FREE NON-ARCHIMEDEAN TOPOLOGICAL GROUPS

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Abstract. We study free topological groups defined over uniform spaces in some subclasses of the class **NA** of non-archimedean groups. Our descriptions of the corresponding topologies show that for metrizable uniformities the corresponding free balanced, free abelian and free Boolean **NA** groups are also metrizable. Graev type ultra-metrics determine the corresponding free topologies. Such results are in a striking contrast with free balanced and free abelian topological groups cases (in standard varieties).

Another contrasting advantage is that the induced topological group actions on free abelian **NA** groups frequently remain continuous. One of the main applications is: any epimorphism in the category **NA** must be dense. Moreover, the same methods improve the following result of T.H. Fay [13]: the inclusion of a proper open subgroup $H \hookrightarrow G \in \mathbf{TGR}$ is not an epimorphism in the category **TGR** of all Hausdorff topological groups. A key tool in the proofs is Pestov's test of epimorphisms [42].

Our results provide a convenient way to produce surjectively universal **NA** abelian and balanced groups. In particular, we unify and strengthen some recent results of Gao [15] and Gao-Xuan [16] as well as classical results about profinite groups which go back to Iwasawa and Gildenhuys-Lim [17].

Contents

1. Introduction and preliminaries

1.1. **Non-archimedean groups and uniformities.** A topological group G is said to be *non-archimedean* if it has a local base B at the identity consisting of open subgroups. Notation: $G \in \mathbf{NA}$. If in this definition every $H \in B$ is a normal subgroup of G then we obtain the subclass of all *balanced* (or, **SIN**) non-archimedean groups. Notation: $G \in \mathbf{NA_b}$. All prodiscrete (in particular, profinite) groups are in $\mathbf{NA_b}$.

A uniform space is called *non-archimedean* if it possesses a base of equivalence relations. Observe that a topological group is non-archimedean if and only if its left

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(right) uniform structure is non-archimedean. The study of non-archimedean groups and non-archimedean uniformities has great influence on various fields of Mathematics: Functional Analysis, Descriptive Set Theory and Computer Science are only some of them. The reader can get a general impression from [48, 3, 28, 27, 35] and references therein.

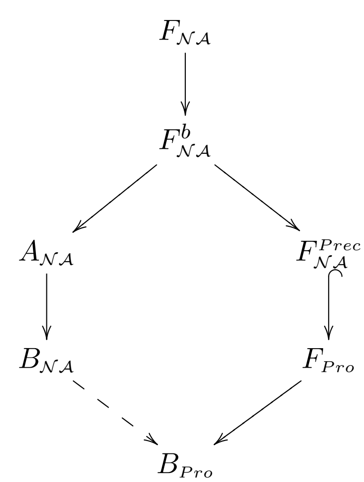
1.2. **Free groups in different contexts.** Recall that according to [29] any continuous map from a Tychonoff space X to a topological group G can be uniquely extended to a continuous homomorphism from the (Markov) free topological group $F(X)$ into G. Moreover, X is a (closed) topological subspace of $F(X)$. There are several descriptions of free topological groups. See for example, [56, 40, 58, 51]. Considering the category of uniform spaces and uniformly continuous maps one obtains the definition of a *uniform free topological group* $F(X,\mathcal{U})$ (see [39]). A description of the topology of this group was given by Pestov [40, 41]. *Free topological G-groups*, the G-space version of the above notions, were introduced in [31].

Let Ω be a class of some Hausdorff topological groups. We study in Section 3 a useful unifying concept of the *Ω-free topological groups.*

Remark 1.1. ('Zoo' of free **NA** groups) Here we give a list of some natural subclasses Ω of **NA** and establish the notation for the corresponding free groups. These groups are well defined by virtue of Theorem 3.4.

(1) $\Omega = \mathbf{NA}$. The free non-archimedean group $F_{\mathcal{NA}}$.
(2) $\Omega = \mathbf{AbNA}$. The *free non-archimedean abelian group* $A_{\mathcal{NA}}$.
(3) $\Omega = \mathbf{NA_b}$. The *free non-archimedean balanced group* $F^b_{\mathcal{NA}}$.
(4) $\Omega = \mathbf{BoolNA}$. The *free non-archimedean Boolean group* $B_{\mathcal{NA}}$.
(5) $\Omega = \mathbf{NA} \cap \mathbf{Prec}$. The *free non-archimedean precompact group* $F^{Prec}_{\mathcal{NA}}$.
(6) $\Omega = \mathbf{Pro}$. The *free profinite group* F_{Pro}.
(7) $\Omega = \mathbf{BoolPro}$. The *free Boolean profinite group* B_{Pro}.

The following diagram demonstrates the interrelation (by the induced homomorphisms) between these free groups defined over the same uniform space $(X,\mathcal{U})$.



$F^{Prec}_{\mathcal{NA}} \hookrightarrow F_{Pro}$ is the completion of the group $F^{Prec}_{\mathcal{NA}}$ and $B_{\mathcal{NA}} \dashrightarrow B_{Pro}$ is a dense injection. Other arrows are onto.

We give descriptions of the topologies of these groups in Sections 4 and 5. These descriptions show that for metrizable uniformities the corresponding free balanced, free abelian and free Boolean non-archimedean groups are also metrizable. The same is true (and is known) for the free profinite group which can be treated as the free compact non-archimedean group over a uniform space. Such results for the subclasses of **NA** are in a striking contrast with the standard classes outside of **NA**. Indeed, it is well known that the usual free topological and free abelian topological groups $F(X)$ and $A(X)$ respectively, are metrizable only for discrete topological spaces X. Similar results are valid for uniform spaces.

In Section 5.1 we discuss the free Boolean profinite group $B_{Pro}(X)$ of a Stone space X which is the Pontryagin dual of the discrete Boolean group of all clopen subsets in X.

In Section 6 we unify and strengthen some recent results of Gao [15] and Gao-Xuan [16] about the existence and the structure of surjectively universal non-archimedean Polish groups for abelian and balanced cases; as well as, results on surjectively universal profinite groups which go back to Iwasawa and Gildenhuys-Lim [17].

1.3. **The actions which come from automorphisms.** Every continuous group action of G on a *Stone space* X (=compact zero-dimensional space) is *automorphizable* in the sense of [31] (see Fact 7.3), that is, X is a G-subspace of a G-*group* Y. This contrasts the case of general compact G-spaces (see [31]). More generally, we study (Theorem 8.2) also metric and uniform versions of automorphizable actions. As a corollary we obtain that every ultra-metric G-space is isometric to a closed G-subset of an ultra-normed Boolean G-group. This result can be treated as a non-archimedean (equivariant) version of the classical Arens-Eells isometric linearization theorem [1].

1.4. **Epimorphisms in topological groups.** A morphism $f : M \to G$ in a category $\mathcal{C}$ is an *epimorphism* if there exists no pair of distinct $g, h : G \to P$ in $\mathcal{C}$ such that $gf = hf$. In the category of Hausdorff topological groups a morphism with a dense range is obviously an epimorphism. K.H. Hofmann asked in the late 1960's whether the converse is true. This *epimorphism problem* was answered by Uspenskij [59] in the negative. Nevertheless, in many natural cases, indeed, the epimorphism $M \to G$ must be dense. For example, in case that the co-domain G is either locally compact or balanced, that is, having the coinciding left and right uniformities (see [38]). Using a criterion of Pestov [42] and the uniform automorphizability of certain actions by non-archimedean groups (see Theorem 7.6) we prove in Theorem 7.9 that any epimorphism in the category **NA** must be dense. Moreover, we show that if a proper closed subgroup H in a Hausdorff topological group G induces a non-archimedean uniformity $\mathcal{U}$ on G/H such that $(G/H, \mathcal{U}) \in \mathrm{Unif}^G$, then the inclusion is not an epimorphism in the category **TGR**. We also improve the following result of T.H. Fay [13]: for a topological group G the inclusion of a proper open subgroup H is not an epimorphism.

1.5. **Graev type ultra-metrics.** In his classical work [18], Graev proved that every metric on $X \cup \{e\}$ admits an extension to a maximal invariant metric on $F(X)$. In the present work we explore (especially see Theorem 8.2) Graev type ultra-metrics and ultra-norms on free Boolean groups which appeared in our previous work [36]).

Graev type ultra-metrics play a major role in several recent papers. In Section 9 we briefly compare two seemingly different constructions: one of Savchenko-Zarichnyi [49] and the other of Gao [15].

1.6. **Preliminaries and notations.** All topological groups and spaces in this paper are assumed to be Hausdorff unless otherwise is stated (for example, in Section 4). The cardinality of a set X is denoted by $|X|$. All cardinal invariants are assumed to be infinite. As usual for a topological space X by $w(X), d(X), \chi(X), l(X), c(X)$ we denote the weight, density, character, Lindelöf degree and the cellularity, respectively. By $N_x(X)$ or N_x we mean the set of all neighborhoods at x.

For every group G we denote the identity element by e (or by 0 for additive groups). A *Boolean group* is a group in which every nonidentity element is of order two. A topological space X with a continuous group action $\pi : G \times X \to X$ of a topological group G is called a *G-space*. If, in addition, X is a topological group and all g-translations, $\pi^g : X \to X,\ x \mapsto gx := \pi(g, x)$, are automorphisms of X then X becomes a *G-group*.

We say that a topological group G is *complete* if it is complete in its two-sided uniformity. For every set X denote by $F(X), A(X)$ and $B(X)$ the free group, the free abelian group and the free Boolean group over X respectively. We reserve the notation $F(X)$ also for the free topological group in the sense of Markov.

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2. Some facts about non-archimedean groups and uniformities

We mostly use the standard definition of a *uniform space* $(X, \mathcal{U})$ by *entourages* (see for example, [12]). An equivalent approach via *coverings* can be found in [23]. We denote the induced topology by $top(\mathcal{U})$ and require it to be Hausdorff, namely, $\cap\{\varepsilon \in \mathcal{U}\} = \triangle$. By **Unif** we denote the category of all uniform spaces.

The subset $\{\varepsilon(a) : \varepsilon \in \mathcal{U}\}$ is a neighborhood base at $a \in X$ in the topological space $(X, top(\mathcal{U}))$, where $\varepsilon(a) = \{x \in X : \ (a, x) \in \varepsilon\}$. For a nonempty subset $A \subset X$ denote $\varepsilon(A) = \cup\{\varepsilon(a) : \ a \in A\}$. We say that a subset $A \subset X$ is *ε-dense* if $\varepsilon(A) = X$.

A subfamily $\alpha \subset \mathcal{U}$ such that each $\varepsilon \in \mathcal{U}$ has a refinement $\delta \subset \varepsilon$ with $\delta \in \alpha$ is said to be a (uniform) *base* of $\mathcal{U}$. The minimal cardinality of a base of $\mathcal{U}$ is called the *weight* of $\mathcal{U}$. Notation: $w(\mathcal{U})$. Recall that $\mathcal{U}$ is metrizable (that is, $\mathcal{U}$ is induced by a metric on X) if and only if the weight is countable, $w(\mathcal{U}) = \aleph_0$.

As usual, $(X, \mathcal{U})$ is *precompact* (or, *totally bounded*) if for every $\varepsilon \in \mathcal{U}$ there exists a finite ε-dense subset. By the *uniform Lindelöf degree* of $(X, \mathcal{U})$ we mean the minimal (infinite) cardinal κ such that for each entourage $\varepsilon \in \mathcal{U}$ there exists an ε-dense subset $A_\varepsilon \subset X$ of cardinality $|A_\varepsilon| \leq \kappa$. Notation: $l(\mathcal{U}) = \kappa$. We write $(X, \mathcal{U}) \in \mathbf{Unif}(\cdot, \kappa)$ whenever $l(\mathcal{U}) \leq \kappa$.

In terms of coverings, $l(\mathcal{U}) \leq \kappa$ means that every uniform covering $\varepsilon \in \mathcal{U}$ has a subcovering δ (equivalently, a subcovering $\delta \in \mathcal{U}$) of cardinality $|\delta| \leq \kappa$. So, we may always choose a base α of $\mathcal{U}$ such that $|\alpha| = w(\mathcal{U})$ and $|\delta| \leq l(\mathcal{U})$ for every $\delta \in \alpha$. Note that $l(\mathcal{U}) \leq \min\{l(X), d(X), c(X)\}$ and $l(\mathcal{U}) \leq w(X) \leq w(\mathcal{U}) \cdot l(\mathcal{U})$, where $X = (X, top(\mathcal{U}))$ is a topological space induced by $\mathcal{U}$.

Remark 2.1. The class $\mathbf{Unif}(\cdot, \kappa)$ is closed under arbitrary products, subspaces and uniformly continuous images. $(X, \mathcal{U}) \in \mathbf{Unif}(\cdot, \kappa)$ if and only if $(X, \mathcal{U})$ can be embedded into a product $\prod_i X_i$ of metrizable uniform spaces $(X_i, \mathcal{U}_i)$ with $w(X_i) = l(\mathcal{U}_i) \leq \kappa$ such that $|I| \leq w(\mathcal{U})$.

Note that $w(\mathcal{U}) = \lambda$, $l(\mathcal{U}) = \kappa$ exactly means that the (uniform) *double weight*, in the sense of [25] is $dw(\mathcal{U}) = (\lambda, \kappa)$. Denote by $\mathbf{Unif}(\lambda, \kappa)$ the class of all uniform spaces with double weight $dw(X, \mathcal{U}) \leq (\lambda, \kappa)$.

Definition 2.2. *Let $\mathcal{K} \subset$ **Unif** be a class of uniform spaces. Let us say that a uniform space X is:*

(a) universal *in $\mathcal{K}$ if $X \in \mathcal{K}$ and $\forall\ \ Y \in \mathcal{K}$ there exists a uniform embedding $Y \hookrightarrow X$.*

(b) co-universal *in $\mathcal{K}$ if $X \in \mathcal{K}$ and for every $Y \in \mathcal{K}$ there exists a uniformly continuous onto map $f : X \to Y$ which is a quotient map of topological spaces.*

In [25] Kulpa proves that there exists a universal uniform space with dimension $\leq n$ and $dw(\mathcal{U}) \leq (\lambda, \kappa)$. Every isometrically universal separable metric space (say, $C[0, 1]$, or the *Urysohn space* $\mathbb{U}$) provides an example of a universal uniform space in the class $\mathbf{Unif}(\aleph_0, \aleph_0)$. For some results about isometrically universal spaces see [24, 57, 22]. However, seemingly it is an open question if there exists a universal uniform space in $\mathbf{Unif}(\lambda, \kappa)$. In fact, it is enough to solve this question for $\mathbf{Unif}(\aleph_0, \kappa)$. Indeed, if $(X, \mathcal{U})$ is universal in $\mathbf{Unif}(\aleph_0, \kappa)$ then the uniform space X^λ is universal in $\mathbf{Unif}(\lambda, \kappa)$. In order to see this recall that $\mathbf{Unif}(\cdot, \kappa)$ is closed under products, subspaces and uniformly continuous images (Remark 2.1).

For a topological group G we have four natural uniformities: *left*, *right*, *two-sided* and *lower*. Notation: $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}_{l\vee r}$, $\mathcal{U}_{l\wedge r}$ respectively. Note that the weight of all these uniformities is equal to $\chi(G)$, the *topological character* of G. Also, $l(\mathcal{U}_l) = l(\mathcal{U}_r) = l(\mathcal{U}_{l\vee r})$. This invariant, the *uniform Lindelöf degree* of G, is denoted by $l^u(G)$. Always, $l(\mathcal{U}_{l\wedge r}) \leq l^u(G)$. Note that $l^u(G) \leq \kappa$ if and only if G is *κ-bounded* in the sense of Guran (*κ-narrow* in other terminology). That is, for every $U \in N_e(G)$ there exists a subset $S \subset G$ such that $US = G$ and $|S| \leq \kappa$.

Lemma 2.3. *Let G be a topological group.*

(1) $w(G) = \chi(G) \cdot l^u(G)$ and $dw(G, \mathcal{U}_{l\vee r}) = (\chi(G), l^u(G)) \leq (w(G), w(G))$.
(2) (Guran, see for example [2, Theorem 5.1.10]*) $l^u(G) \leq \kappa$ if and only if G can be embedded into a product $\prod_i G_i$ of topological groups G_i of topological weight $w(G_i) \leq \kappa$.*
(3) [8] *(see also* [2, p. 292]*) Let $X \subset G$ topologically generate G. Consider the induced uniform subspace $(X, \mathcal{U}_X)$ where $\mathcal{U}_X = \mathcal{U}_{l\vee r}|X$. Then $l^u(G) = l(\mathcal{U}_X)$.*

Every uniform space $(X, \mathcal{U})$ is uniformly embedded into an (abelian) topological group G such that $w(\mathcal{U}) = \chi(G)$ and $l(\mathcal{U}) = l^u(G)$ (i.e., $dw(\mathcal{U}) = dw(G, \mathcal{U}_{l\vee r})$). In order to see this one may use Arens-Eells embedding theorem [1] taking into account Lemma 2.3.3.

