supp } w_{n-(l+j+1)} \right) \quad (j = 0, \ldots, n-l-1)
\]
is \( \left(\frac{1}{2} \left(1 - \frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k} |I| \).

In particular, if we denote by \( A_l \) the union of tail intervals in
\[
[0,1) \cap \left( \text{supp } w_{n-l} \setminus \text{supp } w_{n-(l+1)} \right),
\]
then
\[
(2.10) \quad |A_l| = \left(\frac{1}{2} \left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k} \quad (l = 0, \ldots, n-1).
\]
2.4. Estimate of the maximal operator. In this section, we will prove the first inequality in (2.2). We start with the reduction of this estimate to its triadic version.

Let $\mathcal{T}$ be the standard triadic lattice, that is,

$$\mathcal{T} = \{[3^j n, 3^j (n + 1)), \quad j, n \in \mathbb{Z}\}.$$ 

Denote by $\mathcal{J}$ the family of all unions of two adjacent triadic intervals of equal length.

Our key tool will be the following estimate:

$$(2.11) \quad \|M\|_{L^2(\sigma) \to L^2(\sigma)} \leq 24 \sup_{J \in \mathcal{J}} \left( \frac{1}{w(J)} \int_J (M(w \chi_J))^2 \sigma \right)^{1/2}.$$ 

This estimate is fairly standard and well-known. For reader’s convenience, we supply the proof in the Appendix.

Combining (2.11) with the inequality $p \leq 2^t \leq 6t$, we see that in order to prove the first estimate in (2.2), it suffices to show that there exists $C > 0$ such that for every interval $J \in \mathcal{J}$,

$$(2.12) \quad \int_J (M(w \chi_J))^2 \sigma \leq C p^2 w(J).$$

Define an auxiliary 1-periodic function $\tilde{w}$ by

$$\tilde{w}(x) = \sum_{l=1}^n 2^l \chi_{\text{supp } w_{n-(l-1)} \setminus \text{supp } w_{n-l}}(x) + 2^{n+1} \chi_{\text{supp } w_0}(x).$$

The role of this function is clarified in the following two propositions.

**Proposition 2.3.** For all $x \in \mathbb{R}$,

$$(2.13) \quad M w(x) \leq \frac{9}{2} \tilde{w}(x).$$

**Proof.** First, notice that for $x \in \text{supp } w_0$ the statement is trivial. Indeed, $w \leq \frac{2^{n-1}}{p}$ on the complement of supp $w_0$, and if $I$ carries $w_0$, then on $I$

$$w_0 = \frac{2^n}{\sqrt{p}} \left( (\sqrt{p} + \sqrt{p - 1})\chi_I + \frac{1}{\sqrt{p} + \sqrt{p - 1}} \chi_I - \chi_I - \chi_{I+} \right).$$

Hence,

$$\|w\|_{L^\infty} \leq \frac{2^n (\sqrt{p} + \sqrt{p - 1})}{\sqrt{p}} \leq 2^{n+1},$$

and therefore $\|M w\|_{L^\infty} \leq 2^{n+1}$.

On the other hand, for $x \in \text{supp } w_{n-(l-1)} \setminus \text{supp } w_{n-l}$, the estimate (2.13) follows immediately from the facts that $w \leq \frac{2^{l-1}}{p}$ on the complement of supp $w_{n-l}$ and that, by Proposition 2.2, the average of $w$ over the intersection of any interval $J$ carrying $w_{n-l}$ with an interval not contained in $J$ is at most $\frac{9}{2} \cdot 2^l$. \qed

Recall that, given an interval $I$, we denoted by $I_m$ ($m = 0, \ldots, k - 1$) the interval with the same right endpoint as $I$ of length $|I_m| = \frac{1}{3^m}|I|$. These intervals have already appeared in the definition of $w_\nu(\omega, \sigma, I)$.

