Abstract

On table algebras and applications to finite group theory

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The pair $(A, \mathbf{B})$ is called a *table algebra* if A is a commutative associative $\mathbb{C}$-algebra of dimension k with a distinguished basis $\mathbf{B} = \{1 = b_1, b_2, ..., b_k\}$ which satisfies the following conditions

(a) For all $i, j,\ b_i b_j = \sum_{m=1}^{k} \lambda_{ijm} b_m$ where $\lambda_{ijm} \in \mathbb{R}^+ \cup \{0\}$ (non-negative reals).

(b) There is an algebra isomorphism $\bar{\ } : A \to A$ such that $\overline{\mathbf{B}} = \mathbf{B}$ ans $\overline{\overline{a}} = a$ for all $a \in A$. (If $b_i = \overline{b}_i = b_{\overline{i}}$, then b_i is called *real*).

(c) For all i, j $\lambda_{ij1} \neq 0$ iff $i = \bar{j}$.

Arad and Blau proved that there exists a unique algebra homomorphism $f : A \to \mathbb{C}$ such that $f(b_i) = f(\bar{b}_i) \in \mathbb{R}^+$ for all i. The numbers $\{f(b_i) \,|\, b_i \in \mathbf{B}\}$ are called the *degrees* of $(A, \mathbf{B})$.

Let G be a finite group.

1. $(Z(\mathbb{C}[(G)]), \mathrm{Cla}(G))$ (is a standard ITA). The center of the group algebra with $\mathbb{B}$, the set of sums $\hat{C}$ of G-conjugacy classes. Here $\overline{C} = C^{-1}$ and $f(C) = |C|$ where $C \in \mathrm{Cla}(G)$.

2. $(Ch(G), \mathrm{Irr}(G))$ (a normalized ITA) the algebra of complex valued functions on G with $\mathbf{B} = \mathrm{Irr}(G)$. Here $\bar{\chi}$ is the conjugate of χ and $f(\chi) = \chi(1)$ is the dimension of χ.

In our current research we classified certain table algebras satisfied one of the following 4 conditions:

1) There exist $a, b \in \mathbf{B}$ such that $ab = ma + nb, m, n \in \mathbb{R}$.

2) There exist $a, b \in \mathbf{B}$ such that $ab = ma + n\overline{b}, m, n \in \mathbb{R}$.

3) There exist $a, b \in \mathbf{B}$ such that $ab = m\overline{a} + n\overline{b}, m, n \in \mathbb{R}$.

4) There exist $a \in \mathbb{B}$ a nonreal element of degree 3 and $(A, \mathbf{B})$ is generalized by $a \in \mathbf{B}$.

In certain cases we give the Jordan-Hölder composition series and various properties of these table algebras. We applied our results to finite group theory. For example, as an application of the case 1) we conclude the following

Theorem.
Let G be a finite group and $\chi, \psi \in \mathrm{Irr}(G)$ such that $\chi\psi = m\chi + n\psi, m, n \in \mathbb{N}$. Then either G is Frobenius or G is solvable of even order or there exist elementary abelian 2-subgroup $N \lhd G$ such that $N \leq O_2(G)$ and $G/O_2(G) \cong Sz(q), q = 2^{2d+1}, d \in \mathbb{N}$. Furthermore, (G, N) is Camina pair.

We didn't succeed to construct an example of the third case. The results related to the cases 1)-3) are based on current research of Zvi Arad, Efi Cohen and Misha Muzychuk. The results to the case 4) are based on current research of Zvi Arad, Guiyan Chen and Arisha Haj Ihia Hussam.