2.1. **Non-archimedean uniformities.** Monna (see [48, p.38] for more details) introduced the notion of *non-archimedean* uniform spaces. A uniform space X is non-archimedean if it has a base B consisting of equivalence relations (or, partitions, in the language of coverings) on X. It is also equivalent to say that for such a space the *large uniform dimension* (in the sense of [23, p. 78]) is zero. For a uniform space $(X, \mathcal{U})$ denote by $Eq(\mathcal{U})$ the set of all equivalence relations on X which belong to $\mathcal{U}$.

Recall that every compact space has a unique compatible uniformity. A *Stone space* is a compact zero-dimensional space. It is easy to see that such a space is always non-archimedean. There exist $2^{\aleph_0}$-many nonhomeomorphic metrizable Stone spaces.

A metric space (X, d) is an *ultra-metric space* (or, *isosceles* [27]) if d is an *ultra-metric*, i.e., it satisfies *the strong triangle inequality*

$$d(x, z) \leq max\{d(x, y), d(y, z)\}.$$

Allowing the distance between distinct elements to be zero we obtain the definition of an *ultra-pseudometric*. For every ultra-pseudometric d on X the open balls of radius $\varepsilon > 0$ form a clopen partition of X. So, the uniformity induced by any ultra-pseudometric d on X is non-archimedean. A uniformity is non-archimedean if and only if it is generated by a system $\{d_i\}_{i \in I}$ of *ultra-pseudometrics*.

Let us say that a uniformity $\mathcal{U}$ on X is *discrete* if $\mathcal{U} = P(X \times X)$ (or, equivalently, $\Delta := \{(x, x) : x \in X\} \in \mathcal{U}$).

Denote by κ^λ the power space of the discrete uniform space with cardinality κ.

Lemma 2.4. *(1) The Baire space $B(\kappa) = \kappa^{\aleph_0}$ is a universal uniform space in the class $\mathbf{Unif}_{\mathcal{NA}}(\aleph_0, \kappa)$ of all metrizable non-archimedean uniform spaces $(X, \mathcal{U})$ such that $l(\mathcal{U}) \leq \kappa$.*

(2) *The generalized Baire space* κ^λ *is a universal uniform space in* $\mathbf{Unif}_{\mathcal{NA}}(\lambda,\kappa)$.

Let $(X,\mathcal{U})$ be a non-archimedean uniformity. By a result of R. Ellis [11] for every uniform ultra-pseudometric on a subset $Y \subset X$ there exists an extension to a uniform ultra-pseudometric on X. Another result from [11] shows that, in fact, $B(\kappa)$ is also co-universal in the class $\mathbf{Unif}_{\mathcal{NA}}(\aleph_0,\kappa)$ (see Section 6 below).

2.2. **Non-archimedean groups.** Recall that a topological group is said to be *non-archimedean* if it has a local base at the identity consisting of open subgroups. All **NA** groups are totally disconnected. The converse is not true in general (e.g., the group $\mathbb{Q}$ of all rationals). However, in case that a totally disconnected group G is also locally compact then both G and $Aut(G)$, the group of all automorphisms of G endowed with the *Birkhoff topology*, are **NA** (see Theorems 7.7 and 26.8 in [20]).

The prodiscrete groups (= inverse limits of discrete groups) are in **NA**. Every complete balanced **NA** group (in particular, every profinite group) is prodiscrete.

Example 2.5. We list here some non-archimedean groups:

1. $(\mathbb{Z},\tau_p)$ where τ_p is the p-adic topology on the set of all integers $\mathbb{Z}$.
2. The symmetric topological group S_X with the topology of pointwise convergence. Note that S_X is not balanced for any infinite set X.
3. $\mathrm{Homeo}\,(\{0,1\}^{\aleph_0})$, the homeomorphism group of the Cantor cube, equipped with the compact-open topology.

The **NA** topological groups from Example 2.5 are all *minimal*, that is, each of them does not admit a strictly coarser Hausdorff group topology. By a result of Becker-Kechris [3] every second countable (Polish) **NA** group is topologically isomorphic to a (resp., closed) subgroup of the symmetric group $S_{\mathbb{N}}$. So, $S_{\mathbb{N}}$ is a universal group in the class of all second countable **NA** groups. In fact, a more general result remains true: S_X is a universal group in the class of all **NA** groups G with the topological weight $w(G) \leq |X|$, where $|X|$ is the cardinality of the infinite set X. See for example, [19, 35] and also Fact 2.6 below.

By results of [35] there are many minimal **NA** groups: every **NA** group is a group retract of a minimal **NA** group. See Section 5.1 below and also survey papers on minimal groups [6, 7].

Teleman [55] proved that every topological group is a subgroup of $\mathrm{Homeo}\,(X)$ for some compact X and, it is also a subgroup of $Is(M,d)$, the topological group of isometries of some metric space (M,d) equipped with the pointwise topology (see also [44]). Replacing "compact" with "compact zero-dimensional" and "metric" with "ultra-metric" we obtain characterizations for the class **NA** (see [28] and Fact 2.6 below).

The class **NA** is a *variety* in the sense of [37], i.e., it is closed under taking subgroups, quotients and arbitrary products. Furthermore, **NA** is closed under group extensions (see [19, Theorem 2.7]). In particular, **NA** is stable under semidirect products. We collect here some characterizations of non-archimedean groups, majority of which are known. For details and more results see [28, 43, 35].

Fact 2.6. [35] *The following assertions are equivalent:*

(1) *G is a non-archimedean topological group.*
(2) *The right (left, two-sided, lower) uniformity on G is non-archimedean.*
(3) $\dim \beta_G G = 0$, *where $\beta_G G$ is the* maximal G-compactification [34] *of G.*
(4) *G is a topological subgroup of* $\mathrm{Homeo}\,(X)$ *for some Stone space X (where $w(X) = w(G)$).*
(5) *G is a topological subgroup of the automorphism group (with the pointwise topology) $Aut(V)$ for some discrete Boolean ring V (where $|V| = w(G)$).*

(6) G is a topological subgroup of the group $Is_{Aut}(M)$ of all norm preserving automorphisms of some ultra-normed Boolean group $(M, ||\cdot||)$ (where $w(M) = w(G)$).
(7) G is embedded into the symmetric topological group S_κ (where $\kappa = w(G)$).
(8) G is a topological subgroup of the group $Is(X, d)$ of all isometries of an ultra-metric space (X, d), with the topology of pointwise convergence (where $w(X) = w(G)$).
(9) The right (left) uniformity on G can be generated by a system $\{d_i\})_{i \in I}$ of right (left) invariant ultra-pseudometrics of cardinality $|I| \leq \chi(G)$.
(10) G is a topological subgroup of the automorphism group $Aut(K)$ for some compact abelian group K (with $w(K) = w(G)$).
(11) G is a topological subgroup of the automorphism group $Aut(K)$ for some profinite group K (with $w(K) = w(G)$).

An *ultra-seminorm* on a topological group G is a function $p : G \to \mathbb{R}$ such that
(1) $p(e) = 0$;
(2) $p(x^{-1}) = p(x)$;
(3) $p(xy) \leq max\{p(x), p(y)\}$.

Always, $p(x) \geq 0$. We call p an *ultra-norm* if in addition $p(x) = 0$ implies $x = e$. For ultra-seminorms on an abelian additive group $(G, +)$ we prefer the notation $|| \cdot ||$ rather than p. For every ultra-seminorm on G and every $a \in G$ the function $q(x) := p(axa^{-1})$ is also an ultra-seminorm on G. We say that p is *invariant* if $p(axa^{-1}) = p(x)$ for every $a, x \in G$. We say that a pseudometric d on G is *invariant* if it is left and right invariant.

Lemma 2.7. *(1) For every ultra-seminorm p on G we have:*
(a) $H_\varepsilon := \{g \in G : \ p(g) < \varepsilon\}$ is an open subgroup of G for every $\varepsilon > 0$.
(b) The function $d : G \times G \to \mathbb{R}$ defined by $d(x, y) := p(x^{-1}y)$ is a left invariant ultra-pseudometric on G and $p(x) = d(e, x)$.
If p is invariant then H_ε is a normal subgroup in G and d is invariant.
(2) Let $G \times X \to X$ be an action of a group G on a set X. If d is a G-invariant ultra-pseudometric on X and $x_0 \in X$ is a point in X then $p(g) := d(x_0, gx_0)$ is an ultra-seminorm on G.
(3) As a particular case of (2), for every left invariant ultra-pseudometric d on G we have the ultra-seminorm $p(x) := d(e, x)$. Here p is invariant if and only if d is invariant.
(4) For every topological group G and an open subgroup H of G there exists a continuous ultra-seminorm p on G such that $\{x \in G : \ p(x) < 1\} = H$. If, in addition, H is normal in G then we can assume that p is invariant.
(5) A homomorphism $f : G \to H$ from a topological group G into a non-archimedean group H is continuous if and only if for every continuous ultra-seminorm p on H the ultra-seminorm $q : G \to \mathbb{R}$ defined by $q(g) := p(f(g))$ is continuous.

Proof. (1) (a) Indeed, if $p(x) < \varepsilon$ and $p(y) < \varepsilon$ then

$$p(xy^{-1}) \leq \max\{p(x), p(y^{-1})\} = \max\{p(x), p(y)\} < \varepsilon.$$

If p is invariant then H_ε is normal in G since $p(axa^{-1}) = p(x) < \varepsilon$ for every $a \in G$.
(b) of (1) is trivial and (2), (3) and (5) are straightforward.
(4) Define the ultra-seminorm on G as $p(g) = 0$ for $g \in H$ and $p(g) := 1$ if $g \notin H$. □

A topological group G is *balanced* (or, **SIN**) if its left and right uniform structures coincide (see for example [47]). It is equivalent to say that G has small neighborhoods which are invariant under conjugations. That is, for every $U \in N_e(G)$ there exists

$V \in N_e(G)$ such that $V \subset U$ and $gVg^{-1} = V$ for every $g \in G$. Furthermore, G is balanced if and only if the uniformity on G can be generated by a system of invariant pseudometrics (or invariant seminorms).

Lemma 2.8. *For a balanced group G the following conditions are equivalent:*

(1) $G \in \mathbf{NA}$*;*
(2) *G has a local base at the identity consisting of open normal subgroups;*
(3) *G is embedded into a product $\prod_{i \in I} G_i$ of discrete groups, where $|I| \leq \chi(G)$;*
(4) *the uniformity on G can be generated by a system $\{d_i\}_{i \in I}$ of invariant ultra-pseudometrics (ultra-seminorms), where $|I| \leq \chi(G)$.*

Proof. (1) ⇒ (2): Let V be a neighborhood of e in G. We have to show that there exists an open normal subgroup M of G such that $M \subseteq V$. Since $G \in \mathbf{NA}$, there exists an open subgroup H of G such that $H \subseteq V$. Since G is balanced $N := \cap_{g \in G} gHg^{-1}$ is again a neighborhood of e. Then N is a normal subgroup of G and $N \subseteq V$. Clearly the subgroup N is open because its interior is nonempty.

(2) ⇒ (3): For every open, and hence closed, normal subgroup N of G the corresponding factor-group G/N is discrete.

(3) ⇒ (4): For every discrete group P the usual $\{0,1\}$-ultra-metric is invariant.

(4) ⇒ (1): Let p be an invariant ultra-seminorm on G. Then the set

$$H_\varepsilon := \{g \in G : \ p(g) < \varepsilon\}$$

is an open (normal) subgroup of G by Lemma 2.7.1. □

Every complete balanced **NA** group G is a prodiscrete group as it follows from assertion (3) and standard properties of projective limits (see e.g., [12, Prop. 2.5.6]).

Lemma 2.9. *Let* $G \in \mathbf{NA}$*.*

(1) *G is metrizable iff its right (left) uniformity can be generated by a single right (left) invariant ultra-metric d on G.*
(2) *G is metrizable and balanced iff its right (left) uniformity can be generated by a single invariant ultra-metric d on G.*

Proof. (1): If G is metrizable then the right (left) uniformity of $G \in \mathbf{NA}$ can be generated by a countable system $\{d_n\}_{n \in \mathbb{N}}$ of right (left) invariant ultra-pseudometrics (cf. Fact 2.6.9). One may assume in addition that $d_n \leq 1$. Then the desired right (left) invariant ultra-metric on G can be defined by $d(x,y) := \sup_{n \in \mathbb{N}} \{\frac{1}{2^n} d_n(x,y)\}$.

(2): If G is metrizable and balanced then one may assume in the proof of (1) that each d_n is invariant (see Lemma 2.8). Therefore, d is also invariant.

Conversely, if the right (left) uniformity of G can be generated by an invariant ultra-metric d then clearly, G is metrizable and balanced. □

3. Uniform free **NA** topological groups

By **TGr** we denote the category of all topological groups. By **AbGr**, **Prec**, **Pro** we denote its full subcategories of all abelian, precompact, and *profinite* (= inverse limits of finite groups) groups respectively. Usually we denote a category and its class of all objects by the same symbol.

In this section, unless otherwise is stated, all topological groups are considered with respect to the two sided uniformity $\mathcal{U}_{l \vee r}$. Assigning to every topological group G the uniform space $(G, \mathcal{U}_{l \vee r})$ defines a forgetful functor from the category of all topological groups **TGr** to the category of all uniform spaces **Unif**.

Definition 3.1. *Let Ω be a subclass of* **TGr** *and $(X,\mathcal{U}) \in$* **Unif** *be a uniform space. By an* Ω-free topological group of $(X,\mathcal{U})$ *we mean a pair $(F_\Omega(X,\mathcal{U}), i)$ (or, simply, $F_\Omega(X,\mathcal{U})$, when i is understood), where $F_\Omega(X,\mathcal{U})$ is a topological group from Ω and $i : X \to F_\Omega(X,\mathcal{U})$ is a uniform map satisfying the following universal property. For every uniformly continuous map $\varphi : (X,\mathcal{U}) \to G$ into a topological group $G \in \Omega$ there exists a unique continuous homomorphism $\Phi : F_\Omega(X,\mathcal{U}) \to G$ for which the following diagram commutes:*

$$\begin{array}{ccc} (X,\mathcal{U}) & \xrightarrow{\ i\ } & F_\Omega(X,\mathcal{U}) \\ & \overset{\varphi}{\searrow} & \downarrow \Phi \\ & & G \end{array}$$

If Ω is a full subcategory of **TGr** then a categorical reformulation of this definition is that $i : X \to F_\Omega(X,\mathcal{U})$ is a universal arrow from $(X,\mathcal{U})$ to the forgetful functor $\Omega \to$ **Unif**.

Remark 3.2. Also we use a shorter notation dropping $\mathcal{U}$ (and X) when the uniformity (and the space) is understood. For example, we may write $F_\Omega(X)$ (or, F_Ω) instead of $F_\Omega(X,\mathcal{U})$.

Every Tychonoff space X admits the greatest compatible uniformity, the so-called *fine uniformity*, which we denote by $\mathcal{U}_{max}$. The corresponding free group $F_\Omega(X,\mathcal{U}_{max})$ is denoted by $F_\Omega(X)$ and is called the Ω-*free topological group* of X. For $\Omega =$ **TGr** and $\Omega =$ **AbTGr** we get the classical *free topological group* and *free abelian topological group* (in the sense of Markov) of X keeping the standard notation: $F(X)$ and $A(X)$.

3.1. **The existence.**

Definition 3.3. *A nonempty subclass Ω of* **TGr** *is said to be:*

(1) $\overline{S}C$-variety *(see* [2]*) if Ω is closed under: a) cartesian products; b)* ***closed*** *subgroups.*
(2) SC-variety *if Ω is closed under: a) cartesian products; b) subgroups.*
(3) variety *(see* [37]*) if Ω is closed under: a) cartesian products; b) subgroups; c) quotients.*

Note that while **Pro** is an $\overline{S}C$-variety, all other subclasses Ω of **TGr** from Remark 1.1 are varieties.

Theorem 3.4. *Let Ω be a subclass of* **TGr** *which is an $\overline{S}C$-variety and $(X,\mathcal{U})$ be a uniform space.*

(1) The uniform free topological group $F_\Omega := F_\Omega(X,\mathcal{U})$ exists.
(2) F_Ω is unique up to a topological group isomorphism.
(3) (a) F_Ω is topologically generated by $i(X) \subset F_\Omega$.
(b) For every uniform map $\varphi : (X,\mathcal{U}) \to G$ into a topological group $G \in \Omega$ there exists a continuous homomorphism $\Phi : F_\Omega \to G$ such that $\Phi \circ i = \varphi$.
Moreover, these two properties characterize $F_\Omega(X,\mathcal{U})$.
(4) If Ω is an SC-variety then F_Ω is algebraically generated by $i(X)$.