**Proposition 2.4.** Assume that $I$ carries $w_{n-l}$. Then

$$\int_{I_m} (\tilde{w})^2 \sigma \leq 30p^2 w(I_m) \quad (l = 0, \ldots, n; \ m = 0, \ldots, k - 2).$$

**Proof.** First, notice that the case when $l = n$ is trivial, since $2^{n+1} \leq 4pw(x)$ on any interval $I$ carrying $w_0$, and hence,

$$\int_J (\tilde{w})^2 \sigma \leq 16p^2 w(J) \quad \text{for every } J \subset \text{supp } w_0.$$

Suppose now that $l \leq n - 1$ and consider first the case $m = 0$. Assume that $I$ carries $w_{n-l}$. For $j = 0, \ldots, n - l - 1$ denote

$$F_j = I \cap \left( \text{supp } w_{n-(l+j)} \setminus \text{supp } w_{n-(l+j+1)} \right)$$

and let $E_j$ be the union of the tail intervals contained in $F_j$. Observe that $w = \frac{2^{l+j}}{p}$ on $F_j \setminus E_j$, and hence, $\tilde{w}(x) = 2pw(x)$ for $x \in F_j \setminus E_j$, which implies

$$\int_{\cup_j (F_j \setminus E_j)} (\tilde{w})^2 \sigma \leq 4p^2 w(I).$$

On the other hand, $w = \frac{4\varepsilon}{1+\varepsilon} \frac{2^{l+j}}{p}$ on $E_j$ and, as we have seen in the previous section,

$$|E_j| = \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^j \frac{1}{3} |I| \leq \frac{1}{2^j} \frac{1}{3^k} |I|.$$  

Combining this with Proposition 2.1 yields

$$\int_{\cup_j E_j} (\tilde{w})^2 \sigma \leq 4 \sum_{j=0}^{n-l-1} 2^{2(l+j)} \frac{(1+\varepsilon)p}{4\varepsilon 2^{l+j}} \frac{1}{2^j} \frac{1}{3^k} |I|$$

$$\leq \frac{2p n}{\varepsilon} \frac{2^l |I|}{3^k} \leq 4p^2 2^l |I| = 4p^2 w(I).$$

Further, by (2.14),

$$\int_{I \cap \text{supp } w_0} (\tilde{w})^2 \sigma \leq 16p^2 w(I \cap \text{supp } w_0) \leq 16p^2 w(I).$$

Combining this estimate with (2.15) and (2.16), we obtain

$$\int_I (\tilde{w})^2 \sigma = \int_{\cup_j (F_j \setminus E_j)} (\tilde{w})^2 \sigma + \int_{\cup_j E_j} (\tilde{w})^2 \sigma$$

$$+ \int_{I \cap \text{supp } w_0} (\tilde{w})^2 \sigma \leq 24p^2 w(I),$$
and this completes the proof in the case $m = 0$.

Assume now that $1 \leq m \leq k - 2$. Notice that $I_m \setminus I_{m+1} = I_m^{(1)} \cup I_m^{(2)}$, where $I_m^{(2)}$ carries $w_{n-(l+1)}$, and $\tilde{w}(x) = 2pw(x)$ on $I_m^{(1)}$. Thus, by (2.17),

$$(2.18) \quad \int_{I_m \setminus I_{m+1}} (\tilde{w})^2 \sigma \leq 4p^2 w(I_m^{(1)}) + 24p^2 w(I_m^{(2)}) \leq 24p^2 w(I_m \setminus I_{m+1}).$$

Further,

$$\int_{I_{k-1}} (\tilde{w})^2 \sigma \leq 4(2l)^2 \frac{(1 + \varepsilon)p}{4e2l} |I_{k-1}| \leq 6p2^l |I|.$$  

On the other hand, by Proposition 2.1,

$$w(I_m) \geq w(I_m^{(2)}) = 2l+1 |I_m^{(2)}| = \frac{2l+1}{3^m} |I|,$$

which, combined with the previous estimate, implies

$$\int_{I_{k-1}} (\tilde{w})^2 \sigma \leq p3^{m+2} w(I_m) \leq \frac{p}{\varepsilon} w(I_m) \leq 6p^2 w(I_m).$$

