Minimality conditions in topological groups

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Abstract

This is a survey on the recent progress in minimal topological groups with a particular emphasis on constructions leading to non-abelian minimal groups, as semidirect products, generalized Heisenberg groups and other groups naturally arising in Analysis and Geometry. A special attention is paid to several generalizations of minimality (as local minimality, relative minimality and co-minimality), the relations of minimality to (dis)connectedness and to various level of compactness and completeness.

1 Introduction and historical background

A Hausdorff topological group G is *minimal* (Stephenson [190] and D. Doïtchinov [90]) if it does not admit a strictly coarser Hausdorff group topology, or equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a group embedding. This nice property of the class (continuous bijective morphisms = topological isomorphisms) may suggest the idea of an "algebraic paradise". The stronger version, G is *totally minimal* if all Hausdorff quotients of G are minimal, is nothing else but the *open mapping property* ([66]). This witnesses the deep roots of this notion in Analysis [112], as well as Algebra [153] (via the discrete minimal groups, usually called also non-topologizable groups [118, 155, 185], and more generally, through the algebraic structure of the minimal groups) and Number Theory (this connection comes through the p-adic numbers that are deeply related to minimal abelian groups).

A natural connection of minimal groups to General Topology comes from the standing interest in properties close to compactness. The Hausdorff spaces that are closed in every Hausdorff space Y containing X as a subspace were introduced by Alexandroff and Urysohn [3] under the name *H-closed spaces*. The semiregular H-spaces (X, τ) are known better as *minimal spaces*, since every Hausdorff topology $\tau' \subseteq \tau$ on X coincides with τ. Therefore, one has the implications

$$\text{compact} \quad \Rightarrow \quad \text{minimal} \quad \Rightarrow \quad H\text{-closed} \tag{1}$$

for all Hausdorff spaces. Obviously, minimal groups are defined analogously in the category $\mathcal{H}$ of the Hausdorff topological groups, while the counterpart of H-closed spaces in $\mathcal{H}$ are the complete groups. Compact groups are obviously both complete and minimal, i.e., one has the following obvious (partial) counterpart of (1)

$$\text{compact} \Rightarrow \text{minimal \& complete} \tag{2}$$

for every Hausdorff topological group (see §1.1 for further comments).

Apart from a significant number of publications, several surveys [51, 54, 169] and books [68, 131] appeared in the last thirty years, about minimal groups (the surveys [24, 27, 28] dedicate a large section on minimal groups), as well as dissertations [118, 125, 168, 179, 181, 188, 197].

Nevertheless, it is fair to say that in most of these sources the abelian case was paid more attention (at least as far as deep specific results are concerned). The backbone of this survey is the non-abelian case. The corresponding results and examples are well distributed in the whole text. Many results are new and presented here for the first time. The intersection with the surveys [51, 54, 169] is kept to the reasonable minimum, confined within §3 aiming to make this survey sufficiently self-contained and avoiding substantial overlap with the previous ones.

The survey is organized as follows. In the first section we give some historical background on minimal groups. In §2 we discuss some old and new results around minimality conditions in symmetric groups, while §3 collects the basic structural properties of minimal groups frequently used in the survey. In §4 we examine mostly non-abelian examples and their properties presented as semidirect products (§4.1), groups of homeomorphisms (§4.2) or isometries (§4.3), or groups coming from Analysis and Geometry (e.g., unitary groups of Hilbert spaces, matrix groups, etc. §4.4). Several new results are given in §§4.1 and 4.2 with their proofs, while §§4.3 and 4.4 contain a review of known results.

In §5 we expose, often with complete proofs, older and also new applications of generalized Heisenberg groups to minimal groups theory. The culmination is §5.5, containing the following theorem from [145]: every group is a retract of a minimal group (Theorem 5.38). This section makes it well visible how the examples of minimal groups and the involved technique come in extremely natural ways.

Another line of this survey is dedicated to several recent generalizations of minimality. In §6 we treat relative minimality and co-minimality of subgroups. For some naturally defined groups (like generalized Heisenberg groups or some matrix groups) certain non-minimal subgroups still might be relatively minimal or co-minimal in the whole group. Some applications to group representations theory are presented in §6.1. In §7 we focus on locally minimal groups – a common generalization of minimal groups, locally compact groups, subgroups of Banach-Lie groups and UFSS groups. In particular, we study the permanence properties of locally minimal groups (with respect to taking closed or dense subgroup, direct products and quotients), as well as their cardinal invariants and algebraic structure. In §8 the connection between minimality and (dis)connectedness is studied. This section contains also a large subsection 8.3 dedicated to the sequentially complete minimal groups and their structure, as well as a subsection 8.4 dedicated to a question of Arhangel′skiĭ. In §9.1 the minimal groups are observed under the looking glass of (the presence of) convergent sequences. In §9.2 the compact abelian groups containing proper dense (totally) minimal subgroups with some additional compact-like property are studied. In §9.3 we investigate the minimal groups admitting some special kinds of topological generators. Section 9 ends up with a subsection where we collect miscellaneous facts and we give a brief account of what we could not include in the survey, providing references for further reading.

We pay attention, as much as possible, to keep a chronological exposition and we give in §1.1 some historical facts showing how this fascinating topic was established and developed in its first decade.

Many mathematicians contributed to minimal groups theory. We shall mention here only a few names since the list is long and rapidly growing: Prodanov, Doïtchinov, Stephenson, Stoyanov, Banaschewski can be considered as pioneers, the role of Prodanov remaining exceptionally influential (even if not sufficiently known). The field enjoyed also the active participation (by posing challenging open problems, survey's, etc) of Comfort and Arhangel′skiĭ, among others. More details can be found in §1.1.

Contents

Notation and terminology

We denote by $\mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{T}, \mathbb{Z}_m$ the naturals, the primes, the integers, the rationals, the reals, the circle group and the cyclic group of size m respectively. For a prime p we denote by $\mathbb{J}_p$ the compact group (and ring) of p-adic integers. The cardinality of the continuum is denoted by $\mathfrak{c}$, while $|X|$ denotes the cardinality of a set X.

For a set X we denote by $S(X)$ the permutation group of X. For a group G, the center of G is denoted by $Z(G)$. We recall that G' denotes the derived subgroup of a group G, and that the *derived series* $G^{(n)}$ of a group G is defined by: $G^{(0)} = G$ and $G^{(n+1)}$ is the derived group of $G^{(n)}$. The group G is *solvable* if $G^{(n)} = \{1\}$ for some integer n, G is *metabelian* if $G^{(2)} = \{1\}$. As usual, G is *nilpotent* means that its lower central series terminates in the trivial subgroup after finitely many steps. If the length of such series is 2 (i.e., if $[[G,G],G] = \{e\}$) then G is said to be *2-step nilpotent*.

The socle $soc(G)$ of G is the subgroup generated by all normal subgroups of G of prime order. A group G is *bounded* if there exists a positive integer m such that, for every $g \in G$ we have $g^m = e$, the smallest such m is called *exponent* of G. Let G be an abelian group. The *torsion part* of G will be denoted by $tor(G)$, its *free-rank* by $r(G)$, its *p-rank* by $r_p(G)$ and for $m \in \mathbb{N}$ we let $G[m] = \{x \in G : mx = 0\}$.

For a topological space X denote by $w(X)$, $\chi(X)$, $\psi(X)$, $t(X)$ the *weight*, the *character*, the *pseudo-character* and the *tightness* of X. A space X is *Polish* if it is separable and completely metrizable. All topological groups and all group topologies will be assumed Hausdorff, unless otherwise stated.

By $\mathscr{V}(e_G)$ we denote the filter of all neighborhoods of the neutral element e_G in a topological group G. A group G is *monothetic* if G contains a cyclic dense subgroup; *ω-bounded* if every countable subset of G is contained in a compact subset of G; (*locally*) *precompact*, if it is isomorphic to a subgroup of a (locally) compact group; *non-archimedean*, if $\mathscr{V}(e_G)$ has a base of open subgroups. For every topological group G there exists a compact Hausdorff group bG and a continuous homomorphism $b : G \to bG$ onto a dense subgroup of bG with the following universal property: for every continuous homomorphism $f : G \to K$ into a compact group there is a continuous homomorphism $f_b : bG \to K$ such that $f = f_b \circ b$. This homomorphism $b : G \to bG$ is called the *Bohr compactification* of G, the groups G for which it is injective are called *maximally almost periodic* (or MAP groups). If b is trivial then G is said to be *minimally almost periodic*. The latter is equivalent to saying that G has only trivial finite dimensional continuous unitary representations.

For every locally compact abelian group G denote by $G^\wedge$ its *Pontryagin dual*. That is the topological group $Hom(G, \mathbb{T})$ of all continuous characters endowed with the compact open topology. For a topological group G we denote by $c(G)$ the *connected component* of G. We say that G is *complete* if G is complete with respect to its two-sided uniformity and we denote by $\widetilde{G}$ the *completion* of a topological group G with respect to this uniformity (some authors use the terms Raĭkov complete and Raĭkov completion, respectively). Moreover, we say that G is *h-complete* if every Hausdorff continuous homomorphic image of G is complete. Let p be a prime number, and let G be a topological abelian group. The *topological p-component* of G is $G_p = \{x \in G : p^n x \to 0 \text{ in } G \text{ where } n \in \mathbb{N}\}$.

For undefined terms or notation see [7, 68, 120, 119].

1.1 Some history: *Minimal groups in Sofia 1971-1984*

This subsection contains a brief historical review of (roughly) the first decade of the development of the area witnessed by the first named author.

The minimal groups appeared for the first time at the Topology Seminar of Sofia University in January 1971 with D. Doïtchinov's results from his still unpublished paper [90]. This paper was triggered by G. Choquet's question whether minimality is preserved under products and contained the following relevant facts:

Example 1.1 *(a) The subgroup $\mathbb{Q}/\mathbb{Z}$ of the circle group $\mathbb{T}$ is minimal (see Example 3.7).*

(b) $\mathbb{Z}$ with the 2-adic topology τ_2 is minimal, but $(\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$ is not minimal, answering negatively Choquet's question (see Remark 3.2 (b) for an argument).

(c) If G is a minimal group and H is a group that is either compact or minimal torsion, then $G \times H$ is minimal.

For a class $\mathcal{P}$ of topological spaces, a space $X \in \mathcal{P}$ is $\mathcal{P}$-*closed*, if X is closed in every $Y \in \mathcal{P}$ containing X as a subspace; $(X, \tau) \in \mathcal{P}$ is $\mathcal{P}$-*minimal*, if every topology $\tau' \leq \tau$ on X with $(X, \tau') \in \mathcal{P}$ coincides with τ. For $\mathcal{P}$ the class of all Hausdorff topological spaces one obtains the notions of H-closed spaces and minimal space, respectively. The $\mathcal{P}$-closed spaces and the $\mathcal{P}$-space were widely studied (they can be characterized via appropriate open covers or open filters, see [18, 191] for more detail). If $\mathcal{P}$ consists of only Hausdorff spaces, then one has the implication

$$\text{compact } \mathcal{P}\text{-space} \quad \Rightarrow \quad \mathcal{P}\text{-minimal \& } \mathcal{P}\text{-closed}. \tag{3}$$

Clearly, the minimal topological groups can be obtained by taking $\mathcal{P} = \mathcal{H}$, so that the implication (2) in the introduction follows from (3). Moreover, many questions concerning the permanence properties of minimal groups and complete groups are inspired by their counterparts in the realm of $\mathcal{P}$-closed spaces and $\mathcal{P}$-minimal spaces (e.g., the above mentioned question of Choquet, as well as Doïtchinov's paper [90], were motivated by the fact that products of minimal spaces are minimal [121, 89]).

In the spring of 1971 Prodanov [163, 162] used a simple idea for describing the minimal precompact topologies on an abelian group G via the character group $G^\wedge$. More precisely, according to a theorem of Comfort and Ross [35], the precompact group topologies τ of G have the form $\tau = T_H$, where H is a dense subgroup of the compact dual $G^\wedge$ and T_H denotes the initial topology of the family of characters $\chi \in H$; moreover, the correspondence $H \mapsto T_H$ is a monotone bijection.

Theorem 1.2 [163, Theorem 2] *The minimal precompact topologies on G have the form T_H, where H is a minimal (with respect to inclusion) dense subgroups of the compact dual group $G^\wedge$.*

Clearly, the topologies obtained in this way are just minimal, since precompactness is preserved by taking coarser topologies. Using this simple characterization, one can easily see that all p-adic topologies τ_p on $\mathbb{Z}$ are minimal [163]; moreover, these are the only minimal precompact topologies on $\mathbb{Z}$. The original proof of Example 1.1(b) was only using the definition of a group topology and was tailored for the case $p = 2$. Similarly, one can prove that $\mathbb{Z}(p^\infty)$ admits no minimal precompact topologies, since none of the dense subgroups of its dual $\mathbb{J}_p$ is minimal with respect to inclusion (see also 3.3 for alternative proofs and stronger results).

This fact suggested Prodanov to conjecture the precompactness of all minimal topologies of the abelian groups (see also [91]). This problem turned out to be quite hard and was resolved only in 1983 (see §3.2 for more details). Since the completion of a minimal groups is still minimal, according to Theorem 3.1, the positive answer to this conjecture is equivalent to the inversion of the implication (2) for abelian groups.

Chronologically, the first paper on minimal groups was that of Stephenson, [190], where he proved that locally compact abelian groups are minimal precisely when they are compact (e.g., $\mathbb{R}$ is not minimal) and produced some examples of non-compact minimal groups by means of a useful criterion for minimality of dense subgroups of compact groups (see Theorem 3.1). This criterion appeared in a much more general form in a paper of Banaschewski [15] on minimal topological algebraic structures. We came to know about these two papers somewhat late, anyway after the publication of [90] and [162] (the latter contained Stephenson's minimality criterion in the abelian case).

For $\mathbb{Z}$ equipped with any minimal topology, all Hausdorff quotients are finite, hence obviously minimal. But a quotient of a minimal group need not be minimal in general (see Example 3.7). This suggested the introduction of the smaller class, of totally minimal groups in [66] (see §3 for more detail). This paper provides also a criterion for total minimality of a dense subgroup H of Hausdorff group G (see Theorem 3.6). This criterion was inspired by Grant's results on the open mapping theorem [109], who kindly sent us his manuscript long before publication.

In the first several years (up to 1975) minimal groups were considered exclusively in the abelian case, where the above mentioned precompactness conjecture of Prodanov was the corner stone. In the autumn of 1975 Doïtchinov came up by an example of a minimal non-precompact group, that was non-abelian. It was the symmetric group $S(X)$ of an infinite (discrete) set with the topology of pointwise convergence. Unfortunately, a gap was found in his argument during his talk at our seminar. Shortly afterwards, Prodanov got a letter from Susanne Dierolf and Ulrich Schwanengel, containing the manuscript of [38], where they were proving the same theorem (actually, in a slightly stronger version, namely that $S(X)$ is totally minimal). This was happening roughly *ten years after* Gaughan's paper [104] and nobody was aware of the stronger theorem Gaughan had proved (namely, the topology of pointwise convergence on $S(X)$ is the coarsest Hausdorff group topology on $S(X)$). For many years, Gaughan's stronger theorem remained unknown in the area of minimal groups.

In 1978 Arhangel′skiĭ visited Sofia University and proposed an interesting problem inspired by the lack of productivity of the class of minimal groups (see §3.4 for more details on Arhangel′skiĭ's problem). Shortly afterwards he formulated the following problem that contributed still more to the development of new ideas and techniques in this area (see §2 for the solution of this problem and Question 5.40.1 for a stronger version of this question).

Question 1.3 [4, Problem 2] *Do the character and the pseudocharacter of a minimal group coincide?*

Around that time was circulating a draft of the survey of Comfort and Grant [27] containing a large part on minimal groups.

In 1984 Wis Comfort, Dieter Remus and Volker Eberhard visited Bulgaria in the framework of a Topology Conference at Primorsko (on the Black Sea coast). Since then Dieter got deeply interested in minimal groups [173, 174]. He discovered at some point (around 1986) Gaughan's paper [104] and used it to find an easy and elegant solution to a problem of Markov on connected topologization of groups (see [28, §3.5] for more details).

In 1984 Dima Shakhmatov sent us the unpublished manuscript of his remarkable paper [183], resolving the above mentioned Arhangel′skiĭ's problem (1.3) in the negative, with the biggest possible gap between the character and the pseudocharacter of a minimal group. This was the first step of a long and fruitful collaboration between him and the first named author ([70, 71, 72]).