Proof. (1) : Existence. We give here a standard categorical construction (with some minor adaptations) which goes back to Samuel and Kakutani. Denote $\mathfrak{m} := \max(|X|, \aleph_0)$. Let $\mathfrak{F}$ be a subclass of Ω such that $|G| \leq 2^{2^{\mathfrak{m}}}$ for $G \in \mathfrak{F}$, distinct members of $\mathfrak{F}$ are not topologically isomorphic, and every topological group H for which $|H| \leq 2^{2^{\mathfrak{m}}}$ is topologically isomorphic with some $G \in \mathfrak{F}$. Let $\{(G_j, \varphi_j)\}_{j \in J}$ consist of all pairs (G_j, φ_j) where $G_j \in \mathfrak{F}$ and φ_j is a uniformly continuous mapping of X into G_j. It is easy to see that $\mathfrak{F}$ is a set. Then J is a set as well. If $H \in \Omega$ is a topological group, $|H| \leq 2^{2^{\mathfrak{m}}}$, and φ is a uniformly continuous mapping of X into H, then there is $j_0 \in J$ and a topological isomorphism $\tau : G_{j_0} \to H$ such that $\tau \circ \varphi_{j_0} = \varphi$. In such a case we identify the pair (H, ϕ) with the pair (G_{j_0}, φ_{j_0}). Let $M = \prod_{j \in J} G_j$. For $x \in X$, define $i(x) \in M$ by $i(x)_j = \varphi_j(x)$. Finally, let $F_\Omega := F_\Omega(X, \mathcal{U})$ be the closed subgroup of M topologically generated by $i(X)$. Since the class Ω is an $\overline{S}C$-variety, both M and F_Ω are in Ω by conditions (a) and (b) of Definition 3.3.1. Clearly, $i : (X, \mathcal{U}) \to F_\Omega$ is uniformly continuous. Now, if φ is a uniformly continuous mapping of X into any topological group $G \in \Omega$, the image $\varphi(X)$ in G is contained in the subgroup $P := cl(< \varphi(X) >)$ of G, where $< \varphi(X) >$ is the subgroup of G algebraically generated by $\varphi(X)$. Since $| < \varphi(X) > | \leq \mathfrak{m} = \max\{|X|, \aleph_0\}$ and P is Hausdorff we have $|P| \leq 2^{2^{\mathfrak{m}}}$. Thus, by our assumption on $\mathfrak{F}$, the pair (P, φ) is isomorphic to a pair (G_{j_0}, φ_{j_0}) for some $j_0 \in J$. Let $\pi_{j_0} : F_\Omega \to G_{j_0}$ be the restriction on $F_\Omega \subseteq \prod_{j \in J} G_j$ of the projection onto the j_0-th axis. Then $\varphi = \Phi \circ i$, where $\Phi := \tau \circ \pi_{j_0}$. Finally, note that Φ is unique since $< i(X) >$ is a dense subgroup of F_Ω and G is Hausdorff.

(2) : Uniqueness. Assume that there exist Hausdorff topological groups F_1, F_2 and uniformly continuous maps $i : X \to F_1,\ j : X \to F_2$ such that the pairs (i, F_1) and (j, F_2) satisfy the universal property. Then by Definition 3.1 there exist *unique* continuous homomorphisms $\Phi_1 : F_2 \to F_1, \Phi_2 : F_1 \to F_2$ such that $\Phi_2 \circ i = j,\ \Phi_1 \circ j = i$. For $\Phi \in \{\Phi_1 \circ \Phi_2, Id_{F_1}\}$ we have $\Phi \circ i = i$, and thus $\Phi_1 \circ \Phi_2 = Id_{F_1}$. Similarly, $\Phi_2 \circ \Phi_1 = Id_{F_2}$. Therefore, $\Phi_2 : F_1 \to F_2$ is a topological group isomorphism.

(3) Assertion (a) follows from the constructive description of F_Ω given in the proof of (1), and from (2). Property (b) is a part of the definition of F_Ω. These two properties characterize F_Ω since the latter group is Hausdorff.

(4) As an SC-variety Ω is closed under (not necessarily closed) subgroups. So in the constructive description appearing in the proof of (1) we may define F_Ω as the subgroup *algebraically* generated by $i(X)$. Apply (2) to conclude the proof. □

The completion of G with respect to the two-sided uniformity is denoted by $\widehat{G}$. The proof of the following observation is straightforward.

Lemma 3.5. *Let Ω be an $\overline{S}C$-variety. Denote by Ω_C its subclass of all complete groups from Ω. Then $F_{\Omega_C} = \widehat{F_\Omega}$.*

3.2. **Classical constructions.** For $\Omega = \mathbf{TGr}$ the universal object $F_\Omega(X, \mathcal{U})$ is the *uniform free topological group* of $(X, \mathcal{U})$. Notation: $F(X, \mathcal{U})$. This was invented by Nakayama and studied by Numella [39] and Pestov [40, 41].

In particular, Pestov described the topology of $F(X, \mathcal{U})$ ([40], see also Remark 4.17 below). If $\Omega = \mathbf{AbGr}$ then $F_\Omega(X, \mathcal{U})$ is the *uniform free abelian topological group* of $(X, \mathcal{U})$. Notation: $A(X, \mathcal{U})$. In [51] Sipacheva used Pestov's description of the free topological group $F(X)$ to generate a description of the free abelian topological group $A(X)$. Similarly, one can prove the following:

Theorem 3.6. *(compare with* [51, page 5779]*) Let $(X, \mathcal{U})$ be a uniform space. For each $n \in \mathbb{N}$, we fix an arbitrary entourage $W_n \in \mathcal{U}$ of the diagonal in $X \times X$ and set $W = \{W_n\}_{n \in \mathbb{N}}$,*

$$U(W_n) = \{\epsilon x - \epsilon y : (x, y) \in W_n, \epsilon = \pm 1\},$$

and

$$\widetilde{U}(W) = \bigcup_{n \in \mathbb{N}} (U(W_1) + U(W_2) + \cdots + U(W_n)).$$

The sets $\widetilde{U}(W)$, where W are all sequences of uniform entourages of the diagonal, form a neighborhood base at zero for the topology of the uniform free abelian topological group $A(X,\mathcal{U})$.

Recall a classical result concerning the (non)metrizability of free topological groups.

Theorem 3.7. [2, Theorem 7.1.20] *If a Tychonoff space X is non-discrete, then neither $F(X)$ nor $A(X)$ are metrizable.*

Theorem 3.7 has a uniform modification. In fact, we can mimic the proof of Theorem 3.7 to obtain the following:

Theorem 3.8. *Let $\mathcal{U}$ be a non-discrete uniformity on X and*

$$\Omega \in \{\mathbf{TGr}, \mathbf{Prec}, \mathbf{SIN}, \mathbf{AbGr}\}.$$

Then $F_\Omega(X,\mathcal{U})$ is not metrizable.

Contrast this result with Theorem 4.16, where we show that for some natural subclasses of $\Omega = \mathbf{NA}$ the free group $F_\Omega(X,\mathcal{U})$ is metrizable whenever $(X,\mathcal{U})$ is metrizable.

3.3. **Free groups in some subclasses of NA.** In Remark 1.1 we gave a list of some classes Ω and the corresponding free groups. We keep the corresponding notations.

Theorem 3.9. *Let $(X,\mathcal{U})$ be a non-archimedean uniform space and*

$$G \in \{F_{\mathcal{NA}}, F^b_{\mathcal{NA}}, A_{\mathcal{NA}}, B_{\mathcal{NA}}\}.$$

Then:

(1) *The universal morphism $i : (X,\mathcal{U}) \to G$ is a uniform embedding.*
(2) *If $G \in \{F_{\mathcal{NA}}, F^b_{\mathcal{NA}}\}$ then G is algebraically free over $i(X)$.*
If $G = A_{\mathcal{NA}}$ or $G = B_{\mathcal{NA}}$ then G is algebraically isomorphic to $A(X)$, or $B(X)$, respectively.
(3) *$i(X)$ is a closed subspace of G.*

Proof. (1) : It suffices to prove that the universal morphism $i : X \to B_{\mathcal{NA}}$ is a uniform embedding. We show the existence of a Hausdorff **NA** group topology τ on the free Boolean group $B(X)$, and a uniform embedding $\iota : (X,\mathcal{U}) \to (B(X), \tau)$, that clearly will imply that $i : (X,\mathcal{U}) \to B_{\mathcal{NA}}$ is a uniform embedding.

Consider the natural set embedding

$$\iota : X \hookrightarrow B(X), \ \iota(x) = \{x\}.$$

We identify $x \in X$ with $\iota(x) = \{x\} \in B(X)$. Let $\mathcal{B} := \{< \varepsilon >\}_{\varepsilon \in Eq(\mathcal{U})}$, where $Eq(\mathcal{U})$ is the set of equivalence relations from $\mathcal{U}$ and $< \varepsilon >$ is the subgroup of $B(X)$ algebraically generated by the set

$$\{x + y \in B(X) : (x, y) \in \varepsilon\}.$$

Now, $\mathcal{B}$ is a filter base on $B(X)$ and $\forall b \in B(X)\ \forall \varepsilon \in Eq(\mathcal{U})$ we have

$$< \varepsilon > + < \varepsilon >= - < \varepsilon >= b+ < \varepsilon > +b.$$

It follows that there exists a **NA** group topology τ for which $\mathcal{B}$ is a local base at the identity. To prove that this topology is indeed Hausdorff, we have to show that if $u \neq 0$ is of the form $u = \sum_{i=1}^{2n} a_i$ where $n \in \mathbb{N}$ and $a_i \in X\ \forall 1 \leq i \leq 2n$, then there exists $\varepsilon \in Eq(\mathcal{U})$ such that $u \notin < \varepsilon >$. Since $(X,\mathcal{U})$ is a Hausdorff uniform space there exists $\varepsilon \in \mathcal{U}$ such that $(a_i, a_j) \notin \varepsilon$ for every $i \neq j$. Assuming the contrary, let $u \in < \varepsilon >$. Then there exists a minimal $m \in \mathbb{N}$ such that $u = \sum_{i=1}^{m} (x_i + y_i)$ where $(x_i, y_i) \in \varepsilon\ \forall 1 \leq i \leq m$. Without loss of generality we may assume that there exists $1 \leq i_0 \leq m$ such that $a_1 = x_{i_0}$. Note that $y_{i_0} \neq a_j$ for every $1 \leq j \leq 2n$, since otherwise we obtain a contradiction to the

minimality of m or to the definition of ε. Since $B(X)$ is the free Boolean group over X and ε is symmetric we can assume without loss of generality that there exists $r \neq i$ such that $y_{i_0} = x_r$. It follows that $(x_{i_0} + y_{i_0}) + (x_r + y_r) = x_{i_0} + y_r$. Since ε is transitive we also have $(x_{i_0}, y_r) \in \varepsilon$ and we obtain a contradiction to the minimality of m. Therefore, τ is Hausdorff.
We show that $\iota : (X,\mathcal{U}) \to (B(X),\tau)$ is uniformly continuous. To see this observe that if $(x,y) \in \varepsilon$ then $x+y \in <\varepsilon>$. Finally, assume that $(x,y) \in X \times X$ such that $x+y \in <\varepsilon>$ where ε is an equivalence relation. We show that $(x,y) \in \varepsilon$ and conclude that $\iota : (X,\mathcal{U}) \to (B(X),\tau)$ is a uniform embedding. Since $x+y \in <\varepsilon>$ there exists a natural number n such that $x+y = \sum_{i=1}^{n}(a_i+b_i)$, where $(a_i,b_i) \in \varepsilon$ $\forall 1 \le i \le n$. Moreover, n may be chosen to be minimal. By the definition of $B(X)$ and the fact that ε is symmetric we may assume without loss of generality that there exists $1 \le i_0 \le n$ such that $a_{i_0} = x$. The case $b_{i_0} = y$ is trivial. So we can assume that $b_{i_0} \neq y$. Since $B(X)$ is the free Boolean group over X there exists $i_1 \neq i_0$ such that either $b_{i_0} = a_{i_1}$ or $b_{i_0} = b_{i_1}$. In the former case we have $(a_{i_0} + b_{i_0}) + (a_{i_1} + b_{i_1}) = a_{i_0} + b_{i_1}$ and $(a_{i_0}, b_{i_1}) \in \varepsilon$, since ε is transitive, which contradicts the minimality of n. The latter case yields a similar contradiction.
(2) : We can now identify X with $i(X)$. Denote by X/ε the quotient set equipped with the discrete uniformity. The function $f_\varepsilon : X \to X/\varepsilon$ is the uniformly continuous map which maps every $x \in X$ to the equivalence class $[x]_\varepsilon$. We first deal with the case $G = F_{\mathcal{NA}}$. We reserve the notation f_ε also for the homomorphic extension from $F_{\mathcal{NA}}$ to the (discrete) group $F(X/\varepsilon)$. This allows us to define $[w]_\varepsilon := f_\varepsilon(w)$ for every $w \in F_{\mathcal{NA}}$. Let $w = x_1^{t_1} \cdots x_k^{t_k} \in F_{\mathcal{NA}}$ where $t_i \in \mathbb{Z} \setminus \{0\}$ for every $1 \le i \le k$ and $x_i \neq x_{i+1}$ for every $1 \le i \le k-1$. If w is of the form x^t, one can consider the extension $\overline{f} : F_{\mathcal{NA}} \to \mathbb{Z}$ of the constant function $f : (X,\mathcal{U}) \to \mathbb{Z},\ f \equiv 1$. We have

$$\overline{f}(w) = t \neq 0.$$

Otherwise, assume that w is not of the form x^t. Since $(X,\mathcal{U})$ is a non-archimedean Hausdorff space there exists an equivalence relation $\varepsilon \in \mathcal{U}$ such that

$$(x_i, x_{i+1}) \notin \varepsilon\ \forall i \in \{1,2,\ldots,k-1\}.$$

Since $F(X/\varepsilon)$ is algebraically free it follows that $f_\varepsilon(w) \neq e_{F(X/\varepsilon)}$. The groups $F(X/\varepsilon)$ and $\mathbb{Z}$ (being discrete groups) are both non-archimedean Hausdorff. So we can conclude that $F_{\mathcal{NA}}$ is algebraically free over X.
For the case $G = F^b_{\mathcal{NA}}$ use the fact that $\mathbb{Z}$ and $F(X/\varepsilon)$ are also balanced.
For the case $G = A_{\mathcal{NA}}$ one may use the fact that $\mathbb{Z}$ is also abelian and replace $F(X/\varepsilon)$ with $A(X/\varepsilon)$. Up to minor changes the proof is similar to the proof of the case $G = F^b_{\mathcal{NA}}$. For the Boolean case replace $\mathbb{Z}$ with $\mathbb{Z}_2$ and $F(X/\varepsilon)$ with $B(X/\varepsilon)$.

(3) : In case $G = F_{\mathcal{NA}}$, let $w \in F_{\mathcal{NA}} \setminus X$. Assume first that w is either the identity element of $F_{\mathcal{NA}}$ or has the form x^{-1} where $x \in X$. Then

$$\overline{f}(w) \neq \overline{f}(y) = 1\ \forall y \in X$$

where $\overline{f} : F_{\mathcal{NA}} \to \mathbb{Z}$ is the extension of the constant function $f : (X,\mathcal{U}) \to \mathbb{Z},\ f \equiv 1$. Since $\mathbb{Z}$ is discrete the set $O := \{z \in F_{\mathcal{NA}} |\ \overline{f}(z) \neq 1\}$ is clearly an open subset of $F_{\mathcal{NA}}$ and we have $w \in O \subseteq X^C$. Let $k > 1$ and $w = x_1^{t_1} \cdots x_k^{t_k}$ where $t_i \in \mathbb{Z} \setminus \{0\}$ for every $1 \le i \le k$ and $x_i \neq x_{i+1}$ for every $1 \le i \le k-1$. Then there exists an equivalence relation $\varepsilon \in \mathcal{U}$ such that

$$(x_i, x_{i+1}) \notin \varepsilon\ \forall i \in \{1,2,\ldots,k-1\}.$$

Since $F(X/\varepsilon)$ is algebraically free it follows that $[w]_\varepsilon \neq [x]_\varepsilon\ \forall x \in X$. Since $F(X/\varepsilon)$ is discrete the set $U := \{z \in F_{\mathcal{NA}} |\ f_\varepsilon(z) = [w]_\varepsilon\}$ is an open subset of $F_{\mathcal{NA}}$ and we also have $w \in U \subseteq X^C$. This implies that $(X,\mathcal{U})$ is a closed subspace of $F_{\mathcal{NA}}$.
For $G \neq F_{\mathcal{NA}}$ we may use the same modifications appearing in the proof of (2). □

Remark 3.10. It is clear that if the universal morphism $i : (X,\mathcal{U}) \to G$ is a uniform embedding, where G is non-archimedean, then $(X,\mathcal{U})$ is non-archimedean.