Therefore, using (2.18), we obtain

$$\int_{I_m} (\tilde{w})^2 \sigma = \sum_{j=m}^{k-2} \int_{I_j \setminus I_{j+1}} (\tilde{w})^2 \sigma + \int_{I_{k-1}} (\tilde{w})^2 \sigma \leq 30p^2 w(I_m),$$

which completes the proof. \qed

We now turn to the proof of (2.12). Let $J \in J$. First consider the simple case when $|J| \geq 1$. In this case, $|J| = k$ for some $k \in \mathbb{N}$. Using that $w$ and $\tilde{w}$ are 1-periodic along with the fact that $\int_0^1 w = 1$, and combining Propositions 2.3 and 2.4, we obtain

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq 25 \int_0^1 (\tilde{w})^2 \sigma \leq 25 \cdot 30p^2.$$  

Suppose that $|J| < 1$. We can represent $J$ as the union of two triadic intervals $J = J_- \cup J_+$, where $J_-, J_+ \in \mathcal{T}$ are the left and the right halves of $J$ respectively. Since $J_-$ is triadic, we must have that $|J_-| \leq \frac{1}{3}$. Also, by the 1-periodicity of $w$, one can assume that $J_- \subset [0, 1)$.

Consider the case when $J$ contains an interval carrying $\text{supp} w_{n-(l+1)}$ for some $l$. Out of all such intervals choose the longest one. Note that since $|J| \leq \frac{2}{3}$, we must have $l \geq 0$ in this case. Thus, the interval in question must be of the kind $R_m^{(2)}$ where $R$ is an interval carrying $w_{n-l}$. Since neither $R = R_0$, nor $R_m^{(2)}$ (if $m \geq 1$) is contained in $J$, there are only three possible options:

- $J_- = R_m^{(2)}$, $0 \leq m \leq k - 2$;
• $J_+ = R_m^{(2)}$, $0 \leq m \leq k - 2$;

• $J_- = R_m$, $1 \leq m \leq k - 2$.

Suppose first that $J_- = R_m^{(2)}$ or $J_+ = R_m^{(2)}$. Then $J \subset R_m$. By (2.9), $w(R_m) \leq 3 \cdot 2^{l-1} |R_m|$. On the other hand, since $R_m^{(2)}$ carries $w_{n-(l+1)}$, by Proposition 2.1,

\[ w(J) \geq w(R_m^{(2)}) = 2^{l+1} |R_m^{(2)}| = 2^{l+1} \frac{|R_m|}{3}, \tag{2.19} \]

which implies $w(R_m) \leq \frac{2}{9} w(J)$. Therefore, by Propositions 2.3 and 2.4,

\[
\int_J (M(w \chi_J))^2 \sigma \leq 25 \int_{R_m} (\tilde{w})^2 \sigma \leq 25 \cdot 30 p^2 w(R_m) \leq 25 \cdot 75 p^2 w(J).
\]

Assume now that $J_- = R_m$, $1 \leq m \leq k - 2$. Then $w \equiv \frac{2^{l-1}}{p}$ on $J_+$ if $l > 0$ and $w \equiv \frac{2^l}{p}$ on $J_+$ if $l = 0$. In either case, $\tilde{w} = 2pw$ on $J_+$, so

\[
\int_{J_+} (\tilde{w})^2 \sigma = 4p^2 w(J_+),
\]

and thus, by Propositions 2.3 and 2.4,

\[
\int_J (M(w \chi_J))^2 \sigma \leq 25 \left( \int_{R_m} (\tilde{w})^2 \sigma + \int_{J_+} (\tilde{w})^2 \sigma \right) \\
\leq 25 \left( 30p^2 w(R_m) + 4p^2 w(J_+) \right) \leq 25 \cdot 30 p^2 w(J).
\]

It remains to consider the case when $J$ does not contain an interval carrying $w_{n-(l+1)}$ for any $0 \leq l \leq n - 1$. Denote by $E$ the union of all tail intervals appearing in the definition of $w$. Notice that if $x \not\in E$, then

\[
\sum_{l=1}^{n} 2^l \chi_{\text{supp } w_{n-(l-1)} \text{ supp } w_{n-l}}(x) = 2pw(x) \chi_{\mathbb{R} \setminus \text{supp } w_0}. \]