The untimely death of Prodanov in the Spring of 1985 broke this paradise. The book [68] collects the results achieved by that time.

2 The symmetric group and its subgroups

The remarkable property of the symmetric group $S(X)$ discovered by Gaughan can be formulated in the general case using the so called *Markov topology*, introduced implicitly by Markov [134] and explicitly in [76] (see also [74]). Namely, for any group G this is the intersection $\mathfrak{M}_G$ of all Hausdorff group topologies on G. This is a T_1 topology that need not be a group topology, yet all translations, as well as taking the inverse, are continuous. For any group G the closed sets of $(G, \mathfrak{M}_G)$ are precisely the so called *unconditionally closed* sets of G (i.e., the sets that are closed in every Hausdorff group topology on G). The following notion appeared in [84] under the name $\mathfrak{M}$-*Hausdorff group*:

Definition 2.1 *A group G is called an a-*minimal *group if G admits a coarsest Hausdorff group topology (equivalently, if $\mathfrak{M}_G$ is a topological group topology). We shall use the term a-minimal group also for the topological group $(G, \mathfrak{M}_G)$.*

For every a-minimal group G, $\mathfrak{M}_G$ necessarily is a Hausdorff group topology (since, $\mathfrak{M}_G$ is always a T_1 topology).

One can show that $\mathfrak{M}_G$ is non-Hausdorff for any infinite abelian group G ([76]), hence G cannot be a-minimal. Since for every group G the inclusion $(Z(G), \mathfrak{M}_{Z(G)}) \hookrightarrow (G, \mathfrak{M}_G)$ is continuous ([75, Corollary 3.6, Lemma 3.7]), one can conclude that the center of an a-minimal group must be finite. Obviously, an a-group has precisely one minimal topology (namely, $\mathfrak{M}_G$), but the converse implication fails. In Example 3.12(a) one can find examples of infinite abelian (so non-a-minimal) groups with precisely one minimal topology.

In these new terms, one can announce Gaughan's theorem by saying that *the group $S(X)$ is a-minimal.*

This setting may hopefully let the reader see better how $S(X)$ and minimality interplay. The connection is remarkable and far non-trivial. A lot of authors worked on this, succeeding to see some of these properties at a time. For example, to answer Arhangel′skiĭ's question 1.3, Shakhmatov [183] used appropriate dense embedding of free groups F into $S(X)$ inducing on F minimal topologies (due to the minimality criterion 3.1 and the topological simplicity of $S(X)$) with countable pseudocharacter and arbitrarily large character. Arhangel′skiĭ's problem was resolved also by Pestov [159] and Guran [114, 113], although without such an impressive gap between the character and the pseudocharacter. Results in the positive direction were obtained by Grant and Comfort [111] (see also [68, §§7.5, 7.7], [131, Corollary] and [131, §3.4] for further details). These positive results, as well as the fact that the examples were not complete, motivated Arhangel′skiĭ to re-formulate later his question for *complete* groups (see Question 5.40.1).

As we mentioned, the groups $S(X)$ are a-minimal. In fact also totally minimal, because this group is topologically simple for every infinite X. Since $S(X)$ is also complete, we deduce now that $S(X)$ is h-complete as well.

Let $S_\omega(X)$ denote the subgroup of $S(X)$ formed by all permutations of finite support. Recently, Banakh, Guran and Protasov [13] proved the following remarkable extension of Gaughan's theorem:

Theorem 2.2 [13] *Every subgroup of $S(X)$, containing $S_\omega(X)$, is a-minimal.*

Intuitively, the subgroups of $S(\mathbb{N})$ which have highly transitive properties would be closer to having some minimality properties. A closed subgroup of $S(\mathbb{N})$ is *oligomorphic* if its action on $\mathbb{N}^n$ $(n \geq 1)$ has only finitely many orbits. Equivalently, by a theorem of Engeler, Ryll-Nardzewski and Svenonius, these are the automorphism groups of $\aleph_0$-categorical (Fraissé) structures. The class of oligomorphic groups are important in model theory, combinatorial enumeration and dynamical systems. Some examples are: $S(\mathbb{N})$, the automorphism group of the countable dense linear order, the homeomorphism group of the Cantor space Homeo$\,(2^\omega)$ (see for instance [21, 122, 123]).

Question 2.3 *Which oligomorphic subgroups of $S(\mathbb{N})$ are a-minimal (or, at least, minimal) ?*

The class of all topological subgroups of the symmetric groups (up to the topological isomorphism) is exactly the class of *non-archimedean groups.* Equivalently, topological subgroups of homeomorphism groups Homeo$\,(X)$, where X are zero-dimensional compact spaces. Non-archimedean Polish groups are just the closed subgroups of $S(\mathbb{N})$. By [148] (see Theorem 5.30) every non-archimedean group is a group retract of a minimal non-archimedean group.

Problem 2.4 *Describe the infinite a-minimal groups G that are non-archimedean.*

A class of groups with this property was pointed out in [84]. If F_i is a center-free finite non-trivial group for every $i \in I$ from an infinite set I, then $G = \prod_{i\in I} F_i$ has the required property, namely $\mathfrak{M}_G$ is a group topology, coinciding with the usual product topology of G. It is second countable precisely when I is countable. For a stronger result see Proposition 3.21 and Corollary 3.22.

2.1 Non-topologizable groups

Here we discuss a special class of a-minimal groups, namely the groups G with *discrete* $\mathfrak{M}_G$. Clearly, these are the discrete minimal groups, i.e., the non-topologizable groups (admitting no Hausdorff group topology beyond the discrete one). The problem of existence of infinite non-topologizable groups was raised by Markov. The first example of such a group, under the assumption of CH, was given by Shelah [185], later Hesse [118] eliminated the use of CH in his argument. Ol′shanskii [155] produced an elegant example of a countable non-topologizable group in ZFC, making use of appropriate quotients of the Adian groups $A(m, n)$, [1]. Further examples were produced in [127, 202, 203] (for more detail see [28, 68, 74, 75, 84]). Following [132], call *hereditarily non-topologizable* the groups G that are non-topologizable along with all quotients of their subgroups, i.e., the discrete groups G such that all subgroups of G are totally minimal. Clearly, all abelian (actually, all resolvable) subgroups of a hereditarily non-topologizable group G are finite (in particular, G is torsion). This groups will be briefly discussed in §9.4.

3 Minimality - basic properties

3.1 Subgroups and quotients of minimal groups

The following criterion for minimality of dense subgroups is due to Stephenson, [190] in the case of a compact group G. Call a subgroup H of a topological group G *essential* if the intersection $H \cap N$ is non-trivial in N for every closed non-trivial normal subgroup N of G.

Theorem 3.1 *A dense subgroup H of a topological group G is minimal if and only if G is minimal and H is essential in G.*

Remark 3.2 *If G is a topological group, then every dense minimal subgroup of G, being essential, contains $soc(Z(G))$. In case G is compact and abelian, a dense subgroup H of G is essential (hence, minimal) if and only if H contains $soc(G)$ and H non-trivially meets every closed subgroup of G isomorphic to the group $\mathbb{J}_p$ of p-adic integers for some prime p ([68, Theorem 4.3.7]). In particular:*

(a) *every subgroup of the circle group $\mathbb{T}$ containing $soc(\mathbb{T})$ (in particular, the rational circle $\mathbb{Q}/\mathbb{Z}$) is essential in $\mathbb{T}$.*

(b) *[164] every dense minimal subgroup of $\mathbb{J}_p^2$, $p \in \mathbb{P}$, has size $\mathfrak{c}$ (since $\mathbb{J}_p^2$ has $\mathfrak{c}$-many closed subgroups of the form $\mathbb{J}_p(\xi, 1)$ $(\xi \in \mathbb{J}_p)$, with pairwise trivial intersection), hence $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p)$ is not minimal (see Example 1.1 (b)).*

Minimality is preserved under taking closed central subgroups

Proposition 3.3 [68, Proposition 2.5.7] *A closed central subgroup of a minimal group is minimal.*

While every topological group can be a closed topological subgroup of a minimal group (see Theorem 5.38 for a much stronger result), not every *abelian* topological group G can be a *central* closed topological subgroup of a minimal group. Indeed, according to Proposition 3.3, this would be equivalent to ask G to be minimal.

It is easy to see that minimality is preserved also under taking direct summands. Indeed, if $G = H \times K$ is minimal, then both H and K are minimal (as otherwise one can easily get a coarser Hausdorff group topology on G by taking on one of the factors a coarser Hausdorff group topology). It would be reasonable to unify these two instances of preservation of the minimality by taking closed subgroups by a single wider class of subgroups that includes both central subgroups and direct summands. Namely, call *super-normal* a subgroup H of a group G such that $H \cdot c_G(H) = G$, where $c_G(H)$ is the centralizer of H in G. In other words, these are those normal subgroups of G such that the restriction on H of internal automorphisms of G are internal automorphisms of H. Obviously, both central subgroups and direct summands are super-normal.

Question 3.4 *Is it true that every closed super-normal subgroup of a minimal group is still minimal?*

An abelian subgroup is super-normal if and only if it is central. We shall see an example of an index-2 open abelian subgroup H of a minimal group G that is not minimal (Example 4.7). Since H is not minimal, it cannot be central, so it is not super-normal either (being abelian). So it seems reasonable to impose the stronger assumption "super-normal" (rather than only normal) in the above question.

In the case of nilpotent or center-free groups one can obtain the following variant of the minimality criterion 3.1:

Theorem 3.5 *Let G be a dense subgroup of a group K. Then the implications (d) $\leftarrow$ (a) $\leftrightarrow$ (b) $\rightarrow$ (c) hold for the conditions:*

(a) *G is minimal;*

(b) *K is minimal and G is essential in K;*

(c) *K is minimal and $Z(G)$ is an essential subgroup of $Z(K)$;*

(d) K is minimal and G non-trivially meets all closed normal subgroups of K contained in K'.

If K is nilpotent, then (a), (b) and (c) are equivalent. If $Z(K) = \{e\}$, then (a), (b) and (d) are equivalent.

Proof. The equivalence of (a) and (b) is the minimality criterion.

For the rest of the proof, it will be relevant to note that $Z(K) \cap G = Z(G)$, as G is dense in K.

(b) $\to$ (d) is trvial. To check (b) $\to$ (c), take a closed subgroup N of $Z(K)$ with $N \cap Z(G) = \{e\}$. Then $\{e\} = N \cap Z(G) = N \cap (Z(K) \cap G) = N \cap G$. Since N is a normal closed subgroup of K, our assumption (b) yields $N = \{e\}$.

Assume that K is nilpotent in order to prove (c) $\to$ (b). Let N be a closed non-trivial normal subgroup of K. It is a well known fact that every non-trivial normal subgroup of a nilpotent group non-trivially meets the center. Hence, $N_1 = N \cap Z(K)$ is non-trivial, so by our hypothesis (c), N_1 (hence also N) non-trivially meets G.

Assume that $Z(K) = \{e\}$ in order to prove (d) $\to$ (b). Take a non-trivial closed normal subgroup N of G. If x is a non-trivial element of N, then $Z(K) = \{e\}$ implies that x is not central. So $z = [x, y] \neq e$ for some $y \in G$. Since $z \in N$ as well, we conclude that $N_1 = N \cap K'$ is a non-trivial closed normal subgroup of G contained in G', so $N_1 \cap G \neq \{e\}$. This proves that $N_1 \cap G \neq \{e\}$. □

Making use of Proposition 3.3 (and the fact that essentiality of $Z(G)$ in $Z(K)$ implies the essentiality of $Z(G)$ in $\mathrm{cl}(Z(G))$), one can deduce that (c) implies the following weaker condition

(c*) K and $Z(G)$ are minimal.

We shall see below (see §5.1, Example 5.5), that (c*) does not imply minimality of G even when G is a two-step nilpotent precompact group (i.e., K is compact).

It is not clear whether (d) implies the stronger condition

(d*) K is minimal and $G \cap K'$ is an essential subgroup of K'.

Due to the equality $Z(G) = Z(K) \cap G$, (d*) looks the correct counterpart of (c), when $Z(K)$ is replaced by K'.

Totally minimal groups are defined in [66] as those Hausdorff groups G such that all Hausdorff quotients are minimal. Later these groups were studied also by Schwanengel [182] under the name *q-minimal groups*. Call a subgroup H of a Hausdorff group G *totally dense* in G if the intersection $H \cap N$ is dense in N for every closed normal subgroup N of G. This notion allows one to produce a counterpart of Theorem 3.1 for total minimality:

Theorem 3.6 [66] *A dense subgroup H of a Hausdorff group G is totally minimal if and only if G is totally minimal and H is* totally dense *in G.*

Example 3.7 *Using the criterion 3.6 one can see that if a dense subgroup G of a topological group K is totally minimal, then G contains $tor(Z(K))$. Since $\mathbb{Q}/\mathbb{Z} = tor(\mathbb{T})$ is totally dense in $\mathbb{T}$, one obtains:*

- *a dense subgroup G of $\mathbb{T}$ is totally minimal if and only if G contains $\mathbb{Q}/\mathbb{Z}$ (cf. Example 1.1);*
- *if $soc(\mathbb{Q}/\mathbb{Z}) \leq G < \mathbb{Q}/\mathbb{Z}$, then G is minimal but not totally minimal (see Remark 3.2 (a)).*

Dierolf and Schwanengel presented every discrete group as a retract (hence, also as a quotient group) of a locally compact minimal group ([39]). See below the generalization for non-archimedean groups, Theorem 5.30.

In the sequel we shall see when a special quotient of a minimal abelian group remains minimal (item (d) of Fact 3.8). For a topological group G denote by $o(G)$ the intersection of all open subgroups of G. Clearly, $o(G)$ is trivial in a non-archimedean group G, but a group with trivial $o(G)$ need not be non-archimedean, it only admits a coarser non-archimedean topology.

Fact 3.8 *(a) Obviously, $o(G)$ contains the connected component $c(G)$ of G. If G is locally compact, then $o(G) = c(G)$ (see §8 for more detail).*

(b) If H is a dense subgroup of G, then $o(H) = o(G) \cap H$.

(c) $o(G) = o(\widetilde{G}) \cap G = c(\widetilde{G}) \cap G$ for a locally precompact group G.

(d) For a minimal abelian group G the quotient $G/o(G)$ is minimal if and only if $o(G)$ is dense in $c(\widetilde{G})$ ([68, Exercise 4.5.15]).

3.2 The precompactness problem for the minimal abelian group

The next theorem shows that complete minimal abelian groups are compact, i.e., the implication (2) from the introduction becomes an equivalence for abelian groups. In view of Theorem 3.1, this statement is equivalent to the statement we give below:

Theorem 3.9 [170, 68] *Every abelian minimal group is precompact.*

Let us give a brief history of this remarkable theorem, more detail can be found in [68, 51, 54].

In 1977 Prodanov proved that every totally minimal abelian group is precompact [166]. Around that time and motivated by Prodanov's theorem, Dierolf and Schwanengel [38] found the first example of a non-precompact (complete) totally minimal group: the symmetric group $S(X)$ of any infinite set X (see §2). Using *maximal* topologies (see §9.4 for more detail) Prodanov succeeded to prove that a huge part of every minimal abelian group must be precompact and he showed in [164] that minimal countable abelian groups are precompact. Stoyanov [193] proved that the minimal metrizable torsion abelian groups are precompact using Følner's theorem as well as results of [67] and [165]. Later he established precompactness of all minimal abelian groups G satisfying $|G/(D(G) + tor(G))| < \mathfrak{c}$, where $D(G)$ is the maximal divisible subgroup of G [196]. The final positive solution of the problem of precompactness was given by Prodanov and Stoyanov in 1983 [170]. This is, undoubtedly, one of the most impressing major result obtained in the field of minimal groups. A somewhat simplified proof can be found in [68, Theorem 2.7.7].

Theorems 3.1 and 3.9 reduced the study of (totally) minimal abelian group to that of dense essential (resp., totally dense) subgroups of the compact abelian groups.

Every MAP minimal group is precompact. Hence the non-precompact minimal groups are necessarily non-MAP. The topologically simple non-precompact minimal groups are actually minimally almost periodic.