Lemma 3.11. *Let $\mathcal{U}$ be the discrete uniformity on a finite set X. Then $F^{Prec}_{\mathcal{NA}}$ algebraically is the free group $F(X)$ over X.*

Proof. It suffices to find a Hausdorff non-archimedean precompact group topology τ on the abstract free group $F(X)$. Consider the group topology τ generated by the filter base $\{N \lhd F(X) :\ [F(X) : N] < \infty\}$. Clearly, τ is a non-archimedean precompact group topology on $F(X)$. To see that τ is Hausdorff recall that every free group is residually finite, that is, the intersection of all normal subgroups of finite index is trivial. □

Theorem 3.12. *Let $(X,\mathcal{U})$ be a non-archimedean precompact uniform space and*

$$G \in \{F^{Prec}_{\mathcal{NA}}, F_{Pro}\}.$$

Then:

1. *The universal morphism $i : (X,\mathcal{U}) \to G$ is a uniform embedding.*
2. *$F^{Prec}_{\mathcal{NA}}$ is algebraically free over $i(X)$.*
3. *$i(X)$ is a closed subspace of $F^{Prec}_{\mathcal{NA}}$.*

Proof. (1) : $(X,\mathcal{U})$ is a uniform subspace of its compact zero dimensional completion $(\widehat{X},\widehat{\mathcal{U}})$. Consider the compact group $\mathbb{Z}_2^{w(\widehat{X})}$ where $w(\widehat{X})$ is the topological weight of $\widehat{X}$. Then it is clear that $(\widehat{X},\widehat{\mathcal{U}})$ is uniformly embedded in $\mathbb{Z}_2^{w(\widehat{X})}$. Now, since $\mathbb{Z}_2^{w(\widehat{X})}$ is a profinite group then each of the universal morphisms is a uniform embedding.
(2) : We use similar ideas to those appearing in the proof of Theorem 3.9.2. This time due to the precompactness assumption the set X/ε is finite. By Lemma 3.11, $F^{Prec}_{\mathcal{NA}}(X/\varepsilon)$ is algebraically free over the set X/ε. Thus we may replace $F_{\mathcal{NA}}(X/\varepsilon)$ with $F^{Prec}_{\mathcal{NA}}(X/\varepsilon)$, and also the discrete topology on $\mathbb{Z}$ with its Hausdorff topology generated by all of its finite-index subgroups, to conclude that $F^{Prec}_{\mathcal{NA}}$ is algebraically free over $i(X)$.
(3) : Very similar to the proof of Theorem 3.9.3. Just observe that

$$O := \{z \in F^{Prec}_{\mathcal{NA}} |\ \overline{f}(z) \neq 1\}$$

is an open subset of $F^{Prec}_{\mathcal{NA}}$, since the group topology on $\mathbb{Z}$ which we consider this time remains Hausdorff. Moreover, the set

$$U := \{z \in F^{Prec}_{\mathcal{NA}} |\ f_\varepsilon(z) \notin \{[x]_\varepsilon : x \in X\}\}$$

is also an open subset of $F^{Prec}_{\mathcal{NA}}$, since $F^{Prec}_{\mathcal{NA}}(X/\varepsilon)$ is Hausdorff and $\{[x]_\varepsilon : x \in X\}$ is finite. □

4. Final non-archimedean group topologies

In this section the topological groups are not necessarily Hausdorff. Recall that the description of $F(X,\mathcal{U})$, given by Pestov in [40] (see also Remark 4.17.1 below), was based on final group topologies, which were studied by Dierolf and Roelcke [4, Chapter 4]. Here we study final non-archimedean group topologies.

In the sequel we present a non-archimedean modification of final group topologies. The general structure of final non-archimedean group topologies is then used to find descriptions of the topologies for the free **NA** groups from Remark 1.1.

We also provide a new description of the topology of $F^b(X,\mathcal{U})$, the uniform free balanced group of a uniform space $(X,\mathcal{U})$.

Definition 4.1. *Let P be a group, α a filter base on P and $\Omega \subset$ **TGr** an SC-variety. Assume that there exists a group topology τ on P such that:*

(1) $(P,\tau) \in \Omega$, and
(2) the filter α converges to e (notation: $\alpha \to e$) in (P,τ).

Then among all group topologies on P satisfying properties (1) *and* (2) *there is a finest one. We call it the* Ω-group topology generated by α *and denote it by* $\langle\alpha\rangle_\Omega$.

Definition 4.2. [4, Chapter 4]*If P is a group and $(B_n)_{n\in\mathbb{N}}$ a sequence of subsets of P, let*

$$[(B_n)] := \bigcup_{n\in\mathbb{N}} \bigcup_{\pi\in S_n} B_{\pi(1)}B_{\pi(2)}\cdots B_{\pi(n)}.$$

Remark 4.3. Note that if $(B_n)_{n\in\mathbb{N}}$ is a constant sequence such that

$$B_1 = B_2 = \cdots = B_n = \cdots = B$$

then $[(B_n)] = \bigcup_{n\in\mathbb{N}} B^n$. In this case we write $[B]$ instead of $[(B_n)]$. It is easy to see that if $B = B^{-1}$ then $[B]$ is simply the subgroup generated by B.

Lemma 4.4. *Let P be a non-archimedean topological group and $\mathcal{L}$ a base of $N_e(P)$. Then the set $\{[B] : B \in \mathcal{L}\}$ is also a base of $N_e(P)$.*

Proof. For every $B \in \mathcal{L}$ we have $[B] \in N_e(P)$ since $B \subseteq [B]$. Let $V \in N_e(P)$. We have to show that there exists $B \in \mathcal{L}$ such that $[B] \subseteq V$. Since G is non-archimedean, there exists an open subgroup H such that $H \subseteq V$ and $H \in N_e(P)$. On the other hand, $\mathcal{L}$ is a base of $N_e(P)$ and therefore there exists $B \in \mathcal{L}$ such that $B \subseteq H$. From the fact that H is a subgroup we conclude that

$$[B] = \bigcup_{n\in\mathbb{N}} B^n \subseteq H \subseteq V.$$

□

Lemma 4.5. *(Compare with* [4, Remark 4.27]*) Let P be a group, α a filter base on P and Ω an SC-variety. Then:*

(1)

$$\mathcal{L} := \Big\{ \bigcup_{p\in P} (pA_pp^{-1} \cup pA_p^{-1}p^{-1}) : \ A_p \in \alpha \ (p \in P) \Big\}$$

is also a filter base on P and if $\langle\alpha\rangle_\Omega$ exists then $\langle\alpha\rangle_\Omega = \langle\mathcal{L}\rangle_\Omega$.

(2) If, in addition, all topological groups belonging to Ω are balanced then

$$\mathcal{M} := \{ \bigcup_{p\in P} (pVp^{-1} \cup pV^{-1}p^{-1}) : V \in \alpha\}$$

is a filter base on P and if $\langle\alpha\rangle_\Omega$ exists then $\langle\alpha\rangle_\Omega = \langle\mathcal{M}\rangle_\Omega$.

Proof. (1) This follows from the fact that for every group topology τ on P such that $(P,\tau) \in \Omega$ the filter base α converges to e in (P,τ) if and only if $\mathcal{L}$ converges to e in (P,τ). Note that $\mathcal{L}$ satisfies the following properties:

(a) $\forall A \in \mathcal{L} \ \ A = A^{-1}$,
(b) $\forall A \in \mathcal{L}\ \forall p \in P\ \exists B \in \mathcal{L} \ \ pBp^{-1} \subseteq A$.

(2) This follows from the fact that for every group topology τ on P such that $(P,\tau) \in \Omega$ the filter base α converges to e in (P,τ) if and only if $\mathcal{M}$ converges to e in (P,τ). Note that $\mathcal{M}$ satisfies the following stronger properties:

(a*) $\forall A \in \mathcal{M}\ A = A^{-1}$,
(b*) $\forall A \in \mathcal{M}\ \ \exists B \in \mathcal{M}\ \forall p \in P\ pBp^{-1} \subseteq A$.

□

Lemma 4.6. *Let P be a group, α a filter base on P and Ω an SC-variety.*

(1) If $\Omega = \mathbf{NA}$ and α satisfies the following property:
(a) $\forall A \in \alpha\ \forall p \in P\ \exists B \in \alpha\ pBp^{-1} \subseteq A$,
then a base of $N_e(P, \langle\alpha\rangle_\Omega)$ is formed by the sets $[A]$, where $A \in \alpha$.

(2) If

$$\Omega \in \{\mathbf{NA}, \mathbf{NA_b}, \mathbf{AbNA}, \mathbf{BoolNA}\}$$

and α satisfies the stronger property
(a) $\forall A \in \alpha\ \exists B \in \alpha\ \forall p \in P\ \ pBp^{-1} \subseteq A$,*
then the sets $[A]$, where $A \in \alpha$, constitute a base of $N_e(P, \langle\alpha\rangle_\Omega)$.

(3) If $\Omega = \mathbf{NA} \cap \mathbf{Prec}$ and α satisfies property (a^*) of (2) then

$$\{N \lhd P |\ [P : N] < \infty \wedge \exists A \in \alpha\ [A] \subseteq N\}$$

is a local base at the identity element of $(P, \langle\alpha\rangle_\Omega)$.

(4) If $\Omega = \mathbf{SIN}$ and α satisfies the following properties:
(a) $\forall A \in \mathcal{M}\ A = A^{-1}$,*
(b) $\forall A \in \mathcal{M}\ \ \exists B \in \mathcal{M}\ \forall p \in P\ pBp^{-1} \subseteq A$,*
then a base of $N_e(P, \langle\alpha\rangle_\Omega)$ is formed by the sets $[(A_n)]$, where $\forall n \in \mathbb{N}\ A_n \in \alpha$.

Proof. (1) : Clearly $[A]^2 \subseteq [A]$ and $[A]^{-1} = [A]\ \forall A \in \alpha$. Moreover, for every $A \in \alpha$ and for every $p \in P$ there exists $B \in \alpha$ such that $p[B]p^{-1} \subseteq [A]$. Indeed, we can use property (a) to find $B \in \alpha$ such that $pBp^{-1} \subseteq A$. It follows that $p[B]p^{-1} \subseteq [A]$. This proves that there exists a non-archimedean group topology $\mathcal{T}$ such that

$$\{[A] : A \in \alpha\}$$

is a base of $N_e(P, \mathcal{T})$. Clearly, α converges to e with respect to $\mathcal{T}$, and therefore $\mathcal{T} \subseteq \langle\alpha\rangle_\Omega$.

Conversely, let σ be any non-archimedean group topology on P such that

$$\forall U \in N_e(P, \sigma)\ \exists A \in \alpha\ A \subseteq U.$$

To prove that $\sigma \subseteq \mathcal{T}$, let $U \in N_e(P, \sigma)$ be given. By Lemma 4.4 there exists V in $N_e(P, \sigma)$ such that $[V] \subseteq U$, and, by the assumption, there exists a set $A \in \alpha$ such that $A \subseteq V$. Consequently, $[A] \subseteq [V] \subseteq U$, which proves $\sigma \subseteq \mathcal{T}$.

(2) : The proof of the "balanced case" is quite similar. The only difference is the new condition

$$\forall A \in \alpha\ \exists B \in \alpha\ \forall p \in P\ \ pBp^{-1} \subseteq A,$$

which implies that the topology generated by the sets $[A]$ is also balanced.

(3) : Precompact case: clearly there exists a non-archimedean precompact group topology $\mathcal{T}$ on P such that

$$\{N \lhd P |\ [P : N] < \infty \wedge \exists A \in \alpha\ [A] \subseteq N\}$$

is a base of $N_e(P, \mathcal{T})$. It is also trivial to see that α converges to e with respect to $\mathcal{T}$. Then, $\mathcal{T}$ is coarser than $\langle\alpha\rangle_\Omega$. Let σ be any precompact non-archimedean group topology on P such that

$$\forall U \in N_e(P, \sigma)\ \exists A \in \alpha\ A \subseteq U.$$

To prove that $\sigma \subseteq \mathcal{T}$, let $N \in N_e(P, \sigma)$ be given. We can assume that N is a finite-index normal subgroup of P. By Lemma 4.4 there exists V in $N_e(P, \sigma)$ such that $[V] \subseteq U$, and, by the assumption, there exists a set $A \in \alpha$ such that $A \subseteq V$. Consequently, $[A] \subseteq [V] \subseteq N$ which proves $\sigma \subseteq \mathcal{T}$.

(4) : The proof is completely the same as the proof of [4, Proposition 4.28]. Observe that

from condition (b^*) it follows that the group topology, determined by the sets $[(A_n)]$, is also balanced. □

4.1. **The structure of the free NA topological groups.** Let $(X,\mathcal{U})$ be a non-archimedean uniform space, $Eq(\mathcal{U})$ be the set of equivalence relations from $\mathcal{U}$. Denote by j_2 the mapping $(x,y)\mapsto x^{-1}y$ from X^2 to either $F(X), A(X)$ or $B(X)$ and by j_2^* the mapping $(x,y)\mapsto xy^{-1}$.

Lemma 4.7. *Let $(X,\mathcal{U})$ be non-archimedean and let $\mathcal{B}\subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$.*

(1) The topology of $F_{\mathcal{NA}}$ is the strongest among all non-archimedean Hausdorff group topologies on $F(X)$ in which the filter base

$$\mathcal{F}=\{j_2(V)\cup j_2^*(V)|\ V\in\mathcal{B}\}$$

converges to e.

(2) For

$$\Omega\in\{\mathbf{NA_b},\mathbf{NA}\cap\mathbf{Prec},\mathbf{AbNA},\mathbf{BoolNA}\}$$

the topology of F_Ω is $\langle\mathcal{F}\rangle_\Omega$ where

$$\mathcal{F}=\{j_2(V)|\ V\in\mathcal{B}\}.$$

Proof. (1) : First recall that $F_{\mathcal{NA}}$ is algebraically the abstract free group $F(X)$ (see Theorem 3.9.2). Let τ be a non-archimedean group topology on $F(X)$. We show that $Id:(X,\mathcal{U})\to(F(X),\tau)$ is uniformly continuous if and only if $\mathcal{F}$ converges to e with respect to τ.

The map $Id:(X,\mathcal{U})\to(F(X),\tau)$ is uniformly continuous if and only if for every $U\in N_e(F(X),\tau)$ there exists $V\in\mathcal{B}$ such that

$$V\subseteq\tilde{U}=\{(x,y):x^{-1}y\in U\wedge xy^{-1}\in U\}.$$

The latter is equivalent to the following condition: there exists $V\in\mathcal{B}$ such that

$$j_2(V)\cup j_2^*(V)\subseteq U.$$

Thus, $Id:(X,\mathcal{U})\to(F(X),\tau)$ is uniformly continuous if and only if $\mathcal{F}$ converges to e with respect to τ. Clearly, the topology of $F_{\mathcal{NA}}$ is a non-archimedean Hausdorff group topology on $F(X)$ in which the filter base $\mathcal{F}$ converges to e. Moreover, for every non-archimedean Hausdorff group topology τ on $F(X)$ in which $\mathcal{F}$ converges to e, the map $Id:(X,\mathcal{U})\to(F(X),\tau)$ is uniformly continuous. Therefore $Id:F_{\mathcal{NA}}\to(F(X),\tau)$ is uniformly continuous. This completes the proof of (1).
(2) : The proof is very similar to the previous case. This time we can consider the filter base $\{j_2(V)|\ V\in\mathcal{B}\}$ instead of $\{j_2(V)\cup j_2^*(V)|\ V\in\mathcal{B}\}$ since all the groups F_Ω are balanced. □

Lemma 4.8. *Let $(X,\mathcal{U})$ be a uniform space. For $\Omega=\mathbf{SIN}$ the topology of F_Ω is $\langle\mathcal{F}\rangle_\Omega$ where*

$$\mathcal{F}=\{j_2(V)|\ V\in\mathcal{B}\}.$$

Proof. Use the same arguments as those appearing in the proof of Lemma 4.7.2. □

Definition 4.9. *(1) Following* [40], *for every $\psi\in\mathcal{U}^{F(X)}$ let*

$$V_\psi:=\bigcup_{w\in F(X)}w(j_2(\psi(w))\cup j_2^*(\psi(w)))w^{-1}.$$

(2) As a particular case in which every ψ is a constant function we obtain the set

$$\tilde{\varepsilon}:=\bigcup_{w\in F(X)}w(j_2(\varepsilon)\cup j_2^*(\varepsilon))w^{-1}.$$

Remark 4.10. Note that if $\varepsilon \in Eq(\mathcal{U})$ then $(j_2(\varepsilon))^{-1} = j_2(\varepsilon)$, $(j_2^*(\varepsilon))^{-1} = (j_2^*(\varepsilon))$ and

$$\tilde{\varepsilon} = \bigcup_{w \in F(X)} w(j_2(\varepsilon) \cup j_2^*(\varepsilon))w^{-1} = \bigcup_{w \in F(X)} wj_2(\varepsilon)w^{-1}.$$

Indeed, this follows from the equality $wts^{-1}w^{-1} = (ws)s^{-1}t(ws)^{-1}$.