Also, $2^{n+1} \leq 4pw(x) \chi_{\text{supp } w_0}$. From this and from Proposition 2.3,

\[ Mw(x) \leq 18pw(x) \quad (x \not\in E). \]

Therefore, if $J \cap E = \emptyset$,

\[ \frac{1}{w(J)} \int_J (M(w \chi_J))^2 \sigma \leq 18^2 p^2. \tag{2.20} \]

Suppose that $J \cap E \neq \emptyset$. Then there exists $R$ carrying $w_{n-l}$ for some $0 \leq l \leq n - 1$ such that $J \cap R_{k-1}^{(3)} \neq \emptyset$. Since $J_-, J_+$ and $R_{k-1}^{(3)}$ are triadic, we have that either one half of $J$ is contained in $R_{k-1}^{(3)}$ or
$B_{k-1}^{(3)} \subset J$. Since $J$ cannot contain any interval carrying $\text{supp } w_{n-(l+1)}$, in both cases we obtain that $w$ can take only three possible values

$$\frac{2^l}{p} \frac{4\varepsilon 2^l}{(1+\varepsilon)p} \frac{2^{l-1}}{p}$$

on $J$ and therefore,

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq \left( \sup_J w / \inf_J w \right)^2 \leq \left( \frac{1+\varepsilon}{4\varepsilon} \right)^2 \leq 9p^2,$$

which along with (2.20) implies

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq 18^2 p^2.$$

This completes the proof of (2.12), and therefore the first estimate in (2.2) is proved.

2.5. Estimate of the Hilbert transform. The goal of this section is to prove the second estimate in (2.2).

Denote by $A^*_l$, $l = 0, \ldots, n - 1$, the union of all intervals $\frac{1}{2}I$ where $I$ is a tail interval contained in

$$[0, 1) \cap \left( \text{supp } w_{n-l} \setminus \text{supp } w_{n-(l+1)} \right).$$

In other words, $A^*_l$ is the union of all intervals $\frac{1}{2}J_{k-1}^{(3)}$ where $J \subset [0, 1)$ carries $w_{n-l}$. Then, by (2.10),

$$(2.21) \quad |A^*_l| = \frac{1}{2} |A_l| = \frac{1}{2} \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^l \frac{1}{3^k} \quad (l = 0, \ldots, n - 1).$$

The sets $A^*_l$ plays the central role in establishing the lower bound for $H(w\chi_{[0,1)})$, as the following proposition shows.

Proposition 2.5. There exists an absolute $C > 0$ such that for all $l = 0, \ldots, n - 1$ and for every $x \in A^*_l$

$$(2.22) \quad |H(w\chi_{[0,1]})(x)| \geq Ck 2^l.$$

Let us first show how to derive the second estimate in (2.2) from here. By (2.21) and (2.22),

$$\int_{A^*_l} |H(w\chi_{[0,1]})|^2 \sigma \geq C^2 k^2 2^{2l} \frac{1+\varepsilon}{4\varepsilon} \frac{p}{2^l} \frac{1}{2} \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^l \frac{1}{3^k}.$$
Therefore,

\[
\|H(w\chi_{[0,1]})\|_{L^2(\sigma)}^2 \geq \sum_{l=0}^{n-1} \int_{A_l^t} |H(w\chi_{[0,1]})|^2 \sigma \\
\geq \frac{C^2}{8} k^2 p \sum_{l=0}^{n-1} \left(1 - \frac{1}{3^{k-1}}\right)^l = \frac{C^2}{24} k^2 3^k p \left(1 - \left(1 - \frac{1}{3^{k-1}}\right)^n \right).
\]

Since \( n = 3^{k-1} \) and \((1 - 1/n)^n < 1/e\), we obtain

\[
\|H(w\chi_{[0,1]})\|_{L^2(\sigma)}^2 \geq \frac{C^2(1 - 1/e)}{24} k^2 3^k p \geq \frac{C^2(1 - 1/e)}{144(\log 3)^2} t^2 \log^2 t.
\]