Theorem 3.9 motivates the question, on whether nilpotent minimal groups are precompact:

Question 3.10 [51, Question 3.5] *(a) Are minimal nilpotent groups precompact? Is this true for nilpotent groups of class 2 ?*

(b) Are solvable (at least metabelian) totally minimal groups precompact?

In item 2 of Remark 5.13 we see that the answer to item (a) is negative. Nevertheless, (b) still remains open.

On the other hand, from Theorem 3.5 and Proposition 3.3 one obtains:

Theorem 3.11 [51, Proposition 3.4] *Every nilpotent totally minimal group is precompact.*

Using Theorem 3.6, one can reformulate this theorem in the following equivalent form that matches better the implication (2) from the introduction: *complete totally minimal nilpotent groups are compact.*

3.3 On the structure of the minimal groups

Halmos [117] noticed that $\mathbb{R} \cong \mathbb{Q}^{(\mathfrak{c})} \cong \mathbb{Q}^\wedge$ algebraically, hence the underlying group of $\mathbb{R}$ admits a compact group topology. He also raised the general problem of describing the algebraic structure of the compact abelian groups. Here we recall some basic facts about the algebraic structure of minimal groups. In the sequel we give various examples of abelian groups admitting no minimal group topologies (e.g., $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$, $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$, among others).

Example 3.12 *For every prime p let $\mathbb{Z}_{(p)}$ denote the group of all rationals having denominators co-prime to p (i.e, the localization of the ring $\mathbb{Z}$ at p). For a non-empty set π of primes let $\mathbb{Z}_\pi = \bigcap_{p\in\pi} \mathbb{Z}_{(p)}$ and for completeness let $\mathbb{Z}_\emptyset = \mathbb{Q}$.*

- *(a) [163] $\mathbb{Z}_{(p)}$ admits a unique minimal topology, namely, the p-adic one, having as a local base of neighborhoods of 0 the subgroups $p^n\mathbb{Z}_{(p)}$. This minimal topology is not a bottom element in the poset of all Hausdorff group topologies on $\mathbb{Z}_{(p)}$, i.e., does not coincide with the Markov topology $\mathfrak{M}_{\mathbb{Z}_p}$ (the latter coincides with the co-finite topology of $\mathbb{Z}_{(p)}$ [76]).*
- *(b) More generally, for $\pi \neq \emptyset$, $\mathbb{Z}_\pi$ admits $|\pi|$-many minimal topology, namely, the p-adic ones, with $p \in \pi$ [163].*
- *(c) The group $\mathbb{Q} = \mathbb{Z}_\emptyset$ admits no minimal group topologies.*

The proof of (a), (b) and (c) use Theorems 3.1, 3.9 and Remark 3.2: if G is any torsion-free minimal abelian group, then its completion K is a torsion-free compact abelian group, so has the form $K = (\mathbb{Q}^\wedge)^\kappa \times \prod_{p\in\mathbb{P}} \mathbb{J}_p^{\kappa_p}$ for some cardinals κ, κ_p, $p \in \mathbb{P}$. Since $\mathbb{Q}^\wedge$ has a closed subgroup isomorphic to $\prod_{p\in\mathbb{P}} \mathbb{J}_p$, this implies $\kappa = 0$, if $r(G) < \infty$. Moreover, by Remark 3.2(b), $\kappa_p \leq 1$ for all $p \in \mathbb{P}$. Finally, if $r(G) < \infty$, then $\pi = \{p \in \mathbb{P} : \kappa_p = 1\}$ is finite and $|\pi| \leq r(G)$. In particular, if $r(G) = 1$ (as in both cases above), then $K \cong \mathbb{J}_p$, for some $p \in \mathbb{P}$. In case $G = \mathbb{Z}_\pi$, necessarily $p \in \pi$, as $\mathbb{Z}_\pi$ cannot be embedded in $\mathbb{J}_p$ of $p \notin \pi$. To conclude, let us note that all non-zero closed subgroups of $\mathbb{J}_p$ are open, therefore, every dense subgroup of $\mathbb{J}_p$ is also totally dense. Hence, all minimal topologies on the groups $\mathbb{Z}_\pi$ are also totally minimal.

Recall that by $r_p(G)$ we denote the p-rank of G. Using again Remark 3.2, one can see that $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$ does not admit any minimal group topology. More precisely,

Example 3.13 *(a) An infinite abelian group G of prime exponent p admits a minimal group topology if and only if G admits a compact group topology, i.e., $G \cong \mathbb{Z}_p^\kappa$ for some κ, that will be the weight of that compact group topology. Hence, algebraically, $G \cong \bigoplus_{2^\kappa} \mathbb{Z}_p$.*

(b) *Similar arguments work in the more general case of an abelian group of finite exponent* m*. If* $m = p_1^{k_1} \dots p_s^{k_s}$*, then the completion* K *of* G *will be a compact abelian group of exponent* m*, so* $K = \prod_{i=1}^s K_{p_i}$*. By Remark 3.2,* G *contains* $soc(K) = \prod_{i=1}^s soc(K_{p_i})$*. This yields*

$$r_{p_i}(G) = r_{p_i}(K) = r_{p_i}(K_{p_i}) = 2^{w(K_{p_i})},$$

in case K_{p_i} *is infinite. Therefore,* G *admits a minimal topology if and only if its* p_i*-torsion part admits a minimal topology, if and only if* $r_{p_i}(G)$ *is either finite or an exponential cardinal.*

According to (b), a countably infinite abelian group of finite exponent does not admit minimal topologies. According to Lemma 5.16 (C), there exist minimal countably infinite nilpotent groups of exponent m^2 *for every* $m > 1$*.*

The more general case of minimal torsion abelian groups is described in [67], where the class of abelian groups having a dense minimal torsion part (called, *exotic tori*) was thoroughly studied to this end.

Using Example 3.7, one can see that the totally minimal abelian groups of finite exponent are compact. This can easily be extended to the nilpotent case by induction on the nilpotency class. Therefore, a minimal bounded nilpotent group is totally minimal if and only if it is compact. An ample source of non-compact nilpotent (class 2) minimal locally precompact groups of finite exponent is supplied by Lemma 5.16. If one removes nilpotency, many examples of non-compact totally minimal countably compact groups are available which are of finite exponent (so, zero-dimensional [27] (or Example 3.23 (b))).

The structure of the minimal countable abelian groups is described in [165]. In the next example we recall the description of the (compact) completions of these groups.

Example 3.14 *The class of* $\mathcal{P}$ *of the compact abelian groups that admit a dense countable minimal group topology was introduced and described by Prodanov [165]. It consists of the compact abelian groups* G *such that* $r_p(G) < \infty$ *for every prime* p *and* G *contains no copies of* $\mathbb{J}_p^2$ *(the necessity of this condition follows from Remark 3.2). He proved that this is equivalent also imposing on* G *the following condition:*

(a) $n = \dim G < \infty$*; and*

(b) *every continuous surjective homomorphism* $f : G \to \mathbb{T}^n$ *has* $\ker f = \prod_p \mathbb{J}_p^{e_p} \times F_p$*, where each* F_p *is a finite* p*-group and* $0 \leq e_p \leq 1$*.*

The free abelian groups admitting minimal topologies were described by Stoyanov [194] (see §7.2 for more details as well as some more general results).

The next example treats the minimal topologies on divisible abelian groups.

Example 3.15 (a) *It was proved by Prodanov [68, Chap. 2] that* $\mathbb{Q}^n$ *admits no minimal group topology (this can be deduced from the argument given at the end of Example 3.12).*

(b) *Let* π *be a non-empty set of primes. It was proved in [67] that the group* $G_\pi := \mathbb{Z}_\pi/\mathbb{Z} = \bigoplus_{p\in\pi} \mathbb{Z}(p^\infty)$*, admits no minimal group topology if* $\pi \neq \mathbb{P}$*.*

(c) *For a non-exponential cardinal* $\kappa > \mathfrak{c}$ *the group* $G = \mathbb{Q}^{(\kappa)}$ *does not admit minimal group topologies, according to [68].*

The structure of the minimal divisible abelian groups in the general case is described in [46]: a divisible abelian group G *admits a minimal group topology if and only if either* G *admits a compact group topology or there exist* $n \in \mathbb{N}$ *and* $\pi \subseteq \mathbb{P}$ *such that* $|\pi| \leq r(G) < \mathfrak{c}$ *and* $tor(G) = (\mathbb{Q}/\mathbb{Z})^n \oplus \mathbb{Z}_\pi/\mathbb{Z}$*. In particular, a torsion divisible abelian group* D *admits a minimal topology if and only if* $G \cong (\mathbb{Q}/\mathbb{Z})^n$ *for some* $n \in \mathbb{N}$*, [67].*

The minimal topologies on the divisible abelian groups are always totally minimal [46]. Non-abelian divisible minimal groups need not be totally minimal even in the case of nilpotent groups of class 2 (see Remarks 5.12.1 and 5.13.1).

The structure of the minimal abelian groups of free-rank $< \mathfrak{c}$ is described in [44, 48]. The major advance in the general problem of description of the algebraic structure of the minimal abelian groups was achieved by F. Schinkel [179] (see [51, §4] for a detailed description of his results).

Under CH, free abelian groups of size $\mathfrak{c}$ admit countably compact group topologies ([200]), while the free groups do not admit countably compact group topologies ([73]). Shakhmatov [183] proved that on every free group G of infinite rank there exists a totally minimal group topology τ. This construction provides a group (G, τ) without Weil completion. The question on the existence of minimal precompact topologies on the free groups was raised by Stoyanov ([28, Question 3.3.5]). Remus [173] shows that on every countable free group there exist infinitely many non-isomorphic precompact totally minimal group topologies. He also proved that every free group F with cardinality α admits a precompact group topology of weight $\log \alpha$.

A topological group (G, τ) is (*totally*) *minimizable* if G admits a coarser (totally) minimal group topology. This trend was triggered by Prodanov's observation that $\mathbb{R}$ is not minimizable [163] (see also [69, 45] for further progress in this direction). Let us recall two questions from [28, §3.3E] that still remain open.

Question 3.16 (a) ([28, 3.3.3. Question]) *What is the structure of the abelian topological groups which are minimizable?*

(b) ([28, 3.3.4. Question]) *What is the structure of the abelian topological groups which are totally minimizable?*

Clearly, the groups in (a) are MAP and the question can be easily reduced to the case of precompact groups.

3.4 Direct products and minimality

After the first paper [90] on products, appeared Stoyanov's paper [195], containing the important notion of a *perfectly minimal group* (this is a group G such that $G \times H$ is minimal for every minimal group H).

Example 3.17 *According to Example 1.1(c), the compact groups and the torsion minimal groups are perfectly minimal. This was generalized later by Stephenson [191, Theorem 9] to minimal groups covered by compact subgroups.*

The following theorem of Stoyanov emphasizes the special role of the groups $(\mathbb{Z}, \tau_p)$ in connection with productivity:

Theorem 3.18 [195] *G is perfectly minimal if and only if $G \times (\mathbb{Z}, \tau_p)$ is minimal for every prime p.*

The following test for minimality of finite products shows that non-productivity of minimal groups is deeply rooted in the abelian case.

Theorem 3.19 [137]

1. *Let G and H be minimal groups. Then $G \times H$ is minimal if and only if $Z(G \times H)$ is minimal.*
2. *A minimal group G is perfectly minimal if and only if $Z(G)$ is perfectly minimal.*

According to item 2 of the theorem, minimal groups with compact (in particular, finite) center are perfectly minimal. Therefore, due to Theorem 3.9, complete minimal groups are perfectly minimal. We shall see now a stronger version of this property.

A topological group G is said to be *sequentially complete* if it is sequentially closed in its completion, that is, every Cauchy sequence in G converges. Obviously, complete groups and countably compact groups are sequentially complete. The next theorem was announced in [54, Theorem 6.1] (and in [51, Theorem 2.17], in the case of countably compact groups) without proof. Here we provide a complete proof.

Theorem 3.20 *Every sequentially complete minimal group is perfectly minimal.*

Proof. According to the above observation, it suffices to check that $Z(G)$ is perfectly minimal. Since $Z(G)$ is a closed subgroup of G, it is both minimal and sequentially complete. Hence it is enough to assume that $G = Z(G)$ is abelian. According to Theorem 3.18, it is enough to check that $G \times (\mathbb{Z}, \tau_p)$ is minimal for every prime p.

Let K be the (compact, by Theorem 3.9) completion of G. Then $K \times \mathbb{J}_p$ will be the completion of $G \times (\mathbb{Z}, \tau_p)$. Let $\pi : K \times \mathbb{J}_p \to K$ be the projection. Obviously, $soc(K \times \mathbb{J}_p) = soc(K \times \{0\})$ is contained in $G \times \{0\}$, by Remark 3.2 applied to the minimal group G. By the same remark, it suffices to check that every closed subgroup $N \cong \mathbb{J}_p$ of $K \times \mathbb{J}_p$ non-trivially meets $G \times (\mathbb{Z}, \tau_p)$.

In the sequel we identify, in the obvious way, K and $\mathbb{J}_p$ with the subgroups $K \times \{0\}$ and $\{0\} \times \mathbb{J}_p$, respectively, of the product $K \times \mathbb{J}_p$. If $N \cap K \neq \{0\}$ or $N \cap \mathbb{J}_p \neq \{0\}$, then Theorem 3.1 applies (in the former case one uses the hypothesis that G is minimal, in the latter case the fact that $\mathbb{Z}$ is essential in $\mathbb{J}_p$). Therefore, from now we assume that $N \cap K = \{0\}$ and $N \cap \mathbb{J}_p = \{0\}$. Then $\pi\restriction_N : N \to N_1 := \pi(N)$ is an isomorphism. So $N_1 \cong \mathbb{J}_p$ and $N_1 \times \mathbb{J}_p \cong \mathbb{J}_p^2$ has a natural structure of a $\mathbb{J}_p$-module.

Applying Theorem 3.1, one can find a non-zero element $x \in G \cap N_1$. Let $x = \pi(z)$, with $z = (x, y) \in N$. Then $z \neq 0$, hence $y \neq 0$ as well by our assumption $N \cap K = \{0\}$. The closed subgroup of N_1 (hence of K as well) generated by x, has the form $p^k N_1$ for some k, so it is metrizable. By the sequential completeness of G, the closed subgroup $G \cap N_1$ of G is sequentially complete. Being also metrizable, it is simply complete, hence compact. This proves that $p^k N_1 \subseteq G$. Find a non-zero $\xi \in \mathbb{J}_p$ such that $\xi y \in \mathbb{Z}$. Then

$$0 \neq \xi z = \xi(x, y) = (\xi x, \xi y) \in (N_1 \times \mathbb{Z}) \cap N,$$

so $0 \neq p^k \xi z \in (p^k N_1 \times \mathbb{Z}) \cap N \subseteq (G \times \mathbb{Z}) \cap N$. Therefore, $(G \times \mathbb{Z}) \cap N \neq \{0\}$. □

A careful look at the above proof shows that the sequential completeness of the group G was used only to show that some special elements of the group (e.g., like $x \in G \cap N_1$) are contained in a compact subgroup of G. Therefore, this proof provides also a proof of the perfect minimality result of Stephenson the mentioned in Example 3.17.

Proposition 3.21 *Let $\{G_i : i \in I\}$ be a family of topological groups such that for every $j \in I$ the subproduct $G_{(j)} = \prod_{i \in I \setminus \{j\}} G_i$ is unconditionally closed in $G = \prod_{i \in I} G_i$. If all G_i are (a-)minimal groups, then G is (a-)minimal as well.*

Proof. The proof in the option "minimal" goes as the proof of [137, Theorem 1.15]. Now assume that each G_i is a-minimal, i.e., carries its Markov topology $\mathfrak{M}_{G_i}$. Let $\mathfrak{T}$ be a Hausdorff group topology on G and let τ denote the product topology. To prove that $\mathfrak{T} \geq \tau$ it suffices to check that each projection $p_j : (G, \mathfrak{T}) \to G_j$ $(j \in I)$ is continuous. Since $G_{(j)}$ is unconditionally closed in G, it is also $\mathfrak{T}$-closed. So the quotient topology $\overline{\mathfrak{T}}$ on $G/G_{(j)} = G_j$ is Hausdorff, so $\overline{\mathfrak{T}} \geq \mathfrak{M}_{G_j}$, this yields that $p_j : (G, \mathfrak{T}) \to G_j$ $(j \in I)$ is continuous. □

The hypothesis of Theorem 3.21 is fulfilled when all groups of the family $\{G_i : i \in I\}$ are center-free (as $G_{(j)}$ is the centralizer of G_j, so unconditionally closed for every $j \in I$). Hence we obtain the following corollary, where only the option "a-minimal" is new (the "minimal" part was proved in [137, Theorem 1.15]).