Note also that the subgroup $[\tilde{\varepsilon}]$ (see Remark 4.3) generated by ε is normal in $F(X)$.

The proof of the following lemma is straightforward.

Lemma 4.11. *Let ε be an equivalent relation on a set X. Consider the function $f_\varepsilon : X \to X/\varepsilon$. Then $ker(\overline{f_\varepsilon}) = [\tilde{\varepsilon}]$, where $\overline{f_\varepsilon} : F(X) \to F(X/\varepsilon)$ is the induced onto homomorphism.*

Theorem 4.12. *Let $(X,\mathcal{U})$ be a uniform space. Then $\{[(\tilde{\varepsilon}_n)] : \ \varepsilon_n \in \mathcal{U} \ \forall n \in \mathbb{N}\}$ is a base of $N_e(F^b)$.*

Proof. By Lemma 4.8 the topology of F^b is $\langle \mathcal{F} \rangle_\Omega$ where

$$\mathcal{F} = \{j_2(\varepsilon) | \ \varepsilon \in \mathcal{B}\}$$

and Ω is the class of all balanced topological groups. According to Lemma 4.5.2 we have $\langle \mathcal{F} \rangle_\Omega = \langle \mathcal{M} \rangle_\Omega$, where

$$\mathcal{M} := \Big\{ \bigcup_{w \in F(X)} (wAw^{-1} \cup wA^{-1}w^{-1}) : \ A \in \mathcal{F} \Big\}.$$

In particular,

$$N_e(F^b) = N_e(F(X), \langle \mathcal{M} \rangle_\Omega).$$

By the description of the sets $\tilde{\varepsilon}$ in Definition 4.9.2 and Remark 4.10 we have $\mathcal{M} = \{\tilde{\varepsilon} : \varepsilon \in \mathcal{B}\}$. Finally, use Lemma 4.5.2 and Lemma 4.6.4 to complete the proof. ☐

Theorem 4.13. *Let $(X,\mathcal{U})$ be non-archimedean and let $\mathcal{B} \subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$. Then:*

(1) The family (of subgroups) $\{[\mathcal{V}_\psi] : \ \psi \in \mathcal{B}^{F(X)}\}$ is a base of $N_e(F_{\mathcal{NA}})$.
(2) (a) The family (of normal subgroups) $\{[\tilde{\varepsilon}] : \ \varepsilon \in \mathcal{B}\}$ is a base of $N_e(F^b_{\mathcal{NA}})$.
(b) The topology of $F^b_{\mathcal{NA}}$ is the weak topology generated by the system of homomorphisms $\{\overline{f_\varepsilon} : F(X) \to F(X/\varepsilon)\}_{\varepsilon \in \mathcal{B}}$ on discrete groups $F(X/\varepsilon)$.

Proof. (1) : By Lemma 4.7.1 the topology of $F_{\mathcal{NA}}$ is $\langle \mathcal{F} \rangle_\Omega$ where

$$\mathcal{F} = \{j_2(\varepsilon) \cup j_2^*(\varepsilon) | \ \varepsilon \in \mathcal{B}\}$$

and Ω is the class of all non-archimedean topological groups. According to Lemma 4.5.1 $\langle \mathcal{F} \rangle_\Omega = \langle \mathcal{L} \rangle_\Omega$ where

$$\mathcal{L} := \Big\{ \bigcup_{w \in F(X)} (wA_w w^{-1} \cup wA_w^{-1} w^{-1}) : \ A_w \in \mathcal{F}, \ \ w \in F(X) \Big\}.$$

In particular,

$$N_e(F_{\mathcal{NA}}) = N_e(F(X), \langle \mathcal{L} \rangle_\Omega).$$

By the description of the sets $\mathcal{V}_\psi$ in Definition 4.9.1 and Remark 4.10 we have

$$\mathcal{L} = \{\mathcal{V}_\psi : \ \psi \in \mathcal{B}^{F(X)}\}.$$

Finally, use Lemma 4.5.1 and Lemma 4.6.1 to conclude the proof.
$(2.a)$: By Lemma 4.7.2 the topology of $F^b_{\mathcal{NA}}$ is $\langle \mathcal{F} \rangle_\Omega$ where

$$\mathcal{F} = \{j_2(\varepsilon) | \ \varepsilon \in \mathcal{B}\}$$

and Ω is the class of all non-archimedean balanced topological groups. According to Lemma 4.5.2 $\langle\mathcal{F}\rangle_\Omega = \langle\mathcal{M}\rangle_\Omega$ where

$$\mathcal{M} := \Big\{ \bigcup_{w\in F(X)} (wAw^{-1} \cup wA^{-1}w^{-1}) : \ A \in \mathcal{F}\Big\}.$$

In particular,

$$N_e(F^b_{\mathcal{NA}}) = N_e(F(X), \langle\mathcal{M}\rangle_\Omega).$$

By the description of the sets $\tilde{\varepsilon}$ in Definition 4.9.2 and Remark 4.10 $\mathcal{M} = \{\tilde{\varepsilon} : \varepsilon \in \mathcal{B}\}$. Finally, use Lemma 4.5.2 and Lemma 4.6.2.

$(2.b)$: Use $(2.a)$ and observe that $ker(\overline{f_\varepsilon}) = [\tilde{\varepsilon}]$ by Lemma 4.11. □

Theorem 4.14. *Let $(X,\mathcal{U})$ be non-archimedean and let $\mathcal{B} \subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$.*

(1) (abelian case) For every $\varepsilon \in \mathcal{B}$ denote by $<\varepsilon>$ the subgroup of $A(X)$ algebraically generated by the set

$$\{x - y \in A(X) : (x, y) \in \varepsilon\},$$

then $\{<\varepsilon>\}_{\varepsilon\in\mathcal{B}}$ is a base of $N_0(A_{\mathcal{NA}})$.

(2) (Boolean case) If $<\varepsilon>$ denotes the subgroup of $B(X)$ algebraically generated by

$$\{x - y \in B(X) : (x, y) \in \varepsilon\},$$

then $\{<\varepsilon>\}_{\varepsilon\in\mathcal{B}}$ is a base of $N_0(B_{\mathcal{NA}})$.

Proof. (1) : By Lemma 4.7.2 the topology of $A_{\mathcal{NA}}$ is $\langle\mathcal{F}\rangle_{\mathcal{NA}}$ where

$$\mathcal{F} = \{j_2(\varepsilon)|\ \varepsilon \in \mathcal{B}\}.$$

Therefore,

$$N_0(A_{\mathcal{NA}}) = N_0(A(X), \langle\mathcal{F}\rangle_{\mathcal{NA}}).$$

By Remark 4.10 and Lemma 4.6.2 a base of $N_0(A(X), \langle\mathcal{F}\rangle_{\mathcal{NA}})$ is formed by the sets $[j_2(\varepsilon)]$, where $\varepsilon \in \mathcal{B}$ and $[j_2(\varepsilon)]$ is the subgroup generated by $j_2(\varepsilon)$. Since ε is symmetric we have

$$j_2(\varepsilon) = \{y - x \in A(X) : (x, y) \in \varepsilon\} = \{x - y \in A(X) : (x, y) \in \varepsilon\}.$$

The proof of (2) is similar. □

Remark 4.15. Note that the system $\mathcal{B}$ in Theorem 4.13.2 induces a naturally defined inverse limit $\varprojlim_{\varepsilon\in\mathcal{B}} F(X/\varepsilon)$ (of discrete groups $F(X/\varepsilon)$) which can be identified with the complete group $\widehat{F^b_{\mathcal{NA}}}$. Similarly, $\varprojlim_{\varepsilon\in\mathcal{B}} A(X/\varepsilon)$ and $\varprojlim_{\varepsilon\in\mathcal{B}} B(X/\varepsilon)$ can be identified with the groups $\widehat{A_{\mathcal{NA}}}$ and $\widehat{B_{\mathcal{NA}}}$ respectively.

Theorem 4.16. *Let $X := (X,\mathcal{U})$ be a Hausdorff non-archimedean space and*

$$G \in \{F^b_{\mathcal{NA}}(X), A_{\mathcal{NA}}(X), B_{\mathcal{NA}}(X)\}.$$

Then

(1) $\chi(G) = w(\mathcal{U})$ and $w(G) = w(\mathcal{U}) \cdot l(\mathcal{U})$.

(2) If (X, d) is an ultra-metric space then G is an ultra-normable group of the same topological weight as X.

Proof. (1) One may assume in Theorems 4.13.2 and 4.14 that $\mathcal{B} \subseteq Eq(\mathcal{U})$ is a base of $\mathcal{U}$ of cardinality $w(\mathcal{U})$. This explains $\chi(G) = w(\mathcal{U})$. Since $l^u(G) = l(\mathcal{U})$ (Lemma 2.3.3) we can conclude by Lemma 2.3.1 that $w(G) = w(\mathcal{U}) \cdot l(\mathcal{U})$.

(2) Combine (1) and Lemma 2.9. □

Remark 4.17. (1) Note that Pestov showed (see [40]) that the set $\{[(V_{\psi_n})]\}$, where $\{\psi_n\}$ extends over the family of all possible sequences of elements from $\mathcal{U}^{F(X)}$, is a base of $N_e(F(X,\mathcal{U}))$.

(2) Considering only the sequences of constant functions, we obtain the set $\{[(\tilde{\varepsilon}_n)] : \varepsilon_n \in \mathcal{U}\ \forall n \in \mathbb{N}\}$ which is a base of $N_e(F^b(X,\mathcal{U}))$ by Theorem 4.12.

(3) Let $(X,\mathcal{U})$ be a non-archimedean uniform space. One may take in (1) only the constant sequences and obtain the set $\{[\mathcal{V}_\psi] : \psi \in \mathcal{U}^{F(X)}\}$ which is a base of $N_e(F_{\mathcal{NA}})$ by Theorem 4.13.1.

(4) If, in addition, the functions ψ are all constant we obtain a base of $N_e(F^b_{\mathcal{NA}})$ (see Theorem 4.13.2).

(5) **NA** is closed under products and subgroups. So **NA** is a *reflective subcategory* (see for example, [54, Section 9]) of **TGr**. For every topological group G there exists a universal arrow $f : G \to r_{\mathcal{NA}}(G)$, where $r_{\mathcal{NA}}(G) \in$ **NA**. For every uniform space $(X,\mathcal{U})$ the group $F_{\mathcal{NA}}(X,\mathcal{U})$ is in fact $r_{\mathcal{NA}}(G)$, where $G := F(X,\mathcal{U})$.

4.2. **Noncompleteness of $A_{\mathcal{NA}}(X,\mathcal{U})$.** In this subsection we show that $A_{\mathcal{NA}}(X,\mathcal{U})$ is never complete for non-discrete $\mathcal{U}$.

Definition 4.18. *(1) Let $w = \sum_{i=1}^{n} k_i x_i$ be a nonzero element of $A(X)$, where $n \in \mathbb{N}$, and for all $1 \le i \le n :\ x_i \in X$ and $k_i \in \mathbb{Z} \setminus \{0\}$. Define the length of w to be $\sum_{i=1}^{n} |k_i|$ and denote it by $lh(w)$.*

(2) The length of the zero element is 0.

(3) For a non-negative integer n we denote by B_n the subset of $A(X)$ consisting of all words of length $\le n$.

Lemma 4.19. *For every $n \in \mathbb{N}$ the set B_n is closed in $A_{\mathcal{NA}}$.*

Proof. It suffices to show that for every word w of length $> n$ there exists $\varepsilon \in \mathcal{U}$ such that $w+ < \varepsilon > \cap B_n = \emptyset$. Since $\mathcal{U}$ is Hausdorff there exists $\varepsilon \in \mathcal{U}$ such that for every $x \neq y \in supp(w)$ we have $(x,y) \notin \varepsilon$. It follows that for every $(x,y) \in \varepsilon$ we have either $lh(w+(x-y)) = lh(w)$ or $lh(w+(x-y)) = lh(w)+1$. Therefore, $w+ < \varepsilon > \cap B_n = \emptyset$. □

Lemma 4.20. *Let $(X,\mathcal{U})$ be a non-archimedean non-discrete uniform space. Then, for every $n \in \mathbb{N}$, $int(B_n) = \emptyset$.*

Proof. Let $w \in B_n$. Since $(X,\mathcal{U})$ is non-discrete and Hausdorff, every symmetric entourage $\varepsilon \in \mathcal{U}$ contains infinitely many elements of the form (x,y), where $x \neq y$. It follows that there exists $(x,y) \in \varepsilon$ such that $x \notin supp(w)$. Now, if $y \notin supp(w)$ then $lh(w+x-y) = lh(w)+2$. Otherwise, we have $lh(w+x-y) = lh(w)+2$ or $lh(w+y-x) = lh(w)+2$, and either one of these cases implies that $w+ < \varepsilon > \not\subseteq B_n$. Therefore, by Theorem 4.14.1, $int(B_n) = \emptyset$. □

Theorem 4.21. *Let $(X,\mathcal{U})$ be a non-archimedean metrizable non-discrete uniform space. Then $A_{\mathcal{NA}}$ is not complete.*

Proof. By Theorem 4.16 $A_{\mathcal{NA}}$ is metrizable. This group is indeed non-complete. Otherwise, by Baire Category Theorem $A_{\mathcal{NA}}$ is not the countable union of nowhere-dense closed sets. This contradicts the fact that $A_{\mathcal{NA}} = \cup_{n\in\mathbb{N}} B_n$, where the sets B_n are nowhere-dense and closed (see Lemmas 4.19 and 4.20). □

As a contrast recall that $A(X,\mathcal{U})$ is complete for every complete uniform space $(X,\mathcal{U})$ (which for $\mathcal{U} = \mathcal{U}_{max}$ gives Tkachenko-Uspenskij theorem). See [2, page 497].

5. Free profinite groups

The free profinite groups (in several subclasses Ω of **Pro**) play a major role in several applications [17, 46, 14]. By Lemma 3.5 the free profinite group F_{Pro} can be identified with the completion $\widehat{F_{\mathcal{NA}}^{Prec}}$ of the free precompact **NA** group $F_{\mathcal{NA}}^{Prec}$. Its description comes from the following result which for Stone spaces is a version of a known result in the theory of profinite groups. See for example [46, Prop. 3.3.2].

Theorem 5.1. *Let $(X,\mathcal{U})$ be a non-archimedean precompact uniform space and let $\mathcal{B} \subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$ of cardinality $w(\mathcal{U})$. Then:*

1. *The set $S := \{H \lhd F(X) : \ [F(X) : H] < \infty, \ \exists\ \varepsilon \in \mathcal{B} \quad [\tilde{\varepsilon}] \subseteq H\}$ is a local base at the identity of $F_{\mathcal{NA}}^{Prec}$.*
2. *Let $G \in \{F_{Pro}, F_{\mathcal{NA}}^{Prec}\}$. Then $\chi(G) = w(G) = w(\mathcal{U}) = w(X)$. In particular, G is metrizable for every metrizable $\mathcal{U}$.*

Proof. (1) : By Lemma 4.7.2 the topology of $F_{\mathcal{NA}}^{Prec}$ is $\langle \mathcal{F} \rangle_\Omega$ where

$$\mathcal{F} = \{j_2(\varepsilon)|\ \varepsilon \in \mathcal{B}\}$$

and $\Omega = \mathbf{NA} \cap \mathbf{Prec}$. According to Lemma 4.5.2 the latter coincides with $\langle \mathcal{M} \rangle_\Omega$ where

$$\mathcal{M} := \Big\{ \bigcup_{w \in F(X)} (wAw^{-1} \cup wA^{-1}w^{-1}) : \ A \in \mathcal{F} \Big\}.$$

In particular,

$$N_e(F_{\mathcal{NA}}^{Prec}) = N_e(F(X), \langle \mathcal{M} \rangle_\Omega).$$

By the description of the sets $\tilde{\varepsilon}$ in Definition 4.9.2 and Remark 4.10, $\mathcal{M} = \{\tilde{\varepsilon} : \varepsilon \in \mathcal{B}\}$. Finally, use Lemmas 4.5.2 and 4.6.3.
(2) : Since U and G are precompact we have $w(\mathcal{U}) = w(X)$ and $\chi(G) = w(G)$. So we have only to show that $\chi(G) = w(\mathcal{U})$. Assume that $|\mathcal{B}| = w(\mathcal{U})$. By the description of S in (1), it suffices to show that for every $\varepsilon \in B$ there are countably many normal finite-index subgroups of $F(X)$ containing $[\tilde{\varepsilon}]$. Consider the function $f_\varepsilon : X \to X/\varepsilon$. Note that $F(X/\varepsilon)$ is a free group with finite number of generators. It is well known that the set of all (normal) finite-index subgroups of $F(X/\varepsilon)$ is countable. By the Correspondence Theorem there are countably many normal finite-index subgroups of $F(X)$ containing $ker(\overline{f_\varepsilon})$, where $\overline{f_\varepsilon} : F(X) \to F(X/\varepsilon)$ is the induced onto homomorphism. Now in order to complete the proof recall that $ker(\overline{f_\varepsilon}) = [\tilde{\varepsilon}]$ by Lemma 4.11. □

Let Ω be an $\overline{S}C$-variety of groups. Following [54] let us say that two compact spaces X and Y are Ω-*equivalent* if their Ω-free groups $F_\Omega(X)$ and $F_\Omega(Y)$ are topologically isomorphic. Notation: $X \cong_\Omega Y$. In particular, we have the classical concepts of M-equivalent (in the honor of Markov) and A-equivalent compact spaces (for $\Omega = \mathbf{TGr}$ and $\Omega = \mathbf{AbGr}$, respectively). For free compact (abelian) groups and the corresponding equivalence see [21].