Let us now turn to the proof of Proposition 2.5. Let \( J = [a, b) \subset [0, 1) \) be an interval carrying \( w_{n-t} \). Assume that \( x \in \frac{1}{2} J_{k-1}^{(3)} \). Write

\[
H(w\chi_{[0,1]})(x) = H(w\chi_{[0,a]})(x) + \sum_{m=0}^{k-2} H(w\chi_{J_{m} \setminus J_{m+1}})(x) + H(w\chi_{J_{k-1} \setminus J_{k-1}^{(3)}})(x) + H(w\chi_{J_{k-1}^{(3)}})(x) + H(w\chi_{[b,1]})(x) \\
\equiv A(x) + B(x) + C(x) + D(x) + E(x).
\]

We will show that there are absolute constants \( C_1 \) and \( C_2 \) such that for all \( x \in \frac{1}{2} J_{k-1}^{(3)} \),

\[
(2.23) \quad |B(x)| \geq C_1 k 2^l \quad \text{and} \quad \max\{|D(x)|, |E(x)|\} \leq C_2 2^l.
\]

Since \( A(x), B(x) \) and \( C(x) \) are positive for all \( x \in \frac{1}{2} J_{k-1}^{(3)} \), we obtain from (2.23) that

\[
|H(w\chi_{[0,1]})(x)| \geq |A(x) + B(x) + C(x)| - |D(x)| - |E(x)| \\
\geq |B(x)| - |D(x)| - |E(x)| \geq \frac{C_1}{2} k 2^l
\]

for \( k > \frac{4C_2}{C_1} \).

Now let us prove the first estimate in (2.23). If \( y \in J_m \setminus J_{m+1} \) and \( x \in \frac{1}{2} J_{k-1}^{(3)} \), then \( 0 \leq x - y \leq |J_m| \). Using also that \( J_{m}^{(3)} \subset J_m \setminus J_{m+1} \), by Proposition 2.1 we obtain

\[
H(w\chi_{J_m \setminus J_{m+1}})(x) = \int_{J_m \setminus J_{m+1}} \frac{w(y)}{x - y} dy \geq \frac{w(J_m \setminus J_{m+1})}{|J_m|} \\
\geq \frac{w(J_m^{(3)})}{|J_m|} = \frac{2}{3} 2^l.
\]

Therefore,

\[
B(x) \geq \frac{2}{3} (k - 1) 2^l.
\]
Turn to the second part of (2.23). Let \( J^{(3)}_{k-1} = [\alpha, b) \). Then for all \( x \in \frac{1}{2} J^{(3)}_{k-1} \),
\[
\left| \int_{J^{(3)}_{k-1}} \frac{w(y)}{x-y} \, dy \right| = \frac{4\varepsilon}{(1+\varepsilon)^p} \frac{2^l}{p} \left| \log \left| \frac{x-\alpha}{x-b} \right| \right| \leq 4(\log 3)\varepsilon \frac{2^l}{p} \leq 4(\log 3)2^l.
\]

It remains to estimate \(|E(x)|\). Take the intervals \( J^i = [a_i, b_i), i = 0, \ldots, l \) such that \( J^i \) carries \( w_{n-i} \) and
\[
J = J^0 \subset J^1 \subset \cdots \subset J^l = [0, 1).
\]
We claim that for every \( i = 1, \ldots, l \) and for all \( x \) such that \( 0 < x \leq b_{i-1} - \frac{|J^{(i-1)}_{k-1}|}{4 \cdot 3^k} \),
\[
|H(w\chi_{[b_{i-1}, b_i]})(x)| \leq 13 \cdot 2^l - i. \tag{2.24}
\]
Notice first that this claim immediately implies the desired estimate for \( E(x) \). Indeed, let \( x \in \frac{1}{2} J^{(3)}_{k-1} \). Then \( 0 < x \leq b - \frac{|J|}{4 \cdot 3^k} \), and hence (2.24) holds for \( i = 1 \). But since \( x \notin (J^i)_{k-1} \) for all \( i = 1, \ldots, l \), we obviously obtain that \( 0 < x \leq b_{i-1} - \frac{|J^{(i-1)}_{k-1}|}{4 \cdot 3^k} \) for all \( i \leq l \). Therefore, by (2.24),
\[
|H(w\chi_{[b_{i-1}, b_i]})(x)| \leq \sum_{i=1}^l |H(w\chi_{[b_{i-1}, b_i]})(x)| \leq 13 \sum_{i=1}^l 2^l - i \leq 13 \cdot 2^l.
\]