Corollary 3.22 *Let $\{G_i : i \in I\}$ be a family of center-free topological groups. If all G_i are (a-)minimal groups, then $G = \prod_{i\in I} G_i$ is (a-)minimal as well.*

In particular, every power $S(\mathbb{N})^\kappa$ of the symmetric group $S(\mathbb{N})$ is a-minimal.

Stephenson [191, Question 10] asked whether $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$ is minimal. Eberhardt and Schwanengel proved that this power is even totally minimal (see [95]). Total minimality of all powers $(\mathbb{Q}/\mathbb{Z})^\kappa$ was established independently by Banaschewski (unpublished) and Grant [110]. Around the same time Arhangel′skiĭ proposed the problem to find classes of minimal abelian groups properly containing the class of all compact abelian groups and closed with respect to taking closed subgroups, products and Hausdorff quotients (briefly, $\mathbb{A}$-*classes*). Several examples of $\mathbb{A}$-classes were given in [78], where it was proved, among others, that the class $\mathfrak{A}$ of all precompact abelian groups G, containing the subgroup $\bigoplus_{p\in\mathbb{P}} \widetilde{G}_p$ of their completion $\widetilde{G}$, is the smallest $\mathbb{A}$-class. Since

$$\bigoplus_{p\in\mathbb{P}} \mathbb{T}_p = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty) = \mathbb{Q}/\mathbb{Z},$$

we conclude that $\mathbb{Q}/\mathbb{Z} \in \mathfrak{A}$, so this provides, as a by-product, an independent proof of the total minimality of all powers of $\mathbb{Q}/\mathbb{Z}$.

In [47] a general criterion for minimality of arbitrary products of minimal abelian groups is provided. This line is extended in [70], providing groups with extremal properties with respect to minimality of powers (e.g., G^n is minimal, but G^{n+1} is not is minimal, etc.), and studied further in [85, 88, 72] in the case of pseudocompact or countably compact minimal groups. Minimality of products is studied also by Dierolf, Eberhardt and Schwanengel in [96, 97, 95].

Example 3.23 *Let $\{G_i : i \in I\}$ be an infinite family of topologically simple compact groups. Then every non-trivial closed normal subgroup N of $G = \prod_{i\in I} G_i$ has the form $N = \prod_{i\in J} G_i$ for some $\emptyset \neq J \subseteq I$. Therefore, the direct sum $H = \bigoplus_{i\in I} G_i$ is totally dense in G. Since G is compact, the subgroup H is totally minimal (by Theorem 3.6) and non-compact. The following special choices ensure additional properties of H:*

- *(a) if all G_i coincide with the same compact connected simple Lie group L, then H is also connected;*
- *(b) if all G_i coincide with the same finite simple group L (e.g., $L = A_5$), then H is of finite exponent and zero-dimensional (as G is zero-dimensional in this case).*

In both cases one can find an involution $a \in L$ giving rise to an involution $\alpha = (a_i) \in L^I$ having all coordinates $a_i = a$. Then obviously $\alpha \notin H$ and α normalizes H (this will be needed in §4.1). Moreover, if I is uncountable, then the Σ-product

$$\Sigma G_i := \{x \in \prod_{i\in I} G_i : |supp\ x| \leq \omega\} \neq G$$

has the same properties as H (being totally dense in G) and in addition ΣG_i is also ω-bounded.

Now we recall a completeness property that is placed between completeness and h-completeness. A topological group G is said to be *totally complete* if G/N is complete for every closed normal subgroup $N \leq G$. Obviously, h-complete groups, as well as locally compact groups, are totally complete. On the other hand, a totally minimal group is totally complete if and only if it is h-complete.

As noted above, complete minimal groups are perfectly minimal. Hence *finite* products of complete minimal groups are minimal. On the other hand, if $\{G_i : i \in I\}$ is a family of totally minimal h-complete groups, then $\prod_i G_i$ is totally minimal [95]. These facts motivate the following:

Question 3.24 (Uspenskij [208]) *Is it true that arbitrary products of complete minimal groups are minimal?*

By Corollary 3.22, arbitrary products of symmetric groups are a-minimal. Since symmetric groups are complete, this can be considered as an evidence that the answer to Question 3.24 might be positive.

Theorem 3.20 would suggest to replace complete by the weaker version "sequentially complete". This weaker version turns out to fail even for the stronger property of countable compactness (which implies even h-sequential completeness). A countable family of minimal countably compact abelian groups with non-minimal product was found in [72]. The more subtle question of whether the countably infinite power G^ω of a minimal countably compact abelian group G is still minimal turned out to be irresolvable in ZFC, since it depends on the existence of measurable cardinals [51, 85, 54].

Minimality of products of locally compact groups was discussed in [174].

4 Some natural constructions and minimality

4.1 Semidirect products and G-minimality

Many examples of non-abelian minimal group are in fact semidirect products. We briefly recall some definitions. Let X and G be topological groups and

$$\alpha : G \times X \to X, \quad \alpha(g, x) = gx = g(x)$$

be a continuous action by group automorphisms. Then X is said to be a G-*group*. Denote by $X \leftthreetimes_\alpha G$ (or, simply by $X \leftthreetimes G$) the topological semidirect product (see for example [175, Section 6] or [68, Ch. 7]) with the multiplication

$$(x_1, g_1) \cdot (x_2, g_2) := (x_1 \cdot g_1(x_2),\ g_1 \cdot g_2).$$

Then $p : X \leftthreetimes_\alpha G \to G,\ p(x, g) = g$ is a continuous group homomorphism and also a retraction with $ker(p) = X := X \times \{e_G\}$. Inner automorphisms induced by $g \in G$ act on X via α, i.e., $gxg^{-1} = \alpha(g, x)$.

Here we discuss some positive and negative results regarding the following natural question:

Question 4.1 *When $X \leftthreetimes_\alpha G$ is minimal ?*

The counterpart of this problem in the case of direct products was completely reduced to the case of direct products of two abelian groups in view of Theorem 3.19. The above question turns out to be quite complex, with somewhat unexpected effects, as we see below. Again, for abelian X, the question is completely reduced to the closely related natural concepts of G-minimality and t-exactness (see Definition 4.8 and Theorem 4.14).

Direct products are a particular case of semidirect products with trivial action α. So, without any reasonable restrictions the immediate answer to general Question 4.1 is "not always" because by Doïtchinov's example already the group $(\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$ is not minimal, whence $(\mathbb{Z}, \tau_2)$ is minimal.

In the positive direction the situation is much better for complete X, as we shall see in Theorem 4.3.

Lemma 4.2 [175, Theorems 12.3 and 12.4]

1. *Let P be a topological group and let X be a normal subgroup of P. If X and P/X are both (Weil) complete, then P is a (Weil) complete group.*
2. *If $P := X \leftthreetimes G$ is a topological semidirect product then P is a (Weil) complete group if and only if X and G are both (Weil) complete.*

This lemma and the following theorem are typical (partial) solutions of the so-called "three space problem". Below, in Theorem 7.45, we give a modified version of Theorem 4.3 for *local minimality.*

Theorem 4.3 [68, Theorem 7.3.1] *Let P be a topological group and let X be a (totally) complete normal subgroup of P. If X and P/X are both (totally) minimal, then P is (totally) minimal, too.*

In particular, if X is complete then $X \leftthreetimes_\alpha G$ is minimal for minimal groups X and G.

Every complete subgroup is closed in any (Hausdorff) group. This fact is used in the proof of Theorem 4.3 exploiting Merson's Lemma 4.4 below which is a useful technical tool in many other occasions.

For every topological group (G, γ) and its subgroup H denote by γ/H the usual quotient topology on the coset space P/H. More precisely, if $pr : G \to G/H$ is the canonical projection then $\gamma/H := \{OH :\ O \in \gamma\} = \{pr(O) :\ O \in \gamma\}$.

Lemma 4.4 (Merson's Lemma, see for example [68, Lemma 7.2.3]) *Let (G, γ) be a not necessarily Hausdorff topological group and H be a not necessarily closed subgroup of G. Assume that $\gamma_1 \subseteq \gamma$ be a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1/H = \gamma/H$. Then $\gamma_1 = \gamma$.*

Since compact groups are perfectly minimal, the direct product $X \times K$ is minimal for every minimal group X and compact group K. In particular, $X \times \mathbb{Z}_2$ is minimal for every minimal group X. Surprisingly, one can find even totally minimal precompact groups X that admit a (necessarily non-trivial) action of the two element group $\mathbb{Z}_2$ on X so that the relative semidirect product $G = X \leftthreetimes \mathbb{Z}_2$ is not minimal. By Theorem 4.3 this cannot happen if X is complete. So, in the following theorem providing a complete description of all minimal groups X with this property, the group X cannot be complete (as items (b) and (c) directly show).

Theorem 4.5 *For a minimal group X with completion K the following three conditions are equivalent:*

(a) there exists an action $\mathbb{Z}_2 \times X \to X$ so that the relative semidirect product $G = X \leftthreetimes \mathbb{Z}_2$ is not minimal;

(b) there exists an involution $\iota \in Aut(X)$ such that the (unique) extension $\tilde{\iota} \in Aut(K)$ is the conjugation by an involution of $K \setminus X$;

(c) there exists an involution $a \in K \setminus X$ that normalizes X (i.e., $a \in N_K(X)$).

Proof. (c) → (b) The conjugation by the involution $a \in K \setminus X$ is the desired involution $\iota \in Aut(X)$.

(b) → (a) Consider the action $\langle \iota \rangle \times X \to X$ defined by ι and its extension $\tilde{\iota} : K \to K$. Let $G = X \leftthreetimes \langle \iota \rangle$ and $C = K \leftthreetimes \langle \tilde{\iota} \rangle$ be the respective semidirect products. By hypothesis, there exists an involution $a \in K$ such that $\tilde{\iota}(y) = a^{-1}ya$ for every $y \in K$. Obviously, $\tilde{\iota}(a) = a$, so $z = (a, \tilde{\iota}) \in Z(C)$. Moreover, $o(z) = 2$, so the cyclic subgroup $N = \langle z \rangle$ is normal and trivially meets G (as $a \notin X$). Hence G is not minimal.

(a) → (c) Extend the involutive automorphism $\sigma : X \to X$ defined by the action to a (necessarily involutive) automorphism $\tilde{\sigma}$ of K and consider the respective semidirect products $G = X \leftthreetimes \langle \iota \rangle$ and $C = K \leftthreetimes \langle \tilde{\iota} \rangle$. Since C is minimal by Theorem 4.3, the minimality criterion provides a closed non-trivial normal subgroup N of C such that $N \cap G$ is trivial. Since G is essential in K, this yields $K \cap N = 1$. Since C/K has two elements, we deduce that $|N| = 2$, so N is generated by an involution $z = (a, \tilde{\sigma}) \in Z(C)$, as N is normal. From this one can easily deduce that $\tilde{\sigma}(a) = a = a^{-1}$ and $\tilde{\sigma}(y) = aya$ for every $y \in K$. Since $\tilde{\sigma} \restriction_X = \sigma$, in particular $\tilde{\sigma}(X) \subseteq X$, so $a \in N_K(X)$. □

Now we can produce many examples of totally minimal precompact groups X with non-minimal $X \leftthreetimes \mathbb{Z}_2$:

Example 4.6 *Take a (totally) minimal precompact group X satisfying item (c) of Theorem 4.5 to obtain an action $\mathbb{Z}_2 \times X \to X$ so that the semidirect product $G = X \leftthreetimes \mathbb{Z}_2$ is not minimal. With the special choice (b) from Example 3.23 (all $G_i = A_5$), we obtain as a particular case an Example of Eberhardt-Dierolf-Schwanengel [96, Example 10].*

In Theorem 4.5 we saw that the semidirect product $G = X \leftthreetimes \mathbb{Z}_2$ may be non-minimal even when X is minimal. Example 4.7 is dedicated to the following related natural question: *can $X \leftthreetimes_\alpha \mathbb{Z}_2$ be minimal with X non-minimal*? If the action α is trivial, this is not possible, since direct summands of minimal groups are minimal. The example shows that the minimality of $X \leftthreetimes_\alpha G$ does not imply in general the minimality of X and this effect might occur in the most extreme way.

Example 4.7 *We build here a non-minimal group X with a t-exact action $\alpha : \mathbb{Z}_2 \times X \to X$, such that the semidirect product $H = X \leftthreetimes_\alpha \mathbb{Z}_2$ is minimal. To this end pick a prime p and let X be the group $\mathbb{Z} \times \mathbb{Z}$ equipped with the product of the p-adic topologies. Define the action $\alpha : \mathbb{Z}_2 \times X \to X$ by $\alpha(\iota, (x, y)) = (y, x)$, where ι is the unique generator of $\mathbb{Z}_2$. Then $H = X \leftthreetimes_\alpha \mathbb{Z}_2$ is minimal (see Proposition 4.10 for a much more general fact). This gives a non-minimal open index-2 (so necessarily, normal) subgroup X of a minimal group H.*

The above examples show that one needs a deeper understanding of the nature of semidirect products in order to obtain a criterion for minimality. To this end we introduce the notions of G-minimality and t-exactness of actions.

Definition 4.8 (See [139, 145]) *Let (G, σ) be a Hausdorff topological group and let (X, τ) be a G-group with respect to the continuous action $\alpha : G \times X \to X$.*

1. *A G-group X is said to be:*
   - G-minimal *if there is no strictly coarser Hausdorff group topology $\tau' \subseteq \tau$ on X such that α is (σ, τ', τ')-continuous.*
   - Strongly G-minimal *if $X \leftthreetimes_\alpha G$ is minimal.*
2. *The action α is topologically exact (*t-exact*, for short) if there is no strictly coarser,* not necessarily Hausdorff, *group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ', τ, τ)-continuous.*

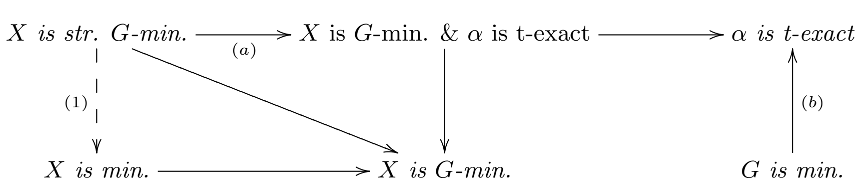
Every strongly G-minimal group X is G-minimal.

Remark 4.9 *Note that every topologically exact action is* algebraically exact*, that is, the kernel of the action $ker(\alpha) := \{g \in G : \ gx = x \ \ \forall \ x \in X\}$ is trivial. Therefore, the reader may keep in mind the following equivalent form of t-exactness of an action α as above:* α is t-exact provided it is algebraically exact and there is no strictly coarser Hausdorff group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ', τ, τ)-continuous. *In particular, in case the action α is algebraically exact:*

(a) if X is strongly G-minimal, then X is G-minimal and α is t-exact;

(b) if the group G is minimal, then the action α is t-exact.

In the sequel we examine these and some other implications (some of them obvious) in the following diagram, where solid arrow indicate implications that always hold true, whereas dotted arrows indicate implications that may fail.



The converse of implication (a) fails, as the Example 4.11 shows (a minimal (so $\mathbb{Z}_2$-minimal) X under a t-exact action that is not strongly $\mathbb{Z}_2$-minimal). The implication (1) fails in both direction already for $\mathbb{Z}_2$-actions, i.e., minimal $\not\Rightarrow$ strongly $\mathbb{Z}_2$-minimal (Example 4.7) and strongly $\mathbb{Z}_2$-minimal $\not\Rightarrow$ minimal (Example 4.6).

The failure of the converse of implication (b) requires a new (stronger) type of example, where the group G is not finite (neither compact). Such an example can be found in Example 4.15, where one has a strongly G-minimal group X, with both X and G non-minimal (according to (a), this yields that the action is t-exact, while the implication (1) fails).

Now we consider a very special case of a t-exact $\mathbb{Z}_2$-actions on an abelian topological group X, when minimality of X implies that X is strongly $\mathbb{Z}_2$-minimal.