Similarly, we get the concepts of **NA**-equivalent, **AbNA**-equivalent and **Pro**-equivalent compact spaces. The **Pro**-equivalence is very rigid as the following remark demonstrates.

Remark 5.2. From Melnikov's result (see [46, Proposition 3.5.12]) it follows that every free profinite group on a compact infinite Stone space X is isomorphic to the free profinite group of the 1-point compactification of a discrete space with cardinality $w(X)$. So two infinite Stone spaces X and Y are **Pro**-equivalent if and only if $w(X) = w(Y)$. This implies that there are **Pro**-equivalent compact spaces which are not M or A-equivalent.

Note that if X is the converging sequence space and Y is the Cantor set then $X \cong_{\mathbf{Pro}} Y$ by Remark 5.2. On the other hand, $X \not\cong_\Omega Y$, where $\Omega = \mathbf{NA} \cap \mathbf{Prec}$, because $F_{\mathcal{NA}}^{Prec}(X)$

is countable in contrast to $F^{Prec}_{\mathcal{NA}}(Y)$. It would be interesting to compare Ω-equivalences on Stone spaces (with the same weight and cardinality) for different subclasses Ω of **NA**.

5.1. **The Heisenberg group associated to a Stone space and free Boolean profinite groups.** To every Stone space X we associate in [35] the natural biadditive mapping

$$w : C(X, \mathbb{Z}_2) \times C(X, \mathbb{Z}_2)^* \to \mathbb{Z}_2 \tag{1}$$

Where $V := C(X, \mathbb{Z}_2)$ can be identified with the discrete group (with respect to symmetric difference) of all clopen subsets in X. Denote by $V^* := \hom(V, \mathbb{T})$ the Pontryagin dual of V. Since V is a Boolean group every character $V \to \mathbb{T}$ can be identified with a homomorphism into the unique 2-element subgroup $\Omega_2 = \{1, -1\}$, a copy of $\mathbb{Z}_2$. The same is true for the characters on V^*, hence the natural evaluation map $w : V \times V^* \to \mathbb{T}$ ($w(x, f) = f(x)$) can be restricted naturally to $V \times V^* \to \mathbb{Z}_2$. Under this identification $V^* := \hom(V, \mathbb{Z}_2)$ is a closed subgroup of the compact group $\mathbb{Z}_2^V$. In particular, V^* is a Boolean profinite group. Similar arguments show that, in general, any Boolean profinite group G is the Pontryagin dual of the discrete Boolean group G^*.

We prove in [35] that for every Stone space X the associated Heisenberg type group $H = (\mathbb{Z}_2 \times V) \leftthreetimes V^*$ is always minimal.

This setting has some additional interesting properties. Note that the natural evaluation map

$$\delta : X \to V^*, \ x \mapsto \delta_x, \quad \delta_x(f) = f(x)$$

is a topological embedding into V^*, where $w(V^*) = w(X)$. Moreover, if X is a G-space then the induced action of G on V^* is continuous and δ is a G-embedding.

The set $\delta(X)$ separates points of V via the biadditive mapping w in (1). Hence the subgroup generated by $\delta(X)$ in V^* is dense. We can say more using additional properties of Pontryagin duality. We thank M. Jibladze and D. Pataraya for the following observation (presented here after some simplifications).

Remark 5.3. (M. Jibladze and D. Pataraya) The Boolean profinite group V^* together with $\delta : X \to V^*$ in fact is the free Boolean profinite group $B_{Pro}(X)$ of X. In order to see this let $f : X \to G$ be a continuous homomorphism into a Boolean profinite group G. Then G is the Pontryagin dual of a discrete Boolean group H. That is, $G = H^*$. Now consider the natural inclusion

$$\nu : G^* = \hom(G, \mathbb{Z}_2) \hookrightarrow V = C(X, \mathbb{Z}_2).$$

Its dual arrow $\nu^* : V^* \to G^{**}$ can be identified with $\nu^* : V^* \to G = G^{**}$ such that $\nu^* \circ \delta = f$. Such extension ν^* of f is uniquely defined because the subgroup generated by $\delta(X) \subset V^*$ is dense.

6. Surjectively universal groups

We already proved in Theorem 4.16 that for metrizable uniformities the corresponding free balanced, free abelian and free Boolean non-archimedean groups are also metrizable. The same is true by Theorem 5.1 for the free profinite group which can be treated as the free compact **NA** group over a uniform space. These results allow us to unify and strengthen some old and recent results about the existence and the structure of surjectively universal **NA** groups.

Let Ω be a class of topological groups. We say that a topological group G is *surjectively universal* (or, *co-universal*) in the class Ω if $G \in \Omega$ and every $H \in \Omega$ is isomorphic to a topological factor group of G.

Remark 6.1. We list some natural classes Ω containing surjectively universal groups:

(1) (Ding [9]) Polish groups. This result answers a long standing question of Kechris.

(2) (Shakhmatov-Pelant-Watson[53])
   (a) The class of all abelian Polish groups. More generally the class of all abelian complete groups with weight $\leq \kappa$.
   (b) The class of all balanced metrizable complete groups with weight $\leq \kappa$.
(3) (See Gao [15] for (a),(b),(c) and also Gao-Xuan [16] for (b),(c))
   (a) **NA** Polish groups.
   (b) **NA** abelian Polish groups.
   (c) **NA** balanced Polish groups.
(4) (Gildenhuys-Lim [17, Lemma 1.11] (see also [46, Thm 3.3.16])) Profinite groups of weight $\leq \kappa$.

Lemma 6.2. *The Baire space $B(\kappa) = (\kappa^{\aleph_0}, \mathcal{U})$ is co-universal in the class of all completely metrizable non-archimedean uniform spaces with topological weight $\leq \kappa$.*

Proof. (First proof) By Shakhmatov-Pelant-Watson [53] there exists a Lipshitz-1 onto open (hence, quotient) map $B(\kappa) \to (X, d)$ for every complete bounded metric space (X, d) with topological weight $\leq \kappa$.

(Second proof) By Ellis [11] for every complete ultra-metric space X and its closed subspace Y there exists a uniformly continuous retraction $r : X \to Y$ (which necessarily is a quotient map by Lemma 6.3). □

Lemma 6.3. *(see for example* [12, Cor. 2.4.5]*) If the composition $f_2 \circ f_1 : X \to Z$ of continuous maps $f_1 : X \to Y$ and $f_2 : Y \to Z$ is a quotient map, then $f_2 : Y \to Z$ is a quotient map.*

Theorem 6.4. *(1) $\widehat{F^b}_{\mathcal{NA}}(\kappa^{\aleph_0}, \mathcal{U})$ is surjectively universal in the class of all balanced* **NA** *metrizable complete groups with weight $\leq \kappa$.*
(2) $\widehat{A}_{\mathcal{NA}}(\kappa^{\aleph_0}, \mathcal{U})$ is surjectively universal in the class of all abelian **NA** *metrizable complete groups with weight $\leq \kappa$.*
(3) $\widehat{B}_{\mathcal{NA}}(\kappa^{\aleph_0}, \mathcal{U})$ is surjectively universal in the class of all Boolean **NA** *metrizable complete groups with weight $\leq \kappa$.*

Proof. We give a proof only for (1) because cases (2) and (3) are very similar.

By Theorem 4.16, $F^b{}_{\mathcal{NA}} := F^b{}_{\mathcal{NA}}(\kappa^{\aleph_0}, \mathcal{U})$ is metrizable. Hence, $\widehat{F^b}_{\mathcal{NA}}$ is metrizable, too. Furthermore, the topological weight of $\widehat{F^b}_{\mathcal{NA}}$ is κ. Indeed, $\kappa^{\aleph_0}$ topologically generates $\widehat{F^b}_{\mathcal{NA}}$. So, by Lemma 2.3, we get

$$w(\widehat{F^b}_{\mathcal{NA}}) = \chi(\widehat{F^b}_{\mathcal{NA}}) \cdot l^u(\widehat{F^b}_{\mathcal{NA}}) = \aleph_0 \cdot l^u(\kappa^{\aleph_0}, \mathcal{U}) = \kappa.$$

Let P be a balanced **NA** metrizable complete group with topological weight $\leq \kappa$. Then its uniformity ρ is both complete (by definition) and non-archimedean (by Fact 2.6.2). By Lemma 6.2 there exists a uniformly continuous onto map $f : \kappa^{\aleph_0} \to (P, \rho)$ which is a quotient map of topological spaces. By the universal property of $\widehat{F^b}_{\mathcal{NA}}$ (Lemma 3.5) there exists a unique continuous homomorphism $\widehat{f} : \widehat{F^b}_{\mathcal{NA}} \to P$ which extends f. That is, $f = \widehat{f} \circ i$, where $i : \kappa^{\aleph_0} \to \widehat{F^b}_{\mathcal{NA}}$ is the universal arrow. Since $f : \kappa^{\aleph_0} \to P$ is a quotient map we obtain by Lemma 6.3 that $\widehat{f}$ is a quotient map. □

If in (1) and (2) we assume that $\kappa = \aleph_0$ then we in fact deal with Polish groups. In this case we get very short proofs of the results mentioned in Remark 6.1 (assertions 3.b and 3.c). A new point in this particular case is that the corresponding universal groups come directly as the free objects. Indeed, recall that by Lemma 3.5 the complete groups $\widehat{F^b}_{\mathcal{NA}}, \widehat{A}_{\mathcal{NA}}, \widehat{B}_{\mathcal{NA}}$ in Theorem 6.4 are Ω_C-free groups for the corresponding classes.

Result (3) in Theorem 6.4 seems to be new.

Question 6.5. *Let $\kappa > \omega$. Is it true that there exists a* ***co-universal*** *space in the class of all complete non-archimedean uniform spaces with $dw(X,\mathcal{U}) \leq (\kappa, \kappa)$?*

A positive solution for Question 6.5 will imply, by Theorem 4.16 and the approach of Theorem 6.4, that there exists a co-universal group in the class of all non-archimedean balanced (abelian, Boolean) groups with topological weight $\leq \kappa$.

The following theorem is known in the theory of profinite groups. Here we provide a very short proof of the existence of surjectively universal profinite groups of weight $\leq \mathfrak{m}$ using Hulanicki's theorem. The first assertion can be derived from [17, Lemma 1.11] (or [46, Thm 3.3.16]). Its version for the case $\mathfrak{m} = \aleph_0$ goes back to Iwasawa. This case was proved also in [16].

Theorem 6.6. *(1) For every infinite cardinal $\mathfrak{m}$ there exists a surjectively universal group in the class $\mathbf{Pro}_{\mathfrak{m}}$ of all profinite groups of weight $\leq \mathfrak{m}$.*

(2) Every free profinite group $F_{Pro}(X)$ over any infinite Stone space X of weight $\mathfrak{m}$ is a surjectively universal group in the class $\mathbf{Pro}_{\mathfrak{m}}$.

Proof. (1) Let $X = \{0,1\}^{\mathfrak{m}}$. By Theorem 5.1.3, $w(F_{Pro}(X)) = w(X)$. So, $F_{Pro}(X) \in \mathbf{Pro}_{\mathfrak{m}}$. By the universal property of $F_{Pro}(X)$ it is enough to show that any $G \in \mathbf{Pro}_{\mathfrak{m}}$ is a continuous image of X. It is obvious for finite G. By a theorem of Hulanicki (see [20, Thm 9.15]) every infinite $G \in \mathbf{Pro}_{\mathfrak{m}}$ is homeomorphic to $\{0,1\}^{\chi(G)} = \{0,1\}^{w(G)}$. Since $w(G) \leq \mathfrak{m}$, there exists a continuous onto map $\phi : \{0,1\}^{\mathfrak{m}} \to G$.

(2) See Remark 5.2. □

7. Automorphizable actions and epimorphisms in topological groups

Resolving a longstanding principal problem introduced by K. Hofmann, Uspenskij [59] showed that in the category of Hausdorff topological groups epimorphisms need not have a dense range. Dikranjan and Tholen [5] gave a rather direct proof of this important result of Uspenskij. Pestov gave a useful criterion [42, 44] (Fact 7.1) which we use below in Theorem 7.8. This test is closely related to the concept of the *free topological G-group* of a (uniform) G-space X introduced in [31]. We denote it by $F_G(X)$. It is a natural G-space version of the usual free topological group. Similarly to Definition 3.1 one may define *Ω-free (uniform) G-group* $F_{G,\Omega}(X,\mathcal{U})$.

A topological (uniform) G-space X is said to be *automorphizable* if X is a topological (uniform) G-subspace of a G-group Y (with its two-sided uniform structure). Equivalently, if the universal morphism $X \to F_G(X)$ of X into the free topological (uniform) G-group $F_G(X)$ of the (uniform) G-space X is an embedding.

Fact 7.1. (Pestov [42, 44]) *Let $f : M \to G$ be a continuous homomorphism between Hausdorff topological groups. Denote by $X := G/H$ the left coset G-space, where H is the closure of the subgroup $f(M)$ in G. The following are equivalent:*

(1) $f : M \to G$ is an epimorphism.

(2) The free topological G-group $F_G(X)$ of the G-space X is trivial.

Triviality in (2) means 'as trivial as possible', that is, $F_G(X)$ is isomorphic to the cyclic discrete group.

Remark 7.2. Let X be the n-dimensional cube $[0,1]^n$ or the n-dimensional sphere $\mathbb{S}_n$. Then by [31] the free topological G-group $F_G(X)$ of the G-space X is trivial for every $n \in \mathbb{N}$, where $G = \mathrm{Homeo}\,(X)$ is the corresponding homeomorphism group. So, one of the possible examples of an epimorphism which is not dense can be constructed as the natural embedding $H \hookrightarrow G$ where $G = \mathrm{Homeo}\,(\mathbb{S}_1)$ and $H = G_z$ is the stabilizer of a point $z \in \mathbb{S}_1$. The same example serves as the original counterexample to the epimorphism problem in the paper of Uspenskij [59].

In contrast, for Stone spaces we have:

Fact 7.3. [35, Lemma 4.3.2] *Every continuous action of a topological group G on a Stone space X is automorphizable (in* **NA***). Hence the canonical G-map $X \to F_G(X)$ is an embedding.*

Roughly speaking the action by conjugations of a subgroup H of a non-archimedean group G on G reflects all possible difficulties of the Stone actions. Below, in Theorem 8.2, we extend Fact 7.3 to a much larger class of actions on non-archimedean uniform spaces, where X need not be compact. This will be used in Theorem 7.8 which deals with epimorphisms into **NA**-groups.

Definition 7.4. [30, 32] *Let $\pi : G \times X \to X$ be an action and $\mathcal{U}$ be a uniformity on X. We say that the action (or, X) is* quasibounded *if for every $\varepsilon \in \mathcal{U}$ there exist: $\delta \in \mathcal{U}$ and a neighborhood O of e in G such that*

$$(gx, gy) \in \varepsilon \quad \forall\ (x, y) \in \delta,\ g \in O.$$

We say that the action on the uniform space $(X,\mathcal{U})$ is π-uniform *if the action is quasibounded and all g-translations are $\mathcal{U}$-uniformly continuous. Equivalently, for every $\varepsilon \in \mathcal{U}$ and $g_0 \in G$ there exist: $\delta \in \mathcal{U}$ and a neighborhood O of g_0 in G such that*

$$(gx, gy) \in \varepsilon \quad \forall\ (x, y) \in \delta,\ g \in O.$$

For a given topological group G denote by Unif^G the triples $(X,\mathcal{U},\pi)$ where $(X,\mathcal{U})$ is a uniform space and $\pi : G \times X \to X$ is a continuous π-uniform action.