It remains to prove the claim. Denote \( x_i = b_{i-1} - \frac{|J^{(i-1)}_{k-1}|}{4 \cdot 3^k} \). Observe that \(|H(w\chi_{[b_{i-1}, b_i]})(x)|\) is an increasing function for \( x < b_{i-1} \). Therefore, it suffices to prove that
\[
|H(w\chi_{[b_{i-1}, b_i]})(x_i)| \leq 13 \cdot 2^l - i. \tag{2.25}
\]
There exists \( 0 \leq m \leq k - 2 \) such that \( J^{i-1} = (J^i_m)^{(2)} \). Then \( [b_{i-1}, b_i] = J^i_{m+1} \). Let \( h = |J^i_{m+1}| \). Split the integral in (2.25) as follows:
\[
\int_{b_{i-1}}^{b_i} \frac{w(y)}{y-x_i} \, dy = \int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i} \, dy + \int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i} \, dy.
\]
Using that \( w \equiv \frac{2^l}{p} \) on \([b_{i-1}, b_{i-1} + h/3]\), we obtain
\[
\int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i} \, dy \leq \frac{2^l \cdot 4 \cdot 3^k}{p} \leq 8 \cdot 2^l.
\]
Next, applying (2.9) yields
\[
\int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i} \, dy \leq \frac{3}{h} \cdot \frac{3}{2} \cdot 2^l \cdot |J^i_{m+1}| = \frac{9}{2} \cdot 2^l - j
\]
which along with the previous estimate proves (2.25).
This proves the claim and so Proposition 2.5. Thus, Theorem 1.1 is completely proved.

3. Appendix

In this section, we will show how to prove (2.11). Let us show first that for every interval $I \subset \mathbb{R}$, there exists an interval $J \in \mathcal{J}$ containing $I$ and such that $|J| \leq 6|I|$. Indeed, let $I = [a, a + h)$. Fix $j \in \mathbb{Z}$ such that $3^{j-1} \leq h < 3^j$, and take $n \in \mathbb{Z}$ such that

$$3^j n \leq a < 3^j(n + 1).$$

Then $I \subset J = [3^j n, 3^j(n + 2))$, and $\frac{|J|}{|I|} \leq \frac{3^j}{3^j-1} = 6$.

It follows immediately from this property that

$$(3.1) \quad Mf(x) \leq 6M^2 f(x),$$

where

$$M^2 f(x) = \sup_{J \ni x, J \in \mathcal{J}} \frac{1}{|J|} \int_J |f|dy.$$

Next, it is easy to see that the intervals from $\mathcal{J}$ can be split into two disjoint triadic lattices, $\mathcal{J} = \mathcal{T}^1 \cup \mathcal{T}^2$ (see Fig. 3 for a geometric illustration of this fact).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The lattices $\mathcal{T}$, $\mathcal{T}^1$ and $\mathcal{T}^2$ shown at two consecutive generations. The unions of blue and adjacent (from the right) red intervals from $\mathcal{T}_j$ form $\mathcal{T}_{j+1}^1$, and the unions of red and adjacent (from the right) blue intervals from $\mathcal{T}_j$ form $\mathcal{T}_{j+1}^2$. In turn, the unions of blue and red children from $\mathcal{T}_{j+1}^1$ form $\mathcal{T}_{j+1}^{11}$, and the unions of red and blue children from $\mathcal{T}_{j+1}^2$ form $\mathcal{T}_{j+1}^{21}$.}
\end{figure}

Therefore, by (3.1),

$$(3.2) \quad \|M\|_{L^2(\sigma) \to L^2(\sigma)} \leq 6\left(\|M^{T^1}\|_{L^2(\sigma) \to L^2(\sigma)} + \|M^{T^2}\|_{L^2(\sigma) \to L^2(\sigma)}\right),$$
In order to estimate the right-hand side of (3.2), we invoke the following proposition.