Proposition 4.10 *Let L be a topological abelian group, $G = L \times L$ the direct product. Then the action $\alpha : \mathbb{Z}_2 \times G \to G$ given by $\alpha(x, y) = (y, x)$ is t-exact. Moreover, G is strongly $\mathbb{Z}_2$-minimal precisely when (L, τ) is minimal.*

Proof. By Remark 4.9, strong $\mathbb{Z}_2$-minimality of G implies that G is $\mathbb{Z}_2$-minimal, one immediately concludes that L is minimal if G is strongly $\mathbb{Z}_2$-minimal. (Indeed, if L is not minimal, then one can easily show that G is not $\mathbb{Z}_2$-minimal.)

Assume that L is minimal. We have to check that the semidirect product $P = G \leftthreetimes \mathbb{Z}_2$ is minimal. According to Theorem 3.9 the completion C of L is compact and $C \times C$ is the completion of G. So the completion K of P is compact and coincides with the semidirect product $K = (C \times C) \leftthreetimes \mathbb{Z}_2$ (for the sake of simplicity, we keep the notation α also for the obvious extension of the action α to $C \times C$).

Let N be a non-trivial closed normal subgroup of K and let $N_1 = N \cap (C \times C)$. If N_1 is trivial, then the subgroup of K generated by $C \times C$ and N coincides with K, as $[K : (C \times C)] = 2$. Since both $C \times C$ and N are normal, this would imply that $K \cong (C \times C) \times N$ is abelian, a contradiction as the action of $\mathbb{Z}_2$ is non-trivial. Therefore the closed normal subgroup N_1 of $C \times C$ is non-trivial. If either $(C \times \{0\}) \cap N_1 \neq \{0\}$ or $(\{0\} \times C) \cap N_1 \neq \{0\}$, then we are done, as L is an essential subgroup of C by Theorem 3.1. So assume $(C \times \{0\}) \cap N_1 = (\{0\} \times C) \cap N_1 = \{0\}$ in the sequel.

The subgroup N_1 of K, being a normal subgroup, is stable under the action of α. Consider the diagonal $\Delta_C = \{(x, x) : x \in C\}$ and the anti-diagonal $\Delta_C^* = \{(x, -x) : x \in C\}$ of $C \times C$. Let us see that either $N_2 = N_1 \cap \Delta_C \neq \{0\}$ or $N_3 = N_1 \cap \Delta_C^* \neq \{0\}$. Indeed, pick a non-zero element $z = (x, y) \in N_1$. Then also $\alpha(z) = (y, x) \in N_1$, so $d = (x + y, y + x) \in N_1 \cap \Delta_C$. If $d = 0$, then $y = -x$, so $0 \neq z \in N_1 \cap \Delta_C^*$. Otherwise, $0 \neq d \in N_1 \cap \Delta_C$.

Now assume that $N_2 \neq \{0\}$. Since the subgroup Δ_L of the diagonal Δ_C is dense and minimal (as $\Delta_L \cong L$), it is essential in Δ_C. Hence $\Delta_L \cap N_2 \neq \{0\}$. Thus $\Delta_L \cap N = \Delta_L \cap N_1 = \Delta_L \cap N_2 \neq \{0\}$.

In case $N_3 \neq \{0\}$ one can argue with Δ_L^* in place of Δ_L above. □

From Theorem 4.5 we obtain the following immediate

Example 4.11 *Let X be a minimal group with completion K and let $\mathbb{Z}_2 \times X \to X$ be an algebraically exact action. Then X is strongly $\mathbb{Z}_2$-minimal if and only if no involution $a \in K \setminus X$, normalizes X. In particular, X is strongly $\mathbb{Z}_2$-minimal if X is abelian or if the completion K of X has no involutions (e.g., if K is torsion-free).*

Definition 4.12 Let $q : X \to Y$ be a retraction of a group X on a subset Y (that need not be a subgroup, so q need not be a group homomorphism). We say that q is *central* if $q \circ \phi|_Y = id_Y (= q|_Y)$ for every internal automorphism ϕ of X.

Note that every retraction $q : X \to Y$ onto a subset $Y \subseteq Z(G)$ is central, while $id_X : X \to X$ is a central retraction precisely when X is abelian (so $X \subseteq Z(X)$), justifying the term central.

Theorem 4.13 [137], [145, Cor. 8.6] *Let (M, γ) be a topological group such that M is* algebraically *a semidirect product $M = X \leftthreetimes_\alpha G$, i.e., the topology on M is not necessarily the product topology of $X \times G$. Assume that there exists a continuous central retraction $q : X \to Y$ on a topological G-subgroup Y of X. Then the action*

$$\alpha|_{G \times Y} : (G, \gamma/X) \times (Y, \gamma|_Y) \to (Y, \gamma|_Y) \tag{4}$$

is continuous.

Proof. Let $pr : M \to G = M/X,\ (x, g) \mapsto g$, denote the canonical projection. By our hypothesis, $M/X = \{X \times \{g\}\}_{g \in G}$, which allows us to identify G with M/X and justifies the notation $(G, \gamma/X)$, so the group topologies γ/X and $pr(\gamma)$ are the same on G, [175, Prop. 6.17(a)]. On the other hand, we identify X with the subgroup $X \times \{e_G\}$ of M (i.e., we often write x in place of $(x, e_G) \in M$).

As each g-translation $(Y, \gamma|_Y) \to (Y, \gamma|_Y)$ is continuous, it suffices to show that the action (4) is continuous at (e_G, y) for every $y \in Y$. Fix an arbitrary $y \in Y$ and a neighborhood O of y in $(Y, \gamma|_Y)$. Since the retraction $q : (X, \gamma|_X) \to (Y, \gamma|_Y)$ is continuous at y there exists a neighborhood U_1 of y in (M, γ) such that

$$q(U_1 \cap X) \subseteq O. \tag{5}$$

Since the conjugation $M \times M \to M,\ (a,b) \to aba^{-1}$ is continuous at (e_M, y), there exist a neighborhood U_2 of y in M and a neighborhood $V \in \mathscr{V}(e_M)$ such that

$$vU_2v^{-1} \subseteq U_1 \qquad \forall\, v \in V. \tag{6}$$

We claim that $\alpha(pr(V) \times (U_2 \cap Y)) \subseteq O$. Indeed, if $v = (x, g) \in V$ and $z \in U_2 \cap Y$ then $vzv^{-1} \in U_1$ by (6). From the normality of X in M we have also $vzv^{-1} \in X$. Thus, $vzv^{-1} \in U_1 \cap X$, so (5) yields $q(vzv^{-1}) \in O$. On the other hand,

$$vzv^{-1} = (x, g)(z, e_G)(x, g)^{-1} = (x\alpha(g, z)x^{-1}, e_G) = x\alpha(g, z)x^{-1}.$$

Thus, $q(x\alpha(g, z)x^{-1}) = q(vzv^{-1}) \in O$. Since q is a central retraction and $\alpha(g, z) \in Y$, we obtain $\alpha(g, z) = q(x\alpha(g, z)x^{-1}) \in O$. Therefore, $\alpha(pr(V) \times (U_2 \cap Y)) \subseteq O$. Since $pr(V) \times (U_2 \cap Y)$ is a neighborhood of (e_G, y) in $(G, \gamma/X) \times (Y, \gamma|_Y)$, we conclude that the action (4) is continuous at (e_G, y). □

As the following results demonstrate, in several important cases, Theorem 4.13 together with the concept of t-exactness are very effective, e.g., the criterion from Remark 4.9(a) becomes an equivalence for abelian groups X.

Theorem 4.14 [139] *Let G be a topological group and let X be an abelian G-group with respect to an algebraically exact action $\alpha : G \times X \to X$. Then the following are equivalent:*

(a) X is strongly G-minimal (i.e., $X \leftthreetimes_\alpha G$ is a minimal group);

(b) the action α is t-exact and X is G-minimal.

Proof. The implication (a) → (b) was established in Remark 4.9(a), so it remains to prove the implication (b) → (a). Let σ and τ be the given topologies on G and X. Denote by γ the original product topology on $X \leftthreetimes G$. Assume that $\gamma_1 \subseteq \gamma$ is a coarser Hausdorff group topology. Then the action $(X \leftthreetimes G) \times (X \leftthreetimes G) \to X \leftthreetimes G$ by conjugations is $(\gamma_1, \gamma_1, \gamma_1)$-continuous. Then the restriction map $\alpha : (G, \gamma_1|_G) \times (X, \gamma_1|_X) \to (X, \gamma_1|_X)$ is also continuous. Clearly, $\gamma_1|_G \subseteq \gamma|_G = \sigma$. Hence α is also continuous with respect to the triple $(\sigma, \gamma_1|_X, \gamma_1|_X)$. Since, the action α is G-minimal we obtain that $\gamma_1|_X = \tau = \gamma|_X$. Since X is abelian the identity map $id : X \to X$ is a central retraction (Definition 4.12). By Theorem 4.13

$$\alpha : (G, \gamma_1/X) \times (X, \gamma_1|_X) \to (X, \gamma_1|_X)$$

is a continuous action. By t-exactness of the action (Definition 4.8) and since $\gamma_1/X \leq \gamma|_G$ on G, γ_1/X coincides with the given topology of G. Therefore, $\gamma_1/X = \gamma/X$. By Merson's Lemma 4.4 we have $\gamma_1 = \gamma$. □

Example 4.6 shows that commutativity of X in Theorem 4.14 is essential.

Dierolf and Schwanengel using semidirect products found the first examples of non-compact locally compact non-abelian minimal groups. This example demonstrates that for the minimality of $X \leftthreetimes_\alpha G$ the minimality of X or G are not necessary. This is one of the effects of non-commutativity which, as other examples show, is quite typical.

Example 4.15 *(Dierolf and Schwanengel [39]) The semidirect product*

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}_+, b \in \mathbb{R} \right\} \cong \mathbb{R} \leftthreetimes \mathbb{R}_+$$

of the group of all reals $\mathbb{R}$ with the multiplicative group $\mathbb{R}_+$ of positive reals is minimal. Here the action $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is given by $(a, x) \mapsto ax$ for $(a, x) \in \mathbb{R}_+ \times \mathbb{R}$. This action is t-exact and $\mathbb{R}$ is $\mathbb{R}_+$-minimal, so Theorem 4.14 applies. On the other hand, neither $\mathbb{R}$ nor $\mathbb{R}_+ \cong \mathbb{R}$ are minimal.

As we mentioned in §3.1, all *closed* subgroups of an abelian minimal group are again minimal. It follows by Example 4.15 that (locally compact) minimal groups may have non-minimal quotients and non-minimal closed subgroups. Arhangel′skiĭ posed the following two natural questions.

Question 4.16 ([6, Problem VI.6], [25, Problem 519], [28, Theorem 3.3F.2], [51, Question 2.9]) *Is every topological group a quotient of a minimal group ?*

Question 4.17 (See [28, Section 3.3F, Question 3.3.1(a)] and [51, page 57]) *Is every topological group G a closed subgroup of a minimal group M ?*

The following stronger conjecture was proposed by Pestov.

Conjecture 4.18 Every topological group is a group retract of a minimal topological group.

Below in Section 5.5 we see the positive solution of Conjecture 4.18 (and hence of Questions 4.16 and 4.17) by showing that for every Hausdorff topological group G there exists a (minimal) topological group X such that the semidirect product $X \leftthreetimes G$ is minimal (see Theorem 5.38).

We consider also the question, switching the roles of X and G.

Definition 4.19 *Let us say that a topological group X is* Aut-minimal *if $X \leftthreetimes_\alpha G$ is minimal for some topological group G and some continuous algebraically exact action $\alpha : G \times X \to X$.*

As it follows from Remark 4.9(a) the action α in Definition 4.19 is necessarily t-exact. If X is abelian then, as it directly follows from Theorem 4.14, X is Aut-minimal if and only if X is G-minimal with respect to some t-exact action $\alpha : G \times X \to X$.

The term "Aut-minimal group" is used in [139] for a LCA group X such that X is $Aut(X)$-minimal.

Trivial examples of Aut-minimal groups are all minimal (in particular, compact) groups. Obviously, $\mathbb{Z}$ is not Aut-minimal. More examples of non-Aut-minimal groups are provided by the following example:

Example 4.20 *According to a result of Shelah [186] (see also earlier results of Fuchs [101]) there exist arbitrarily large torsion-free abelian groups G with $|Aut(G)| = 2$, i.e., id_G and $-id_G$ are the only automorphisms of such G. Clearly, for such a group G, equipped with arbitrary Hausdorff group topology τ, being Aut-minimal is the same as being minimal. Hence, any non-minimal group (G, τ) with underlying group G as above, cannot be Aut-minimal.*

In Section 5.4 we continue to examine the Aut-minimality concept.

4.2 Homeomorphism groups

For every compact space X denote by Homeo$\,(X)$ its homeomorphism group endowed with the compact open topology having as basic neighborhoods of the identity the sets

$$[K, U] := \{f \in \text{Homeo}\,(X) | f(K) \subseteq U\},$$

where $K \subseteq X$ is compact and $U \supseteq K$ is an open set. Then Homeo$\,(X)$ is a topological group and the natural action Homeo$\,(X) \times X \to X$ is continuous.

Definition 4.21 *We say that a compact space K is* M-compact *if the topological group* Homeo$\,(K)$ *is minimal.*

We propose the following general question.

Question 4.22 *Which (notable) compact spaces are M-compact ?*

As one of the main motivations recall the following question of Stoyanov (reformulated in terms of M-compactness).

Question 4.23 (Stoyanov (see [6, 28, 51])) *Is it true that every homogeneous compact space K is M-compact ?*

Since we feel that the question makes sense also in the non-homogeneous case, we start a discussion of this general case. We impose on the compact space K to have a dense Homeo$\,(K)$-invariant subset D (i.e., for every $f \in$ Homeo$\,(K)$ one has $f(D) = D$). Then the restriction $\rho(f) := f|_D \in$ Homeo$\,(D)$ is well defined and completely determines f by the density of D, so defines an *injective* homomorphism $\rho :$ Homeo$\,(K) \to$ Homeo$\,(D)$. This map ρ is continuous when Homeo$\,(D)$ equipped with its own compact open topology. Indeed, for a compact subset $C \subseteq D$ and an open subset O of D with $C \subseteq O$

$$\rho^{-1}[C, O] = \bigcup\{[C, W] : W \text{ open in } K \text{ with } W \cap D = O\}$$

is open. In the case when D is discrete the continuity of ρ follows also from the fact that the subgroup $H = \rho(\text{Homeo}\,(K))$ of $S(D) =$ Homeo$\,(D)$ contains $S_\omega(D)$, so H, as a topological subgroup of $S(D)$, is *a*-minimal. Since ρ is injective, Theorem 2.2 yields that ρ is continuous.

The following example presents some natural instances when a subset D witnessing the above property is available.

Example 4.24 *(a) For every first countable non-compact space D, the remainder $\beta(D) \setminus D$ has no points of first countability (see Čech [22, p. 835]). Therefore, D is a dense* Homeo$\,(K)$*-invariant subset of K. Now the restriction map $\rho :$* Homeo$\,(K) \to$ Homeo$\,(D)$ *is an algebraic isomorphism due to the functorial properties of βD. In particular, when D is discrete, one has a continuous algebraic isomorphism*

$$\rho : \text{Homeo}\,(K) \to \text{Homeo}\,(D) = S(D), \tag{$*$}$$

by the above remark. We shall see below that this isomorphism is not open (i.e., K is not M-compact).

(b) The one-point Alexandrov compactification $K = \alpha D$ of a discrete space D has again a dense discrete Homeo$\,(K)$*-invariant subset, namely D. Now K is M-compact, as $(*)$ is a topological isomorphism, so* Homeo$\,(K)$ *can be identified, as a topological group, with $S(D)$. We see in Theorem 4.25 that this cannot be extended even to two point compactifications of discrete spaces.*

Let us recall that a topological space (X, τ) is *scattered* if every nonempty subspace has an isolated point. For a scattered compact space K the subset D of isolated points is dense and Homeo(K)-invariant, so the continuous injective homomorphism $(*)$ is available again (but now it need not be surjective). In item (b) of the above example we saw that a scattered compact space with a single non-isolated point is M-compact. The next theorem shows that this is the best one may have:

Theorem 4.25 *A scattered compact space K is M-compact if and only if K has at most one non-isolated point.*

Proof. If K has only isolated points, then K is finite, so the assertion if is obvious.