It is an easy observation that if the action $\pi : G \times X \to X$ is $\mathcal{U}$-quasibounded and the orbit maps $\tilde{x} : G \to X$ are continuous then π is continuous.

It is a remarkable fact that a topological G-space X is *G-compactifiable* if and only if X is $\mathcal{U}$-quasibounded with respect to some compatible uniformity $\mathcal{U}$ on X, [30, 31]. This was the main motivation to introduce quasibounded actions. This concept gives a simultaneous generalization of some important classes of actions on uniform spaces.

Fact 7.5. [30, 31] *We list here some examples of actions from* Unif^G*.*

(1) Continuous isometric actions of G on metric spaces.
(2) $\mathrm{Comp}^G \subset \mathrm{Unif}^G$*. Continuous actions on compact spaces (with their unique compatible uniformity).*
(3) $(G/H,\mathcal{U}_r) \in \mathrm{Unif}^G$*. Let $X = G/H$ be the coset G-space and $\mathcal{U}_r$ is the right uniformity on X (which is always compatible with the topology).*
(4) Let X be a G-group. Then $(X,\mathcal{U}) \in \mathrm{Unif}^G$ for every $\mathcal{U} \in \{\mathcal{U}_r,\mathcal{U}_l,\mathcal{U}_{l\vee r},\mathcal{U}_{l\wedge r}\}$.

Recall that the well known Arens-Eells linearization theorem (cf. [1]) asserts that every uniform (metric) space can be (isometrically) embedded into a locally convex vector space (resp., normed space). This theorem on isometric linearization of metric spaces can be naturally extended to the case of non-expansive semigroup actions provided that the metric is bounded [33], or, assuming only that the orbits are bounded [52].

Furthermore, suppose that an action of a *group* G on a metric space (X, d) is only π-uniform in the sense of Definition 7.4 (and not necessarily non-expansive). Then again such an action admits an isometric G-linearization on a normed space.

Here we give a non-archimedean G-version of Arens-Eells type theorem for uniform group actions. Note that we will establish an ultra-metric G-version in Theorem 8.2 below. The assumption $(X,\mathcal{U}) \in \mathrm{Unif}^G$ in Theorems 7.6 and 8.2 is necessary by Fact 7.5.4.

Theorem 7.6. *Let $\pi : G \times X \to X$ be a continuous action of a topological group G on a non-archimedean uniform space $(X,\mathcal{U})$. If $(X,\mathcal{U}) \in \mathrm{Unif}^G$ then the induced action by automorphisms*

$$\overline{\pi} : G \times B_{\mathcal{NA}}(X) \to B_{\mathcal{NA}}(X),\ (g,u) \mapsto gu$$

is continuous and $(X,\mathcal{U})$ is a uniform G-subspace of $B_{\mathcal{NA}}(X)$. Hence, $(X,\mathcal{U})$ is uniformly G-automorphizable (in **NA***).*

Proof. By Theorem 4.14.2 $\{< \varepsilon >\}_{\varepsilon \in Eq(\mathcal{U})}$ is a base of $N_0(B_{\mathcal{NA}})$. By Theorem 3.9, $(X,\mathcal{U})$ is a uniform subspace of $B_{\mathcal{NA}}$. It is easy to see that $(X,\mathcal{U})$ is in fact a uniform G-subspace of $B_{\mathcal{NA}}(X)$. We show now that the action $\overline{\pi}$ of G on $B_{\mathcal{NA}}(X)$ is quasibounded and continuous. The original action on $(X,\mathcal{U})$ is π-quasibounded and continuous. Thus, for every $\varepsilon \in Eq(\mathcal{U})$ and $g_0 \in G$, there exist: $\delta \in Eq(\mathcal{U})$ and a neighborhood $O(g_0)$ of g_0 in G such that for every $(x,y) \in \delta$ and for every $g \in O$ we have

$$(gx, gy) \in \varepsilon.$$

This implies that

$$g < \delta > \subseteq < \varepsilon > \ \ \forall g \in O,$$

which proves that $\overline{\pi}$ is quasibounded. The map $\iota : X \hookrightarrow B_{\mathcal{NA}}(X),\ x \mapsto \{x\}$ is a topological G-embedding. Together with the fact that $\iota(X)$ algebraically spans $B_{\mathcal{NA}}(X)$ this implies that the orbit mappings $G \to B_{\mathcal{NA}}(X),\ g \mapsto gu$ are continuous for all $u \in B_{\mathcal{NA}}(X)$. So we can conclude that $\overline{\pi}$ is continuous (see the remark after Definition 7.4) and $B_{\mathcal{NA}}(X)$ is a G-group. □

Remark 7.7. Let (X,μ) be a non-archimedean uniform space and $\pi : G \times X \to X$ be a continuous action such that $(X,\mu) \in \mathrm{Unif}^G$. The lifted action of G on $B_{\mathcal{NA}}$ is continuous as we proved in Theorem 7.6. This implies that $B_{\mathcal{NA}}$ is the *free topological G-group* of $(X,\mathcal{U})$ in the class Ω of non-archimedean Boolean G-groups. Similarly, one may verify that the same remains true for $F^b_{\mathcal{NA}}$, $A_{\mathcal{NA}}$, $F^{Prec}_{\mathcal{NA}}$, F_{Pro}. The case of $F_{\mathcal{NA}}(X,\mathcal{U})$, however, is unclear.

Recall that the sets

$$\tilde{U} := \{(aH, bH) : bH \subseteq UaH\},$$

where U runs over the neighborhoods of e in G, form a uniformity base on G/H. This uniformity $\mathcal{U}_r$ (called the right uniformity) is compatible with the quotient topology (see, for instance, [4]).

Theorem 7.8. *Let $f : M \to G$ be an epimorphism in the category* **TGr***. Denote by H the closure of the subgroup $f(M)$ in G. Then each of the following conditions implies that $f(M)$ is dense in G.*

(1) The coset uniform space $(G/H,\mathcal{U}_r)$ is non-archimedean.
(2) $G \in$ **NA***.*
(3) (T.H. Fay [13]) *H is open in G.*

Proof. (1) We have to show that $H = G$. Assuming the contrary consider the *nontrivial* Hausdorff coset G-space G/H. By Fact 7.5.3 the natural continuous left action $\pi : G \times G/H \to G/H$ is π-uniform. Hence, we can apply Theorem 7.6 to conclude that the nontrivial G-space $X := G/H$ is G-automorphizable in **NA**. In particular, we obtain that there exists a *nontrivial* equivariant morphism of the G-space X to a Hausdorff G-group E. This implies that the free topological G-group $F_G(X)$ of the G-space X is not trivial. By the criterion of Pestov (Fact 7.1) we conclude that $f : M \to G$ is not an epimorphism.

(2) By (1) it is enough to show that the right uniformity on G/H is non-archimedean. Since G is **NA** there exists a local base $\mathcal{B}$ at e consisting of clopen subgroups. Then it is straightforward to show that $\tilde{\mathcal{B}} := \{\tilde{U} : U \in \mathcal{B}\}$ is a base of the right uniformity of G/H such that its elements are equivalence relations on G/H.

(3) If H is open then $X = G/H$ is topologically discrete. The discrete uniformity $\mathcal{U}$ is compatible and we have $(X, \mathcal{U}) \in \mathrm{Unif}^G$. As in the proof of (1) we apply Theorem 7.6 and Fact 7.1. □

Assertion (3) is just the theorem of Fay [13] mentioned above in Subsection 1.4. In assertion (1) it suffices to assume that the universal non-archimedean image of the coset uniform space G/H is nontrivial.

Note that if $H \to G$ is not an epimorphism in **NA** then, a fortiori, it is not an epimorphism in **TGr**. With a bit more work we can refine assertion (2) of Theorem 7.8 as follows.

Theorem 7.9. *Morphism $f : M \to G$ in the category* **NA** *is an epimorphism in* **NA** *(if and) only if $f(M)$ is dense in G.*

Proof. Assume that $X := G/H$ is non-trivial where $H := cl(f(M))$. It is enough to show that there exists a **NA** group P and a pair of distinct morphisms $g, h : G \to P$ such that $g \circ f = h \circ f$. Theorem 7.6 says not only that the (nontrivial) G-space $X = G/H$ is G-automorphizable but also that it is G-automorphizable in **NA**. By Theorem 7.6, $B_{\mathcal{NA}}(X) \in$ **NA** is a G-group. Now choose the desired P as the corresponding semidirect product of $B_{\mathcal{NA}}(X)$ and G. Since **NA** is closed under semidirect products we obtain that $P \in$ **NA**. According to the approach of [42] there exists a pair of distinct morphisms $g, h : G \to P$ such that $g \circ f = h \circ f$. □

8. Group actions on ultra-metric spaces and Graev type ultra-norms

Lemma 8.1. *Let $f : X \to \mathbb{R}$ be a function on an ultra-metric space (X, d). There exists a one-point ultra-metric extension $X \cup \{b\}$ of X such that f is the distance from b if and only if*

$$|f(x) - f(y)| \leq d(x, y) \leq max\{f(x), f(y)\}$$

for all $x, y \in X$.

Proof. The proof is an ultra-metric modification of the proof in [45, Lemma 5.1.22] which asserts the following. Let $f : X \to \mathbb{R}$ be a function on a metric space (X, d). There exists a one-point metric extension $X \cup \{b\}$ of X such that f is the distance from b if and only if

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for all $x, y \in X$. □

The following result in particular gives a Graev type extension for ultra-metrics on free Boolean groups $B(X)$. To an ultra-metric space (X, d) we assign the Graev type group $B_{Gr}(X, d)$. The latter (ultra-normable) topological group is in fact $B_{\mathcal{NA}}(X, \mathcal{U}_d)$, where $\mathcal{U}_d$ is the uniformity of the metric d.

Theorem 8.2. *Let (X, d) be an ultra-metric space and G a topological group. Let $\pi : G \times X \to X$ be a continuous action such that $(X, \mathcal{U}_d) \in \mathrm{Unif}^G$. Then there exists an ultra-normed Boolean G-group $(E, ||\cdot||)$ and an isometric G-embedding $\iota : X \hookrightarrow E$ (with closed $\iota(X) \subset E$) such that:*

1. *The norm on E comes from the Graev-type ultra-metric extension of d to $B(X)$.*
2. *The topological groups E and $B_{\mathcal{NA}}(X, \mathcal{U}_d)$ (the free Boolean* **NA** *group) are naturally isomorphic.*

Proof. Consider the *free Boolean group* $(B(X),+)$ over the set X. The elements of $B(X)$ are finite subsets of X and the group operation $+$ is the symmetric difference of subsets. We denote by $\mathbf{0}$ the zero element of $B(X)$ (represented by the empty subset of X). Clearly, $u=-u$ for every $u \in B(X)$. Consider the natural set embedding

$$\iota : X \hookrightarrow B(X), \ \ \iota(x)=\{x\}.$$

Sometimes we will identify $x \in X$ with $\iota(x)=\{x\} \in B(X)$.

Fix $x_0 \in X$ and extend the definition of d from X to $\overline{X} := X \cup \{\mathbf{0}\}$ by letting $d(x,\mathbf{0}) = \max\{d(x,x_0),1\}$.

Claim 1: $d : \overline{X} \times \overline{X} \to \mathbb{R}$ is an ultra-metric extending the original ultra-metric d on X.

Proof. The proof can be derived from Lemma 8.1, noting that

$$|f(x)-f(y)| \leq d(x,y) \leq max\{f(x),f(y)\}$$

for $f(x) := \max\{d(x,x_0),1\}$. □

For every nonzero $u=\{x_1,x_2,x_3,\cdots,x_m\} \in B(X)$ define the *support of* u as $supp(u) := u$ if m is even, $supp(u) := u \cup \{\mathbf{0}\}$ if m is odd.

By a *configuration* we mean a finite subset of $\overline{X} \times \overline{X}$ (finite relations). Denote by Conf the set of all configurations. We can think of $\omega \in$ Conf as a finite set of some pairs

$$\omega = \{(x_1,x_2),(x_3,x_4),\cdots,(x_{2n-1},x_{2n})\},$$

where all $\{x_i\}_{i=1}^{2n}$ are (not necessarily distinct) elements of $\overline{X}$. If $x_i \neq x_k$ for all distinct $1 \leq i,k \leq 2n$ then ω is said to be *normal.* For every $\omega \in$ Conf the sum

$$u := \sum_{i=1}^{2n} x_i = \sum_{i=1}^{n}(x_{2i-1}-x_{2i})$$

belongs to $B(X)$ and we say that ω *represents* u or, that ω is a u-*configuration.* Notation: $\omega \in \mathrm{Conf}(u)$. We denote by $\mathrm{Norm}(u)$ the set of all normal configurations of u. If $\omega \in \mathrm{Norm}(u)$ then necessarily $\omega \subseteq supp(u) \times supp(u)$. It follows that $\mathrm{Norm}(u)$ is a finite set for any given $u \in B(X)$.

Our aim is to define a Graev type ultra-norm $||\cdot||$ on the Boolean group $(B(X),+)$ such that $d(x,y)=||x-y||, \ \forall x,y \in X$. For every configuration ω we define its *d-length* by

$$\varphi(\omega) = \max_{1\leq i\leq n} d(x_{2i-1},x_{2i}).$$

Claim 2: For every nonzero element $u \in B(X)$ and every u-configuration

$$\omega = \{(x_1,x_2),(x_3,x_4),\cdots,(x_{2n-1},x_{2n})\}$$

define the following elementary reductions:

(1) Deleting a trivial pair (t,t). That is, deleting the pair (x_{2i-1},x_{2i}) whenever $x_{2i-1}=x_{2i}$.
(2) Define the *trivial inversion at* i of ω as the replacement of (x_{2i-1},x_{2i}) by the pair in the reverse order (x_{2i},x_{2i-1}).
(3) Define the *basic chain reduction rule* as follows. Assume that there exist distinct i and k such that $x_{2i}=x_{2k-1}$. We delete in the configuration ω two pairs (x_{2i-1},x_{2i}), (x_{2k-1},x_{2k}) and add the new pair (x_{2i-1},x_{2k}).

Then, in all three cases, we get again a u-configuration. Reductions (1) and (2) do not change the d-length of the configuration. Reduction (3) cannot exceed the d-length.

Proof. Comes directly from the axioms of ultra-metric. In the proof of (3) observe that

$$x_{2i-1} + x_{2i} + x_{2k-1} + x_{2k} = x_{2i-1} + x_{2k}$$

in $B(X)$. This ensures that the new configuration is again a u-configuration. □

Claim 3: For every nonzero element $u \in B(X)$ and every u-configuration ω there exists a normal u-configuration ν such that $\varphi(\nu) \leq \varphi(\omega)$.

Proof. Using Claim 2 after finitely many reductions of ω we get a normal u-configuration ν such that $\varphi(\nu) \leq \varphi(\omega)$. □

Now we can define the desired ultra-norm $||\cdot||$. For every $u \in B(X)$ define

$$||u|| = \inf_{\omega \in \mathrm{Conf}(u)} \varphi(\omega).$$

Claim 4: For every nonzero $u \in B(X)$ we have

$$||u|| = \min_{\omega \in \mathrm{Norm}(u)} \varphi(\omega).$$

Proof. By Claim 3 it is enough to compute $||u||$ via normal u-configurations. So, since $\mathrm{Norm}(u)$ is finite, we may replace inf by min. □

Claim 5: $||\cdot||$ is an ultra-norm on $B(X)$.

Proof. Clearly, $||u|| \geq 0$ and $||u|| = ||-u||$ (even $u = -u$) for every $u \in B(X)$. For the $\mathbf{0}$-configuration $\{(\mathbf{0}, \mathbf{0})\}$ we obtain that $||\mathbf{0}|| \leq d(\mathbf{0}, \mathbf{0}) = 0$, and so $||\mathbf{0}|| = 0$. Furthermore, if $u \neq \mathbf{0}$ then for every $\omega \in \mathrm{Norm}(u)$ and for each $(t, s) \in \omega$ we have $d(t, s) \neq 0$. We can use Claim 4 to conclude that $||u|| \neq 0$. Finally, we have to show that

$$||u + v|| \leq \max\{||u||, ||v||\} \quad \forall\, u, v \in B(X).$$

Assuming the contrary, there exist configurations $\{(x_i, y_i)\}_{i=1}^n, \{(t_i, s_i)\}_{i=1}^m$ with $u = \sum_{i=1}^n (x_i - y_i),\ v = \sum_{i=1}^m (t_i - s_i)$ such that

$$||u + v|| > c := \max\{\max_{1 \leq i \leq n} d(x_i, y_i), \max_{1 \leq i \leq m} d(t_i, s_i)\}.$$

Since $\omega := \{(x_1, y_1), \cdots, (x_n, y_n), (t_1, s_1), \cdots, (t_m, s_m)\}$ is a configuration of $u + v$ with $||u + v|| > \varphi(\omega) = c$, we obtain a contradiction to the definition of $||\cdot||$. □

Claim 6: $\iota : (X, d) \hookrightarrow E := (B(X), ||\cdot||)$ is an isometric embedding, i.e.

$$||x - y|| = d(x, y) \quad \forall\, x, y \in X.$$

Proof. By Claim 4 we may compute the ultra-norm via normal configurations. For the element $u = x - y \neq \mathbf{0}$ the only possible *normal* configurations are $\{(x, y)\}$ or $\{(y, x)\}$. So $||x - y|| = d(x, y)$. □

One can prove similarly that $d(x, \mathbf{0}) = ||x||$. This observation will be used in the sequel.