**Proposition 3.1.** Let $\mathcal{T}$ be a triadic lattice. Then

$$\|M^\mathcal{T}\|_{L^2(\sigma) \to L^2(\sigma)} \leq 2 \sup_{R \in \mathcal{T}} \left( \frac{1}{w(R)} \int_R (M^\mathcal{T}(w\chi_R))^2 \sigma \right)^{1/2}.$$

**Remark 3.2.** For dyadic lattices this result can be found in [6]. The proof there is closely related to the approach by E. Sawyer [11] in his two-weighted characterization for the maximal operator. For triadic lattices the proof is essentially the same, and we give it for the sake of completeness.

**Proof of Proposition 3.1.** Let $a > 1$. For $k \in \mathbb{Z}$ write the set $\Omega_k = \{M^\mathcal{T}f > a^k\}$ as the union of pairwise disjoint triadic intervals $I^k_j$ such that $\frac{1}{|I^k_j|} \int_{I^k_j} |f| > a^k$. Denote $E^k_j = I^k_j \setminus \Omega_{k+1}$, and set $\alpha_{j,k} = (w(I^k_j)/|I^k_j|)^2 \sigma(E^k_j)$. We have

$$\|M^\mathcal{T}f\|_{L^2(\sigma)}^2 = \sum_{k \in \mathbb{Z}} \int_{\Omega_k \setminus \Omega_{k+1}} (M^\mathcal{T}f)^2 \sigma \leq a^2 \sum_{k,j} \left( \frac{1}{|I^k_j|} \int_{I^k_j} |f| \right)^2 \sigma(E^k_j),$$

(3.3)

$$= a^2 \sum_{k,j} \left( \frac{1}{w(I^k_j)} \int_{I^k_j} |f\sigma|w \right)^2 \alpha_{j,k}.$$

Notice that for every $R \in \mathcal{T}$,

$$\sum_{j,k : I^k_j \subset R} \alpha_{j,k} \leq \int_R (M^\mathcal{T}(w\chi_R))^2 \sigma \leq N^2 w(R),$$

(3.4)

where

$$N = \sup_{R \in \mathcal{T}} \left( \frac{1}{w(R)} \int_R (M^\mathcal{T}(w\chi_R))^2 \sigma \right)^{1/2}.$$

For $\lambda > 0$ set

$$E_\lambda = \left\{ (j,k) : \left( \frac{1}{w(I^k_j)} \int_{I^k_j} |f\sigma|w \right)^2 > \lambda \right\}.$$

Define the weighted maximal operator $M_w^\mathcal{T}$ by

$$M_w^\mathcal{T}f(x) = \sup_{J \ni x, J \in \mathcal{T}} \frac{1}{w(J)} \int_J |f|w dy.$$
Writing the set \( \{ x : M^2_w(f\sigma)^2(x) > \lambda \} \) as the union of the maximal pairwise disjoint triadic intervals \( \bigcup_i R_i \) and applying (3.4), we obtain

\[
\sum_{(j,k) \in E_\lambda} \alpha_{j,k} \leq \sum_{i} \sum_{j,k : I^1_{j,k} \subset R_i} \alpha_{j,k} \leq N^2 w \{ x : M^2_w(f\sigma)^2(x) > \lambda \}.
\]

Therefore,

\[
\sum_{k,j} \left( \frac{1}{w(I^1_j)} \int_{I^1_j} |f\sigma| w \right)^2 \alpha_{j,k} = \int_0^\infty \left( \sum_{(j,k) \in E_\lambda} \alpha_{j,k} \right) d\lambda 
\leq N^2 \| M^2_w(f\sigma) \|_{L^2(w)}^2 \leq 4N^2 \| f\sigma \|_{L^2(w)}^2 = 4N^2 \| f \|_{L^2(\sigma)}^2,
\]

which, along with (3.3), completes the proof. \( \square \)

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