If K has only one isolated point, then K is the one point Alexandrov compactification αD of a discrete space D, so Example 4.24 applies.

Assume that the set $K' = K \setminus D$ of non-isolated points of K has size at least 2 and pick distinct $u, v \in K'$. Since K is scattered, we can choose $u \in K'$ to be an isolated point of K'. Then there are disjoint open neighborhoods O of u and O' of v in K, such that $O \cap K' = \{u\}$. Since K is regular, one can find a closed (hence, compact) neighborhood $W \subseteq O$ of u. Then $W \subseteq D \cup \{u\}$ is compact and $D \cap W$ is discrete, so $W = \alpha(D \cap W)$. Moreover, W is a neighborhood of every point $x \in W$ so W is open. Let us note also that $B = D \setminus W$ is infinite. Indeed, assume for a contradiction that B is finite. Then $W \cap O' = \emptyset$ yields that v has a closed neighborhood $U \subseteq O'$ that misses D, a contradiction since U is scattered.

The set $[W, W]$ is an open neighborhood of the identity in the compact-open topology of Homeo(K) as W is both compact and open. Consider $H = \rho(\text{Homeo}(K))$ as a topological subgroup of $S(D)$. Any neighborhood of identity in H contains some clopen subgroup of the form $G_F = \bigcap_{x \in F} St(x)$, for some finite subset F of D. Let us see that $[W, W]$ contains G_F for no finite $F \subseteq D$. Indeed, since $D \setminus W$ is infinite, for any finite $F \subseteq D$ there exist $w \in W \setminus F$ and $b \in D \setminus (W \cup F)$. Since $F \cup \{w, b\} \subseteq D$ is clopen, there exists $f \in \text{Homeo}(K)$ with $f \in G_F$, $f(w) = b$ and $f(b) = w$. Thus, $f \notin [W, W]$. □

The next theorem was proved recently by Banakh, Guran and Protasov [13]:

Theorem 4.26 *For any infinite discrete set D the compact βD is not M-compact.*

For a brief sketch of the proof notice that the restriction map $(*)$ is a continuous algebraic isomorphism according to Example 4.24. The isomorphism ρ, gives a Hausdorff group topology T_β on $S(D)$ coming from the compact open topology on Homeo(βD). By [13] the normal subgroup $S_\omega(D)$ is closed (and nowhere dense) in $(S(D), T_\beta)$, while $S_\omega(D)$ is pointwise dense in $S(D)$. Hence, that T_β is strictly finer than the usual pointwise topology on $S(D)$.

One of the possible general problems is to establish when some relevant compactifications with easily described homeomorphism group are M-compact. For example, the one-point compactification of a first countable locally compact space X that is not hemicompact shares the same homeomorphism group with X.

Theorem 4.27 (Gamarnik [102])

1. *If $K := [0,1]^n$ for $n \in \mathbb{N}$ then K is M-compact if and only if $n = 1$.*
2. *The Cantor cube $K := 2^\omega$ is M-compact.*

Gartside and Glyn proved in [103] that the group $H[0,1]$ is even a-minimal (see Problem 2.4).

Question 4.28 *Which compact n-dimensional (topological) varieties are M-compact? Which of the following geometric compacts are M-compact: n-dimensional sphere, Möbius band, Klein bottle ?*

The second assertion in the theorem of Gamarnik answers a concrete question from [28]. Recall that a zero-dimensional compact space X is *h-homogeneous* if all non-empty clopen subsets of X are homeomorphic. Uspenskij developed general method for establishing total minimality for some large, in a sense, groups. As a part of this general approach Uspenskij proved the following

Theorem 4.29 (Uspenskij [206]) *For every h-homogeneous compactum X the homeomorphism group* Homeo(X) *is minimal and topologically simple (hence, totally minimal).*

Since Cantor cube $X = 2^\omega$ is h-homogeneous, this theorem extends the result of Gamarnik. It is unclear if h-homogeneous is essential for homogeneous non-metrizable zero-dimensional compact spaces.

Question 4.30 *Is it true that every homogeneous zero-dimensional non-metrizable compact space is M-compact ?*

J. van Mill [149] recently proved that the n-dimensional Menger universal continuum is not M-compact for every $n > 0$. That is, for such n the homeomorphism group of the homogeneous, by a result of Bestvina, n-dimensional Menger universal continuum is not minimal. This answers Question 4.23. Note that the 0-dimensional Menger cube μ^0 is just the Cantor cube which is M-compact by the above mentioned result of Gamarnik. Recall that X *homotopically dominates* Y means that there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity function on Y.

Theorem 4.31 (J. van Mill [149])

1. *n-dimensional Menger universal continuum is not M-compact for every $n > 0$.*
2. *Let X_n be an n-dimensional compact space, $n > 0$, such that for every nonempty open subset U of X_n there is a compact subset A of U that homotopically dominates the n-sphere $\mathbb{S}^n$. Then* Homeo (X_n) *admits a weaker non-archimedean Hausdorff group topology τ_0 whose weight does not exceed the weight of X_n.*

In this theorem the non-archimedean group topology τ_0 is Hausdorff (this part is especially hard) and if $X_n = \mu^n$ is the n-dimensional Menger universal continuum then τ_0 is strictly weaker than the original compact open topology on Homeo (X_n). So, the latter is not minimal. Since this method has a potential to be useful in other cases we briefly describe that new topology τ_n. Let X_n be a compact space as in Theorem 4.31. In addition, let U be a dense subset of $C(X, \mathbb{S}^n)$ with $|U| \leq w(X_n)$. For $u \in U$ consider

$$C_u = \{h \in \text{Homeo}\,(X) : u \circ h \text{ is homotopically equivalent to } u\}.$$

Then for every $u \in U$, C_u is a clopen subgroup of Homeo (X_n). The family of all finite intersections of C_u is a local base at the identity of some Hausdorff group topology τ_0 on Homeo (X_n).

The homeomorphism group of (homogeneous) n-dimensional Menger universal continuum μ^n surprisingly has dimension 1, dim(Homeo $(\mu^n)) = 1$. This can be proved by combining results of Oversteegen-Tymchatyn [156] and Dijkstra [40].

J. van Mill's counterexample solves Stoyanov's question but also suggests further study of those (metrizable) homogeneous "famous" compacta that are M-compact. One of the most important homogeneous compacta is of course the Hilbert cube $[0,1]^\omega$. The question of Uspenskij if the (universal) Polish group Homeo $([0,1]^\omega)$ is minimal remains open.

Question 4.32 (Uspenskij [208]) *Is the Hilbert cube $K = [0,1]^\omega$ M-compact?*

4.3 Isometry groups and Urysohn spaces

For every metric space (M, d) the isometry group Iso (M) is a topological group in the pointwise topology. If V is a Banach space then we denote by $\text{Iso}_{lin}(V)$ the group of all linear onto isometries $V \to V$. Every topological group G is a topological subgroup of some $\text{Iso}_{lin}(V)$. This action induces the dual action of G on V^* and on B_{V^*}, the weak*-compact unit ball of the dual space V^*. The latter action yields an embedding of the topological group G into Homeo (B_{V^*}). It is well known that we can take to this end the Banach space $V := \text{RUC}(G)$ of all right uniformly continuous bounded functions on G. These facts were proved by Teleman (for details see, for example, [158]).

A metric space (X, d) is *metrically homogeneous* if Iso (X) transitively acts on X. Now we ask two general questions.

Question 4.33
1. *For which (metrically homogeneous) metric spaces (X, d) the isometry group* Iso (X) *is minimal ?*
2. *For which reflexive Banach spaces V the topological group* $\text{Iso}_{lin}(V)$ *of all linear isometries is minimal ?*

As to basic motivating examples, we recall the symmetric groups $S(X)$ (treating it as Iso (X, d_0), the 2-valued metric d_0). By Stoyanov's result (see Theorem 4.40) $\text{Iso}_{lin}(V)$ is minimal for every Hilbert space $V = H$.

The *Urysohn universal separable metric space* $\mathbb{U}$ is determined uniquely, up to an isometry, by the following description: $\mathbb{U}$ is a complete separable metric space which contains an isometric copy of every separable metric space and is ω-homogeneous, that is, every isometry between two finite subspaces of $\mathbb{U}$ can be extended to a global self-isometry of the space $\mathbb{U}$. Considering only metric spaces with diameter ≤ 1 one gets the universal Urysohn metric space $\mathbb{U}_1$ of diameter 1. A not necessarily separable metric space M of diameter 1 is said to be, *Urysohn of diameter 1*, [208] if it is *finitely injective* (with respect to spaces of diameter ≤ 1). That is, if it satisfies the following property: if F is a finite metric space of diameter ≤ 1, $K \subseteq F$ and $f : K \to M$ is an isometric embedding, then f can be extended to an isometric embedding of F into M. If, in addition, M is separable then it necessarily is isometric to $\mathbb{U}_1$. Also, $\mathbb{U}_1$ is isometric to the sphere of radius $1/2$ taken in $\mathbb{U}$ around any point. For more information on Urysohn spaces we refer to [160].

A topological group G is said to be *Roelcke-precompact* if the lower uniformity (the infimum of left and right uniformities, [175]) on G is precompact. While a locally compact group is Roelcke-precompact if and only if it is compact, many naturally defined non-abelian groups are Roelcke-precompact. Among others, Iso $(\mathbb{U}_1)$, $U(H)$, Homeo (2^ω), $\text{Homeo}_+[0,1]$, [207] and $S(X)$, Homeo $[0,1]$, [175, p. 169].

Very recently Rosendal and Culler proved in [176] that Homeo $([0;1]^\omega)$ is not Roelcke-precompact negatively answering a question of Uspenskij from [208]. It follows that if Homeo $([0;1]^\omega)$ is minimal (Question 4.32) then it will imply negative answer to Question 4.41.

The following result shows that any topological group can be embedded into Roelcke-precompact minimal group which is (totally) minimal.

Theorem 4.34 (Uspenskij [208, Theorem 1.3]) *Every topological group G is a subgroup of a totally minimal group H. This group H in addition is: a) complete; b) topologically simple; c) Roelcke-precompact and d) preserves the weight of G.*

This theorem follows as a corollary from the following result.

Theorem 4.35 (Uspenskij [208])

1. *For every topological group G there exists a complete ω-homogeneous Urysohn metric space M of diameter 1 of the same wight as G such that G is embedded into* $\mathrm{Iso}\,(M)$.
2. *If M is a complete ω-homogeneous Urysohn space of diameter 1, then the group* $\mathrm{Iso}\,(M)$ *is minimal topologically simple (hence, totally minimal), complete and Roelcke-precompact.*

By Theorem 4.35 the group $\mathrm{Iso}\,(\mathbb{U}_1)$ is a universal Polish group, minimal and Roelcke-precompact. In contrast, the larger group $\mathrm{Iso}\,(\mathbb{U})$ is not Roelcke-precompact [207].

Question 4.36 (Uspenskij [208]) *Is it true that* $\mathrm{Iso}\,(\mathbb{U})$ *is a minimal group ?*

4.4 Minimal groups in analysis

An immediate application of the minimality concept is the fact that a complete MAP group is minimal if and only if it is compact. This was first mentioned by Stephenson [190] for LCA groups.

One of the first motivations of minimality regarding Lie groups was a paper of Goto [112], where the criterion of minimality of connected Lie groups was established using adjoint representations $ad : G \to GL(n, \mathbb{R})$. Whenever $ad(G)$ is closed, G is said to be a *(CA) group*. According to Goto's theorem [112] a connected Lie group is minimal if and only if G is a (CA) group and the center $Z(G)$ is compact. First explicit and systematic consideration of minimal Lie groups comes from Remus and Stoyanov [174]. Using the above mentioned Goto's theorem they proved the following

Theorem 4.37 (Remus and Stoyanov [174]) *A connected semi-simple Lie group is totally minimal if and only if its center is finite.*

So, in particular, the special linear groups $SL(n, \mathbb{R})$ for every $n > 1$ are totally minimal. In the same paper the authors established minimality of the semidirect products $\mathbb{R}^n \leftthreetimes H$, where H is a closed subgroup of $GL(n, \mathbb{R})$ containing all diagonal matrices with positive entries and $n \in \mathbb{N}$. Which of course largely extends the special case of $\mathbb{R} \leftthreetimes \mathbb{R}_+$ (see Example 4.15). Using a technique different from [174, 39] these results were further strengthened in [139] by showing that "scalar matrices" are enough instead of "diagonal matrices". In particular, $\mathbb{R}^n \leftthreetimes \mathbb{R}_+$ is minimal for every natural n. All these results and the importance of matrix groups justify the following general question posed in [139].

Question 4.38 *For which topological fields K and subgroups H of $GL(n, K)$ are the groups $K^n \leftthreetimes H$ minimal ? In particular, when is $K \leftthreetimes K^\star$ minimal (where $K^\star = K \setminus \{0\}$) ?*

This question involves some important classes of topological fields and division rings. A *strictly minimal division ring* in the sense of Nachbin [153] is a topological division ring K which is a minimal module considered as a one-dimensional K-vector space (see also *straight division ring*, [210]). If K is strictly minimal and complete then every finite-dimensional vector K-space is topologically isomorphic to K^n. It is a result of Nachbin largely extending one of the classical results in analysis for the particular case of $K := \mathbb{R}$ which goes back to Tikhonov (1935) and Hausdorff (1931).

Any non-discrete *locally retrobounded division ring* K is strictly minimal. The latter means that retrobounded neighborhoods of zero in K form a fundamental system of neighborhoods of zero. (A subset U of K that contains zero is *retrobounded* if $(K \setminus U)^{-1}$ is bounded. A subset $B \subseteq K$ is *bounded* if for every $U \in \mathscr{V}(0)$ there exists $V \in \mathscr{V}(0)$ such that $VB \cup BV \subseteq U$.) Some particular examples occur in particular in the following cases: a) K is locally compact; b) K is topologized by an absolute value or a valuation; c) K is a linearly ordered field.

Theorem 4.39 [139] *Let K be a non-discrete locally retrobounded complete field and let $H \leq GL(n, K)$. Then $K^n \leftthreetimes H$ is minimal in each of the following cases:*

1. *H contains all scalar matrices with non-zero entries.*
2. *K is linearly ordered and H contains all scalar matrices with positive entries.*

The proof uses among others the G-minimality concept, Definition 4.8 and Theorem 4.14. Theorem 4.39.1 implies that the *affine group* $K^n \leftthreetimes GL(n, K)$ is minimal quite frequently. For $K := \mathbb{R}$ this result comes from the above mentioned result of Remus and Stoyanov. In Section 6 we discuss the affine groups $V \leftthreetimes GL(V)$ for normed spaces V regarding the concept of *relative minimality.*

Total minimality can be reformulated in terms of the open mapping property, a concept traditionally important in many branches of analysis. Surprisingly, many natural groups have this property. We already mentioned the total minimality of the groups $S(X), \mathbb{Q}/\mathbb{Z}, SL(n, \mathbb{R}), (\mathbb{Z}, \tau_p), \mathrm{Iso}\,(\mathbb{U}_1)$. We discuss below further examples. One of the brilliant ones belongs to Stoyanov.

Theorem 4.40 (Stoyanov [198]) *The unitary group $U(H)$ is totally minimal for any Hilbert space H.*

The methods in the proof have their own interest and beauty. Namely, Stoaynov describes equivariant G-compactifications of the G-space S, where $G = U(H)$ and S is the unit sphere of H. In particular, the maximal G-compactification of S is just the weakly compact unit ball B of H. According to [198], a general Tychonov G-space X is said to be *weakly minimal G-space* if every injective G-compactification $\phi : X \to Y$ of X is a homeomorphic embedding. It turns out that the unit sphere S and the projective space P of every Hilbert space H are weakly minimal $U(H)$-spaces. Probably the concept of weakly minimal G-spaces has additional interesting applications that justify its more systematic investigation.

Uspenskij [205] proposed a completely different proof of Stoyanov's $U(H)$ theorem making use of the Roelcke-completion (that is the completion with respect to the lower uniformity) of $U(H)$. The main result of [205] is the following fundamental fact – the Roelcke completion of $U(H)$ is just the compact affine semigroup $\Theta(H)$ (with separately continuous multiplication) of all non-expanding linear operators $\sigma : H \to H, ||\sigma|| \leq 1$. In his proof of total minimality of $U(H)$ Uspenskij uses methods of compact semitopological semigroups and elements of topological dynamics. One of the crucial step is to show that for every non-empty closed subsemigroup $S \subseteq \Theta(H)$ there exists a least (in the natural order of orthogonal projectors) idempotent p and $p = 1$ if and only if $S \subseteq U(H)$.