Claim 7: For any given $u \in B(X)$ with $u \neq \mathbf{0}$ we have

$$||u|| \geq \min\{d(x_i, x_k) : \quad x_i, x_k \in supp(u),\ x_i \neq x_k\}.$$

Proof. Easily deduced from Claims 3 and 4. □

We have the natural group action

$$\overline{\pi} : G \times B(X) \to B(X), (g, u) \mapsto gu$$

induced by the given action $G \times X \to X$. Clearly, $g(u + v) = gu + gv$ for every $(g, u, v) \in G \times B(X) \times B(X)$. So this action is by automorphisms. Clearly $g\mathbf{0} = \mathbf{0}$ for every $g \in G$. It follows that $\iota : X \to B(X)$ is a G-embedding.

Claim 8: The action $\overline{\pi}$ of G on $B(X)$ is quasibounded and continuous.

Proof. The original action on (X, d) is π-quasibounded and continuous. Since $\mathbf{0}$ is an isolated point in $\overline{X}$ then the induced action on $(\overline{X}, d)$ is also continuous and quasibounded. Thus, for every $\varepsilon > 0$ and $g_0 \in G$, there exist: $1 \geq \delta > 0$ and a neighborhood $O(g_0)$ of g_0 in G such that for every $(x, y) \in \overline{X} \times \overline{X}$ with $d(x, y) < \delta$ and for every $g \in O$ we have

$$d(gx, gy) < \varepsilon.$$

By the definition of $||\cdot||$ it is easy to see that

$$||gu|| < \varepsilon \quad \forall\ ||u|| < \delta,\ g \in O.$$

This implies that the action $\overline{\pi}$ of G on $B(X)$ is quasibounded. Claim 5 implies that $\iota : X \hookrightarrow B(X)$ is a topological G-embedding. Since $\iota(X)$ algebraically spans $B(X)$ and $B(X)$ is a topological group, it easily follows that every orbit mapping $G \to B(X),\ g \mapsto gu$ is continuous for every $u \in B(X)$. Thus we can conclude that $\overline{\pi}$ is continuous (see the remark after Definition 7.4) and $B(X)$ is a G-group. □

By Claims 5 and 6 (see also the remark after Claim 6) $||\cdot||$ is an ultra-norm on $B(X)$ which extends the ultra-metric d defined on $\overline{X}$, and it can be viewed as a Graev type ultra-norm. To justify this last remark and the assertion (1) of our theorem observe that $||\cdot||$ satisfies, in addition, the following maximal property.

Claim 9: Let σ be an ultra-norm on $B(X)$ such that

$$\sigma(x - y) = d(x, y)\ \forall x, y \in \overline{X}.$$

Then $||\cdot|| \geq \sigma$.

Proof. Let u be a nonzero element of $B(X)$. By Claim 4 there exists a normal configuration

$$\omega = \{(x_1, x_2), (x_3, x_4), \cdots, (x_{2n-1}, x_{2n})\},$$

such that $u = \sum_{i=1}^{n}(x_{2i-1} - x_{2i})$ and $||u|| = \max_{1 \leq i \leq n} d(x_{2i-1}, x_{2i})$.

Now, σ is an ultra-norm and we also have

$$\sigma(x - y) = d(x, y)\ \forall x, y \in \overline{X}.$$

By induction we obtain that

$$\sigma(u) = \sigma(\sum_{i=1}^{n}(x_{2i-1} - x_{2i})) \leq \max_{1 \leq i \leq n} \sigma(x_{2i-1} - x_{2i}) = \max_{1 \leq i \leq n} d(x_{2i-1}, x_{2i}) = ||u||.$$

□

The proof of assertion (2) in view of the description of $B_{\mathcal{NA}}(X, \mathcal{U}_d)$ given by Theorem 4.14, follows from the fact that for every $0 < \varepsilon < 1$ the subgroup generated by

$$\{x - y \in B(X) : d(x, y) < \varepsilon\}$$

is precisely the ε-neighborhood of $\mathbf{0}$ in E.

Summing up we finish the proof of Theorem 8.2. □

Remark 8.3. Similarly we can assign for an ultra-metric space (X, d) the Graev type groups $A_{Gr}(X, d)$ and $F_{Gr}(X, d)$. Moreover, one may show (making use of Theorems 4.14.1 and 4.13.2) that $A_{Gr}(X, d) = A_{\mathcal{NA}}(X, \mathcal{U}_d)$ and $F_{Gr}(X, d) = F^b_{\mathcal{NA}}(X, \mathcal{U}_d)$.

Corollary 8.4. *Every ultra-metric space is isometric to a closed subset of an ultra-normed Boolean group.*

By a theorem of Schikhof [50], every ultra-metric space can be isometrically embedded into a suitable non-archimedean valued field. Note that every non-archimedean valued field is an ultra-normed abelian group.

8.1. **Continuous actions on Stone spaces.** Assigning to every Stone space X the free profinite group $F_{Pro}(X)$ we get a natural functor γ from the category of Stone G-spaces X to the category of all profinite G-groups P (see Remark 7.7 for the continuity of the lifted action). This functor preserves the topological weight and there exists a canonical G-embedding $j_X : X \hookrightarrow \gamma(X) = F_{Pro}(X)$ where $F_{Pro}(X)$ is metrizable if (and only if) X is metrizable (Theorem 5.1).

This means, in particular, that every Stone G-space is automorphizable in **Pro** and the class of (metrizable) profinite G-groups is at least as complex as the class of (metrizable) Stone G-spaces. In contrast, recall that a compact G-space that is not a Stone space may not be automorphizable (see Remark 7.2).

By Remark 5.2 the profinite groups $j(X)$ and $j(Y)$ are topologically isomorphic for infinite Stone spaces X, Y with the same weight. However, if X and Y are G-spaces (dynamical systems) then the corresponding G-spaces $j(X)$ and $j(Y)$ need not are G-isomorphic.

9. Appendix

By Graev's Extension Theorem (see [18]), for every metric d on $X \cup \{e\}$ there exists a metric δ on $F(X)$ with the following properties:

(1) δ extends d.
(2) δ is a two sided invariant metric on $F(X)$.
(3) δ is maximal among all invariant metrics on $F(X)$ extending d.

Savchenko-Zarichnyi [49] presented an ultra-metrization $\widehat{d}$ of the free group of an ultra-metric space (X, d) with $diam(X) \leq 1$. Gao [15] studied Graev type ultra-metrics δ_u. The metrics $\delta_u, \widehat{d}$ satisfy properties (1) and (2) above. As to the maximal property (3) one may show the following:

Theorem 9.1. *Let d be an ultra-metric on $\overline{X} := X \cup X^{-1} \cup \{e\}$ for which the following conditions hold for every $x, y \in X \cup \{e\}$:*

(1) $d(x^{-1}, y^{-1}) = d(x, y)$.
(2) $d(x^{-1}, y) = d(x, y^{-1})$.

Then:

(a) The Graev ultra-metric δ_u is maximal among all invariant ultra-metrics on $F(X)$ which extend the metric d defined on $\overline{X}$.
(b) If, in addition, $d(x^{-1}, y) = d(x, y^{-1}) = \max\{d(x, e), d(y, e)\}$ then δ_u is maximal among all invariant ultra-metrics on $F(X)$ which extend the metric d defined on $X \cup \{e\}$.
(c) If $diam(X) \leq 1$ and $d(x^{-1}, y) = d(x, y^{-1}) = 1$ then $\delta_u = \widehat{d}$.

References

[1] R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math., **6** (1956), 397-403.

[2] A. Arhangel'skii and M. Tkachenko, *Topological groups and related structures*, v. 1 of Atlantis Studies in Math. Series Editor: J. van Mill. Atlantis Press, World Scientific, Amsterdam-Paris, 2008.

[3] H. Becker and A. Kechris, *The Descriptive Set Theory of Polish group Actions*, London Math. Soc. Lecture Notes Ser. **232**, Cambridge Univ. Press, 1996.

[4] S. Dierolf and W. Roelcke, *Uniform Structures in Topological Groups and their Quotients*, McGraw-Hill, New York, 1981.

[5] D. Dikranjan and W. Tholen, *Categorical Structure of Closure operators with Applications to Topology, Algebra and Discrete Mathematics*, Math. and Appl., 346 Kluwer Academic, Dordrecht (1995).

[6] D. Dikranjan, *Recent advances in minimal topological groups*, Topology Appl., **85** (1998), 53–91.

[7] D. Dikranjan and M. Megrelishvili, *Minimality properties in topological groups*, in Recent Progress in General Topology III (to appear).

[8] D. Dikranjan and M. Tkachenko, *Weakly complete free topological groups*, Topology Appl., **112** (2001), 259-287.

[9] L. Ding, *On surjectively universal Polish groups*, Adv. Math. **231** (2012), no. 5, 2557-2572.

[10] L. Ding and S. Gao, *Graev metric groups and Polishable subgroups*, Adv. Math. **213** (2007) 887-901.

[11] R. Ellis, *Extending uniformly continuous pseudoultrametrics and uniform retracts*, Proc. AMS, **30** (1971), 599-602.

[12] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[13] T.H. Fay, *A note on Hausdorff groups*, Bull. Austral. Math. Soc., **13** (1975), 117-119.

[14] M.D. Fried and M. Jarden, *Field Arithmetic,* A Series of Modern Surveys in Mathematics, v. 11, Third Edition, Springer, 2008.

[15] S. Gao, *Graev ultrametrics and surjectively universal non-Archimedean Polish groups*, Topology Appl., **160** (2013), no. 6, 862-870.

[16] S. Gao and M. Xuan, *On non-Archimedian Polish groups with two-sided invariant metrics*, preprint, 2012.

[17] D. Gildenhuys and C.-K. Lim, *Free pro-C-groups*, Math. Z., **125** (1972), 233-254.

[18] M.I. Graev, *Theory of topological groups I*, (in Russian), Uspekhi, Mat. Nauk **5** (1950), 2-56.

[19] M. Higasikawa, *Topological group with several disconnectedness*, arXiv:math/0106105v1, 2000, 1-13.

[20] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.

[21] K.H. Hofmann and S.A. Morris, *Free compact groups I: Free compact abelian groups,* **23** (1986) 41-64.

[22] M. Husek, *Urysohn universal space, its development and Hausdorff's approach*, Topology Appl., **155** (2008), 1493-1501.

[23] J. Isbell, *Uniform Spaces*, American Mathematical Society, Providence, 1964.

[24] M. Katetov, *On universal metric spaces*, in: Gen. Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topol. Symp. 1986, Z. Frolik, ed., Heldermann Verlag (1988), 323-330.

[25] W. Kulpa, *On uniform universal spaces*, Fund. Math., **69** (1970), 243-251.

[26] A.Yu. Lemin, *Isosceles metric spaces and groups*, in: Cardinal invariants and mappings of topological spaces, Izhevsk, 1984, 26-31.

[27] A.Yu. Lemin, *The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT**, Algebra Universalis, **50** (2003), 35-49.

[28] A.Yu. Lemin and Yu. M. Smirnov, *Groups of isometries of metric and ultrametric spaces and their subgroups*, Russian Math. Surveys **41** (1986), no.6, 213-214.

[29] A.A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945) 3-64.

[30] M. Megrelishvili, *Compactification and Factorization in the Category of G-spaces* in Categorical Topology and its Relation to Analysis, Algebra and Combinatorics, J. Adámek and S. Maclane, editors, World Scientific, Singapore (1989), 220-237.

[31] M. Megrelishvili, *Free topological G-groups*, New Zealand Journal of Mathematics, vol. **25** (1996), no. 1, 59-72.

[32] M. Megrelishvili, *Fragmentability and continuity of semigroup actions*, Semigroup Forum, **57** (1998), 101-126.

[33] M. Megrelishvili, *Compactifications of semigroups and semigroup actions*, Topology Proceedings, **31:2** (2007), 611-650.

[34] M. Megrelishvili and T. Scarr, *The equivariant universality and couniversality of the Cantor cube*, Fund. Math. **167** (2001), no. 3, 269–275.
[35] M. Megrelishvili and M. Shlossberg, *Notes on non-archimedean topological groups*, Top. Appl., **159** (2012), 2497-2505.
[36] M. Megrelishvili and M. Shlossberg, *Notes on non-archimedean topological groups.* First version (2010) of arxiv.org/abs/1010.5987.
[37] S.A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. **1** (1969), 145-160.
[38] E.C. Nummela, *On epimorphisms of topological groups,* Gen. Top. and its Appl. **9** (1978), 155-167.
[39] E.C. Nummela, *Uniform free topological groups and Samuel compactifications*, Topology Appl. **13** (1982), no. 1, 77-83.
[40] V. G. Pestov, *Neighborhoods of identity in free topological groups*, Vestn. Mosk. Univ. Ser. 1. Mat., Mekh., No. 3 (1985), 8–10 .
[41] V. G. Pestov, *Universal arrows to forgetful functors from categories of topological algebras*, Bull. Austral. Math. Soc., **48** (1993), 209-249.
[42] V.G. Pestov, *Epimorphisms of Hausdorff groups by way of topological dynamics*, New Zealand J. of Math. **26** (1997), 257-262.
[43] V.G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. **350** (1998), 4149-4175.
[44] V. Pestov, *Topological groups: where to from here?* Topology Proceedings, **24** (1999), 421-502. http://arXiv.org/abs/math.GN/9910144.
[45] V. Pestov, *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon.* University Lecture Series, **40**. American Mathematical Society, Providence, RI, 2006.
[46] L. Ribes and P.A. Zalesskii, *Profinite Groups*, 2nd ed., Springer 2010.
[47] W. Roelcke and S. Dierolf, *Uniform structures on topological groups and their quotients*, Mc Graw-hill, New York, 1981.
[48] A.C.M. van Rooij, *Non-Archimedean Functional Analysis*, Monographs and Textbooks in Pure and Applied Math. **51**, Marcel Dekker, Inc., New York, 1978.
[49] A. Savchenko and M. Zarichnyi, *Metrization of free groups on ultrametric spaces*, Top. Appl., **157** (2010), 724-729.
[50] W. H. Schikhof, *Isometrical embeddings of ultrametric spaces into non- Archimedean valued fields*, Indag. Math. **46** (1984), 5153.
[51] O.V. Sipacheva, *The topology of a free topological group*, J. Math. Sci. (N. Y.) **131** (2005), no. 4, 5765-5838.
[52] L. Schröder, *Linearizability of non-expansive semigroup actions on metric spaces,* Topology Appl., **155** (2008), 1576-1579.
[53] D. Shakhmatov, J. Pelant and S. Watson, *A universal complete metric abelian group of a given weight*, in Topology with Applications, Szekszárd, 1993, 431-439. Bolyai Mathematical Studies 4. János Bolyai Mathematical Society, Budapest, 1995.
[54] D. Shakhmatov and J. Spěvak, *Group-valued continuous functions with the topology of pointwise convergence,* Top. Appl., **157** (2010), 1518-1540.
[55] S. Teleman, *Sur la représentation linéare des groupes topologiques*, Ann. Sci. Ecole Norm. Sup. **74** (1957), 319-339.
[56] M.G. Tkachenko, *On topologies of free groups*, Czech. Math. J., **34** (1984), 541-551.
[57] V.V. Uspenskii, *On the group of isometries of the Urysohn universal metric space,* Comment. Math. Univ. Carolin. **31** (1990) 181-182.
[58] V.V. Uspenskij, *Free topological groups of metrizable spaces*, Math. USSR Izvestiya, **37** (1991), 657-680.
[59] V.V. Uspenskij, *The epimorphism problem for Hausdorff topological groups,* Topology Appl. **57** (1994), 287–294.
[60] S. Warner, *Topological Fields*, North Holland, Mathematics Studies, vol. **157**, North-Holland-Amsterdam, London, New York, Tokyo, 1993.

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