Uspenskij shows that his general method based on Roelcke-precompactness is quite universal for checking total minimality for several large Polish groups, among others, $U(H), \mathrm{Homeo}\,(2^\omega), \mathrm{Iso}\,(\mathbb{U}_1)$. It applies also in the proof of Theorem 4.35.

We suggest a deeper study of the connection of Roelcke-precompactness to minimality. While many minimal groups fail to be Roelcke-precompact (e.g., all locally compact minimal non-compact groups), it remains unclear even if Roelcke-precompactness combined with some other nice property (like completeness) always imply minimality.

Question 4.41 *Is it true that every Polish Roelcke-precompact group is minimal ?*

Note that this question is closely related to Question 2.3 because any oligomorphic group is Polish and Roelcke-precompact. In Question 4.41 *minimal* can be replaced by *totally minimal*, as it was observed by Uspenskij – because Polish and Roelcke-precompactness both properties are preserved by quotients.

One more important new example comes from a recent work of Glasner [106] which is strongly related to Stoyanov's result about total minimality of the unitary group $U(H)$.

Theorem 4.42 (Glasner [106]) *The Polish group $G = Aut(X, \mu)$ of automorphisms of an atomless standard Borel probability space (X, μ) is Roelcke-precompact and totally minimal.*

To prove the total minimality of $G = Aut(X, \mu)$ Glasner uses a simplified version of Uspenskij's above mentioned scheme (on the unitary group). In order to show that $G = Aut(X, \mu)$ is Roelcke-precompact Glasner identifies the corresponding Roelcke compactification with the space of Markov operators on $L_2(\mu)$. As a major step he shows that the algebra of right and left uniformly continuous functions, the algebra of weakly almost periodic functions, and the algebra of Hilbert functions on G (i.e. functions on G arising from unitary representations) all coincide. A continuous bounded function $f \in C(G)$ is called *weakly almost periodic* (notation: $f \in \mathrm{WAP}(G)$) if the orbit $fG := \{fg\}_{g \in G}$ forms a weakly precompact subset of $C(G)$.

The relevance of (total) minimality in Harmonic analysis was confirmed also in two papers of Mayer [135, 136]). Regarding assertion (2) below recall that a topological group G is *Eberlein* if the uniform closure of the Fourier-Stieltjes algebra $\mathcal{B}(G)$ (algebra spanned by all positive-definite functions) is just the algebra $\mathrm{WAP}(G)$.

Theorem 4.43 (Mayer [135]) *Let G be a connected locally compact group. The following conditions are equivalent:*

1. *G is totally minimal;*
2. *G is an Eberlein group;*
3. *For every closed normal subgroup N of G the center $Z(G/N)$ is compact.*

Proving the equivalence (1) $\Leftrightarrow$ (3), Mayer uses the above mentioned result of Goto and the possibility to approximate G by Lie groups. The latter is a part of Montgomery-Zippin solution of Hilbert's fifth problem [151]. By a result of Veech [209], semi-simple Lie groups with finite center are Eberlein groups. In this way Theorem 4.43 implies the earlier Remus-Stoyanov's Theorem 4.37.

Note that the locally compact nilpotent Eberlein groups are compact (Rudin [177], Chou [23]). Examples of non-locally compact Eberlein groups are the unitary group $U(H)$ and $\mathrm{A}ut\,(\mu)$ (respectively, [146] and [106]). It would be interesting to investigate the relevance of Eberlein groups related to (total) minimality of not necessarily locally compact groups.

Back to Mayer's papers, note that total minimality is an important tool studying ergodic properties of groups as well as the asymptotic behavior of matrix coefficients regarding unitary representations. Connected totally minimal locally compact groups are characterized as groups G of the following form: G has a compact normal subgroup such that the quotient is a semidirect product of a simply connected nilpotent group with a connected reductive group acting without non-trivial fixed points on the nilpotent piece. In particular, the following concrete classical groups are totally minimal: the *Euclidean motion group* $\mathbb{R}^n \leftthreetimes SO_n(\mathbb{R})$, $n \geq 2$ and the *Lorentz groups* $\mathbb{R}^n \leftthreetimes SL_n(\mathbb{R})$.

The following section deals with Heisenberg type groups so it is a natural continuation of the theme of minimal groups which are important in analysis. For further connections to analysis, as a source of many standard examples of locally minimal groups, see Section 7.

5 Generalized Heisenberg groups

Recall that the classical real 3-dimensional Heisenberg group $H(\mathbb{R})$ can be defined as the linear group of all unitriangular matrices:

$$\begin{pmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. As shown below, $H(\mathbb{R})$ is isomorphic to the semidirect product $(\mathbb{R}\oplus\mathbb{R})\leftthreetimes\mathbb{R}$ of $\mathbb{R}\oplus\mathbb{R}$ and $\mathbb{R}$. The group $H(\mathbb{R})$ and its higher-dimensional versions are important in analysis and quantum mechanics. The name "Heisenberg group" is motivated by the fact that the Lie algebra of $H(\mathbb{R})$ gives the Heisenberg commutation relations in quantum mechanics. Below we recall the definition of *generalized Heisenberg groups* which goes back to Hermann Weyl and play a major role in several branches of mathematics. For example, in Harmonic analysis (Reiter [172]), Topological Dynamics (Milnes [150]). Some applications (mainly minimality but not only) of generalized Heisenberg groups can be found in [83, 57, 64, 187, 148, 77].

The definition of the generalized Heisenberg groups is based on biadditive mappings. Namely, let E, F, A be abelian topological groups. A map $w : E \times F \to A$ is said to be *biadditive* if the induced mappings

$$w_x : F \to A, w_f : E \to A, \ w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$. We say that w is *separated* if these homomorphisms separate points of E and F, respectively. For every biadditive w define the induced action of F on $A \oplus E$ by

$$w^{\triangledown} : F \times (A \oplus E) \to (A \oplus E), \quad w^{\triangledown}(f, (a, x)) = (a + f(x), x).$$

Every translation under this action is an automorphism of the direct sum $A \oplus E$. Denote by

$$H(w) = (A \oplus E) \leftthreetimes F$$

the semidirect product of F and the group $A \oplus E$. This means that the group operation is defined by the following rule: for a pair

$$u_1 = (a_1, x_1, f_1) \in H(w), \quad u_2 = (a_2, x_2, f_2) \in H(w) \qquad (*)$$
$$u_1 \cdot u_2 = (a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2)$$

where, $f_1(x_2) = w(x_2, f_1)$. Then $H(w)$ becomes a group which is called the *generalized Heisenberg group* induced by w. We often identify A with $A \times \{0_E\} \times \{0_F\}$, E with $\{0_A\} \times E \times \{0_F\}$, F with $\{0_A\} \times \{0_E\} \times F$, $A \oplus E$ with $A \times E \times \{0_F\}$ and $A \oplus F$ with $A \times \{0_E\} \times F$. Note that $\{0_A\} \times E \times F$ is not a subgroup of $H(w)$. We have the naturally defined group retractions $q_F : H(w) \to F$ and $q_E : H(w) \to E$ with the kernels $A \oplus E$ and $A \oplus F$ respectively. The natural retraction $q_A : H(w) \to A$ is not a group homomorphism.

Lemma 5.1 *1. For $a, a_1, a_2 \in A$, $x, x_1, x_2 \in E$, $f, f_1, f_2 \in E$ and u_1, u_2 as in $(*)$ one has*

$$(a, x, f)^{-1} = (f(x) - a, -x, -f)$$

$$[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1} = (f_1(x_2) - f_2(x_1), 0_E, 0_F).$$

2. *$H(w)$ is a two-step nilpotent group.*
3. *If w is separated then $Z(H(w)) = A$ and the action $w^{\triangledown}$ is algebraically exact.*
4. *The natural projections $q_E : H(w) \to E$, $q_F : H(w) \to F$ and $q_A : H(w) \to A$ are central retractions (Definition 4.12).*
5. *Let E, F, A be topological groups. Then $w^{\triangledown} : F \times (A \oplus E) \to (A \oplus E)$ is continuous if and only if w is continuous.*

Proof. (4) If $u := (c, x, f) \in H(w)$, $y \in E$, $\varphi \in F$ and $b \in A$ then $uyu^{-1} = (f(y), y, 0_F)$, $u\varphi u^{-1} = (-\varphi(x), 0_E, \varphi)$ and $ubu^{-1} = b$. So q_E, q_F and q_A are central retractions.

(5) Easily follows from the identity $w^{\triangledown}(f, (a, x)) - (a, x) = (f(x), 0_E)$. □

Let E, F, A be topological groups and let w be continuous. Then, by Lemma 5.1.5, the group $H(w)$ is a well defined topological semidirect product. Of course this group is Hausdorff if and only if all three groups E, F, A

are Hausdorff. Note that if E, F, A are semitopological groups and w is separately continuous then $H(w)$ is a semitopological group. Similarly, if, E, F, A are paratopological groups and w is continuous then $H(w)$ is a paratopological group.

Intuitively one can describe the group $H(w)$ in the matrix form

$$\begin{pmatrix} I & F & A \\ 0 & I & E \\ 0 & 0 & I \end{pmatrix}$$

Clearly, for the scalar product $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we get the classical $2n+1$-dimensional Heisenberg group $H(w)$. Its factor group $H(w)/\mathbb{Z}$ (with respect to the central subgroup $\mathbb{Z}$) is known under the name *Weyl-Heisenberg group*. It can be identified with the Heisenberg group $H(w_0)$ with respect to the reduced biadditive mapping $w_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Varying the biadditive mappings we obtain a plenty of Heisenberg type groups. For example, for every LCA group G we have the canonical pairing $G \times G^{\wedge} \to \mathbb{T}$, where $G^{\wedge}$ is the Pontryagin dual. The corresponding locally compact group $H(w)$ we write simply as $H(G)$. Such groups have applications in Harmonic analysis, see for example, [172, 150] and references therein. Much more general possibility to create Weyl-Heisenberg type groups is as follows.

Definition 5.2 *Let V be a normed space with the canonical bilinear form $V \times V^* \to \mathbb{R}$, where V^* is the Banach dual.*

1. *By the generalized Heisenberg group (modeled on V) we mean the group $H(w)$, denoting it also by $H(V)$.*
2. *By the generalized Weyl-Heisenberg group (modeled on V) we mean the group $H(w_0)$ where $w_0 : V \times V^* \to \mathbb{T}$ is the corresponding reduced biadditive mapping obtained from w by composing it with the projection $\mathbb{R} \to \mathbb{T}$. Sometimes we write $H_0(V)$ instead of $H(w_0)$.*
3. *The groups $H(w)$ and $H(w_0)$ can be defined in a more general setting – for continuous bilinear mappings $w : E \times F \to \mathbb{R}$ and for the corresponding reduction $w_0 : E \times F \to \mathbb{T}$.*

Frequently we omit the word "generalized". Certainly, the Weyl-Heisenberg group $H(w_0)$ is a particular case of Heisenberg groups. The natural onto homomorphism $H(w) \to H(w_0)$ is open. As in the case of the classical $2n+1$-dimensional group mentioned above, $H(w_0)$ can be identified with the factor group $H(w)/\mathbb{Z}$, where $\mathbb{Z} \le \mathbb{R}$ is a central (in particular, a normal) subgroup of $H(w)$. Note that since the center of $H(w) = H(V)$ is isomorphic to the non-minimal group $\mathbb{R}$, the group $H(w)$ is not minimal (in contrast to the group $H(w_0)$). We explain below why many groups of Heisenberg type are minimal. For example, $H(G)$ for every LCA group G and $H_0(V)$ for every normed space V. This supports the idea we express in Section 4.4 that many *naturally defined* groups are minimal, or have some minimality properties.

Undoubtedly, the classical Heisenberg group $H(\mathbb{R}^n)$, as well as its quotient $H_0(\mathbb{R}^n)$, have some intrinsic beauty due to the fact that in this particular case $(\mathbb{R}^n)^{\wedge} = \mathbb{R}^n$. This suggests to dedicate a particular attention to the class of Heisenberg groups $H(X)$, where X is a self-dual LCA group. Recently they were studied in [161]. For a similar reason, the Heisenberg group $H(l_2)$ and its reduction $H_0(l_2)$ play a distinguished role as well (see also Remark 5.13.3).

If E and F are Banach spaces then for every continuous bilinear mapping $w : E \times F \to \mathbb{R}$ the Heisenberg group $H(w)$ is a Banach-Lie group. Indeed, w is a smooth map by [154, Remark II.2.7(b)]. This implies that the group multiplication and the inversion operation in the group $H(w)$ (defined in Lemma 5.1) are smooth.

5.1 From minimal dualities to minimal groups

Definition 5.3 *Let (E, σ), (F, τ), (A, ν) be abelian Hausdorff groups and*

$$w : (E, \sigma) \times (F, \tau) \to (A, \nu)$$

be a given continuous separated biadditive mapping.

- *(a) a triple $(\sigma_1, \tau_1, \nu_1)$ of coarser group topologies $\sigma_1 \subseteq \sigma$, $\tau_1 \subseteq \tau$, $\nu_1 \subseteq \nu$ on E, F and A, respectively, is called* compatible*, if $w : (E, \sigma_1) \times (F, \tau_1) \to (A, \nu_1)$ is continuous; in case $\nu_1 := \nu$ we call the pair (σ_1, τ_1) a w-*compatible pair*.*
- *(b) w is called* minimal *if $\sigma_1 = \sigma, \tau_1 = \tau$ holds for every w-compatible pair (σ_1, τ_1).*
- *(c) w is called* strongly minimal *if $\sigma_1 = \sigma, \tau_1 = \tau$ holds for every compatible triple $(\sigma_1, \tau_1, \nu_1)$ with Hausdorff ν_1.*

Every compatible triple $(\sigma_1, \tau_1, \nu_1)$ of Hausdorff group topologies gives rise to a coarser Hausdorff group topology on $H(w)$. This fact immediately implies item (a) of the following remark which is one of the motivations of Definition 5.3.

Remark 5.4 *(a) If $H(w)$ is a minimal group then necessarily w is a minimal biadditive mapping and A is a minimal group.*

(b) Observe that asking σ_1 and τ_1 to be Hausdorff in item (a) of the above definition is not restrictive, as far as ν_1 is Hausdorff. Indeed, the assumption that w is continuous and separated, implies that also σ_1 and τ_1 are Hausdorff.

The converse is also true as Theorem 5.8 shows. Clearly, strong minimality and minimality of the biadditive mapping w are equivalent if the group (A, ν) is minimal. However for many strongly minimal biadditive mappings w the group A need not minimal. For example, the multiplication map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is strongly minimal (Lemma 5.7.5). Note that the multiplication map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is minimal but not strongly minimal when $\mathbb{Z}$ carries the discrete topology.

The following example negatively answers the question posed by the first named author [51, Question 3.7] on whether a precompact nilpotent group G with minimal center $Z(G)$ must be minimal.

Example 5.5 *There exists a precompact ring topology σ on $\mathbb{Z}$ strictly finer than τ_2 (e.g., the pro-finite topology of $\mathbb{Z}$). The biadditive mapping $w : (\mathbb{Z}, \sigma) \times (\mathbb{Z}, \sigma) \to (\mathbb{Z}, \tau_2)$ is not minimal because the triple (τ_2, τ_2, τ_2) is compatible. Then the group $H(w)$ is the desired precompact non-minimal (by Remark 5.4) group with minimal center $Z(G) = (\mathbb{Z}, \tau_2)$.*

For every normed space E we denote by B_E its closed unit ball.

Definition 5.6 *Let E and F be normed spaces and $w : E \times F \to \mathbb{R}$ be a bilinear map. We say that*

1. *w is a* left strong duality *if for every norm-unbounded sequence $x_n \in E$ the subset $\{f(x_n) : n \in \mathbb{N},\ f \in B_F\}$ is unbounded in $\mathbb{R}$. This is equivalent to saying that $\{f(x_n) :\ n \in \mathbb{N},\ f \in O\} = \mathbb{R}$ for every neighborhood $O \in \mathscr{V}(0_F)$ (or, even, for every nonempty open subset O in F).*
2. *w is a* right strong duality *if for every norm-unbounded sequence $f_n \in F$ the subset $\{f_n(x) :\ n \in \mathbb{N},\ x \in B_E\}$ is unbounded in $\mathbb{R}$.*
3. *w is a* strong duality *if w is left and right strong.*

We give here some useful examples of strongly minimal biadditive mappings.

Lemma 5.7
1. *$G \times G^{\wedge} \to \mathbb{T}$ is strongly minimal for every LCA group G and its Pontryagin dual $G^{\wedge} := Hom(G, \mathbb{T})$.*
2. *$G \times G^{\wedge} \to \mathbb{Z}_m$ is strongly minimal for every LCA group G of exponent m.*
3. *For every normed space V the canonical bilinear form $V \times V^* \to \mathbb{R}$ is a strong duality.*
4. *Let $w : E \times F \to \mathbb{R}$ be a strong duality. Then w and the canonical reduction $w_0 : E \times F \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ both are strongly minimal biadditive mappings.*
5. *$w : V \times V^* \to \mathbb{R}$ and $w_0 : V \times V^* \to \mathbb{T}$ are strongly minimal for every normed space V.*

Proof. (1) Let $w : G \times G^{\wedge} \to \mathbb{T}$ be the natural (separated) biadditive mapping and let τ and $\tau^{\wedge}$ be the given compact-open topologies of G and $G^{\wedge}$ respectively. Assume that $(\tau_1, \tau_1^{\wedge})$ is an w-compatible pair. For a τ-compact subset $C \subseteq G$ and $\varepsilon > 0$ let $W(C, \varepsilon) = \{\chi \in G^{\wedge} : \chi(C) \subseteq O_\varepsilon\}$, where O_ε is the ε-neighborhood of the zero in $\mathbb{T}$. By the $(\tau_1, \tau_1^{\wedge})$-continuity of w and the τ_1-compactness of C there exists a $\tau_1^{\wedge}$-neighborhood U of 0_G such that $\chi(C) \subseteq O_\varepsilon$ for every $\chi \in U$. Therefore, $U \subseteq W(C, \varepsilon)$, so $\tau_1^{\wedge} = \tau^{\wedge}$. Similarly (or, using the duality) we get $\tau_1 = \tau$.

(2) This case can be reduced to (1). To this end we note first that since G has exponent m, every character $G \to \mathbb{T}$ can be identified with a continuous homomorphism into the unique m-element subgroup of $\mathbb{T}$, a copy of $\mathbb{Z}_m$. The same is true for the characters of $G^{\wedge}$. The evaluation mapping $w : G \times G^{\wedge} \to \mathbb{T}$ can be naturally restricted to $G \times G^{\wedge} \to \mathbb{Z}_m$.

(3) Easily follows using Hahn-Banach theorem.

(4) First of all observe that w is separated, being a strong duality. To see that w_0 is separated, pick a non-zero $x \in E$ and $f \in F$ with $a = f(x) \neq 0$. Let $f_1 = \frac{1}{a^2+1} f \in F$. Then $f_1(x) \neq \mathbb{Z}$, so $w_0(x, f_1) \neq 0$.

Let us check now that w and w_0 are strongly minimal. We consider only the case of w_0, the case of w is similar (and somewhat easier). Let σ, τ and ν_0 be given topologies on E, F and $\mathbb{R}/\mathbb{Z}$. Assume that (σ_1, τ_1) is a compatible triple for $w_0 : E \times F \to \mathbb{R}/\mathbb{Z}$, then $\nu_1 = \nu_0$ since $(\mathbb{R}/\mathbb{Z}, \nu_0)$ is compact, hence minimal. Denote by $q^{-1}(\nu_0)$ the preimage topology on $\mathbb{R}$ with respect to the map $q : \mathbb{R} \to (\mathbb{R}/\mathbb{Z}, \nu_0)$. The topology $q^{-1}(\nu_0)$ is not indiscrete, se one can choose $U \in \mathscr{V}(0)$ in $(\mathbb{R}, q^{-1}(\nu_0))$ with $U \neq \mathbb{R}$. The mapping

$$w : (E, \sigma_1) \times (F, \tau_1) \to (\mathbb{R}, q^{-1}(\nu_0)), \qquad (v, f) \mapsto \langle v, f \rangle = f(v) = w(v, f)$$

is obviously continuous. We show that $\sigma_1 = \sigma$ and $\tau_1 = \tau$. Suppose for a contradiction that σ_1 is strictly coarser than σ. By the $(\sigma_1, \tau_1, q^{-1}(\nu_0))$-continuity of w there exist: $P \in \mathscr{V}(0)$ in (E, σ_1) and $Q \in \mathscr{V}(0)$ in (F, τ_1) such that $\langle P, Q \rangle = \{f(x) : f \in P, x \in Q\} \subseteq U \neq \mathbb{R}$. According to [137, Lemma 3.5], every open subset of (E, σ_1) is norm unbounded because $\sigma_1 \subsetneqq \sigma$ and σ is the norm topology. As P is norm unbounded, Definition 5.6 yields $\langle P, Q \rangle = \mathbb{R}$, a contradiction. Similarly, one proves $\tau_1 = \tau$.

(5) Follows from (4) and (3). □

Theorem 5.8 [137], [64, Theorem 5.1] *Let $w : E \times F \to A$ be a continuous separated biadditive mapping and $H(w)$ be the corresponding Heisenberg group. Then $H(w)$ is a minimal group if and only if A is minimal and w is a minimal biadditive mapping.*

Proof. If $H(w)$ is minimal then the biadditive mapping w is minimal and A is a minimal group by Remark 5.4. The implication in the converse direction comes directly from the following lemma which has its own interest (and will be used later).

Lemma 5.9 *Let $w : E\times F \to A$ be a minimal biadditive mapping and $(H(w), \gamma)$ be the corresponding Heisenberg group. Suppose that $\gamma_1 \subseteq \gamma$ be a coarser Hausdorff group topology such that $\gamma_1|_A = \gamma|_A$. Then $\gamma_1 = \gamma$.*

Proof. By Merson's Lemma 4.4 it suffices to show that $\gamma_1/A = \gamma/A$. Denote by ν, σ, τ the given topologies on A, E and F respectively. Then γ is the product topology $\nu\times\sigma\times\tau$. The topological factor group $(H(w)/A, \gamma/A)$ with respect to the normal subgroup A can be identified with the topological direct sum $(E\oplus F, \sigma\times\tau)$. Therefore in order to verify $\gamma_1/A = \gamma/A$ it is sufficient to check the continuity of the onto homomorphism

$$\pi : (H(w), \gamma_1) \to (E \oplus F, \gamma/A) = (E \oplus F, \sigma \times \tau), \quad (a, x, f) \mapsto (x, f).$$

Since the *subset* $0_A \times E \times F$ is not a subgroup of $H(w)$, we do not identify it with the group $(E \oplus F, \sigma \times \tau)$. We shall make use of the retraction homomorphisms:

$$q_E : H(w) \to E,\ (a, x, f) \mapsto x, \qquad q_F : H(w) \to F,\ (a, x, f) \mapsto f$$

In order to show that π is $(\gamma_1, \sigma \times \tau)$-continuous it is enough to show that q_E and q_F are continuous with respect to the pairs of topologies (γ_1, σ) and (γ_1, τ) respectively. The topological factor groups $(H(w)/A \oplus E, \gamma/(A \oplus F))$ and $(H(w)/A \oplus F, \gamma/(A \oplus F))$ can be identified with (F, τ) and (E, σ), respectively. So we have to check that $\gamma_1/(A \oplus E) = \gamma/(A \oplus E) = \tau$ and $\gamma_1/A \oplus F = \gamma/(A \oplus F) = \sigma$.

First we present the arguments for the case $\gamma_1/(A \oplus E) = \tau$. Clearly, $\gamma_1|_A = \gamma|_A = \nu$, $\gamma_1|_E \subseteq \gamma|_E = \sigma$, $\gamma_1/X \subseteq \gamma/X = \tau$. In order to show that $\gamma_1/A \oplus E = \tau$ it is sufficient to establish the continuity of the map

$$w : (E, \gamma_1|_E) \times (F, \gamma_1/(A \oplus E)) \to (A, \gamma|_A) = (A, \gamma_1|_A) \tag{7}$$

Indeed, if (7) is continuous then the triple $(\gamma_1|_E, \gamma_1/A\oplus E, \gamma|_A)$ is compatible. Since the given biadditive mapping is minimal it will follow that the topology $\gamma_1/(A \oplus E)$ on F coincides with the given topology $\tau = \gamma/A \oplus E$.

We prove the continuity of the map (7) at an arbitrary pair $(x_0, f_0) \in E \times F$. Let O be a neighborhood of $f_0(x_0)$ in $(A, \gamma_1|_A)$. Choose a neighborhood O' of $(f_0(x_0), 0_E, 0_F)$ in $(H(w), \gamma_1)$ such that $O' \cap A = O$. Consider the points $\bar{x_0} := (0_A, x_0, 0_F), \bar{f_0} := (0_A, 0_E, f_0) \in H(w)$. Observe that the commutator $[\bar{f_0}, \bar{x_0}]$ is just $(f_0(x_0), 0_E, 0_F)$. Since $(H(w), \gamma_1)$ is a topological group there exist γ_1-neighborhoods U and V of $\bar{x_0}$ and $\bar{f_0}$ respectively such that $[v, u] \in O'$ for every pair $v \in V, u \in U$. In particular, for every $\bar{y} := (0_A, y, 0_F) \in U\cap E$ and $v := (a, x, f) \in V$ we have $[v, \bar{y}] = (f(y), 0_E, 0_F) \in O' \cap A = O$. We obtain that $f(y) \in O$ for every $f \in q_F(V)$ and $\bar{y} \in U \cap E$. This means that we have the continuity of (*) at (f_0, x_0) because $q_F(V)$ is a neighborhood of f_0 in the space $(F, \gamma_1/(A \times E))$ and $U \cap E$ is a neighborhood of x_0 in $(E, \gamma_1|E)$. This proves the first case. Similarly can be proved the second case of $\gamma_1/(A \oplus F) = \sigma$. □ □

The following two results both follow from Theorem 5.8 and Lemma 5.7. Theorem 5.11 is new.

Theorem 5.10 [137] *The generalized Heisenberg group $H(G)$ is minimal (and locally compact) for every LCA group G.*

Theorem 5.11 *The generalized Weyl-Heisenberg group $H_0(V)$ is minimal for every normed space V. More generally, for every strong duality $w : E \times F \to \mathbb{R}$ the reduction $H(w_0) = H(w)/\mathbb{Z}$ is a minimal group.*

In fact, one may show that every proper central quotient $H(w)/C$ of $H(w)$ is minimal for every non-zero cyclic subgroup C of $\mathbb{R}$. A proof can be obtained from a modification of Lemma 5.7.4 by showing that $w_C : E \times F \to \mathbb{T} = \mathbb{R}/C$ is a strongly minimal biadditive mapping.

Recall that by virtue of Lemma 5.1.2 the groups $H(G)$ and $H(w_0) = H(w)/\mathbb{Z}$ in Theorems 5.10 and 5.11 are 2-step nilpotent.

Remark 5.12 *Here we discuss when $H(G)$ and $H(w_0)$ are totally minimal or (locally) precompact.*

1. *For a LCA group G the following are equivalent:*
   - *G is finite;*
   - *$H(G)$ is totally minimal;*
   - *$H(G)$ is precompact.*
2. *Let V be a normed space.*
   - *The group $H(w_0)$ is totally minimal precisely when V is trivial (i.e., $H(w_0) \cong \mathbb{T}$).*

- *$H(w_0)$ is locally precompact precisely when V is finite-dimensional. In particular, as an illustration of Theorem 5.11 and in contrast with Stoyanov-Prodanov Theorem 3.9, the Weyl-Heisenberg group $H_0(l_2)$ is a minimal 2-step nilpotent Polish which is not locally precompact.*

Remark 5.13 *1. The classical Weyl-Heisenberg group $G := (\mathbb{T} \oplus \mathbb{R}^n) \leftthreetimes \mathbb{R}^n$ is minimal for every $n \in \mathbb{N}$. This follows from both of Theorems 5.10 and 5.11. As it was mentioned in [135] the minimality of G can be derived from Goto's criterion (see Section 4.4). See also direct elementary proof of minimality for G in [204, §14] for the case $n = 1$. This group G is divisible, since both $Z(G) \cong \mathbb{T}$ and $G/Z(G) \cong \mathbb{R}^{2n}$ are divisible.*

2. [65] *There exists a 2-step nilpotent non-abelian 2-dimensional minimal Lie group which is not precompact. Take for example $G := H(\mathbb{T}) = (\mathbb{T} \oplus \mathbb{T}) \leftthreetimes \mathbb{Z}$. This result negatively answers Question 3.10(a) on whether nilpotent minimal groups are precompact.*

3. *Both groups $H(X)$ and $H_0(V)$ (for a LCA group X or a normed space V, respectively) are minimal, nilpotent, having center (isomorphic to) $\mathbb{T}$. Therefore, by Theorem 3.5, a dense subgroup L of any of these groups is minimal (i.e, essential) precisely when L contains $soc(Z(H(X))$ or $soc(Z(H_0(V))$, respectively (in both cases, this socle is isomorphic to $\mathbb{Q}/\mathbb{Z}$).*

As a first application of Theorem 5.11 we recall the solution of Stoyanov's question and give a better example which is 2-step nilpotent using the new Theorem 5.11.

Question 5.14 (Stoyanov [198]) *Is it true that every minimal group is a subgroup of the unitary group $U(H)$ (unitarily representable) ?*

All abelian minimal groups being precompact are embedded into unitary groups. An example of a minimal solvable Polish group which is not unitarily representable was presented in [137] (this is the group $G := H_+(c_0) = H(c_0) \leftthreetimes_\alpha \mathbb{R}_+$ built in §5.4, which is Polish and minimal by Theorem 5.31). One may improve this result by using Theorem 5.11. Weyl-Heisenberg (Polish) group $G := H_0(c_0) = H(c_0)/\mathbb{Z}$ is minimal and 2-step nilpotent. Similar to the case of $H_+(c_0)$ one may show that G is not unitarily representable. Indeed, if G is embedded into $U(l_2)$ then it follows that c_0 as a uniform space is embedded into l_2. Then the *uniform universality* of c_0 (I. Aharoni [2]) would imply that l_2 is a universal uniform space for separable metrizable uniform spaces. The latter is not true by a classical result of P. Enflo [99] because there are countable metrizable uniform spaces not contained uniformly in Hilbert spaces.

5.2 Some locally precompact minimal groups

A topological space X is pseudocompact, if every continuous real-valued function on G is bounded. A topological group G is *locally pseudocompact* if there is a neighborhood U of the identity such that $cl_G U$ is pseudocompact. A locally pseudocompact groups is locally precompact and $\beta G = \widetilde{G}$. Metrizable locally pseudocompact group are actually locally compact.

In the sequel the dimension is used only for locally (pseudo)compact groups G. All three dimensions dim G = ind G = Ind βG coincide when G is locally compact (Pasynkov [157]). In case G is locally pseudocompact, we let $\dim G = \dim \beta G = \dim \widetilde{G}$.

Theorem 5.15 [137] *Every locally (pre)compact abelian group X is a group retract of a locally (pre)compact minimal group; namely of the Heisenberg group $L := \mathbb{T} \oplus X \leftthreetimes \widetilde{X}^\wedge$, where $\widetilde{X}$ is the completion of X.*

Proof. Apply Remark 5.13.3 to the dense subgroup $L := \mathbb{T} \oplus X \leftthreetimes \widetilde{X}^\wedge$ of $H(\widetilde{X})$ to conclude that the locally precompact group L is minimal. It remains to note that the natural projection $L \to X$ is a group retraction. □

Taking in mind some new applications (see Examples 3.13, 7.10, 7.22, 8.24 and 9.2) we need the following useful lemma.

Lemma 5.16 *Let X be an infinite precompact abelian group and $\kappa = w(X)$.*

(A) The locally precompact minimal group (from Theorem 5.15) has the following properties: $L := \mathbb{T} \oplus X \leftthreetimes \widetilde{X}^\wedge$ we have:

(i) $\widetilde{X}^\wedge$ is discrete and $\mathbb{T} \oplus X$ is an open subgroup of L of index κ;

(ii) $|L| = \mathfrak{c} \cdot \kappa \cdot |X|$, $w(L) = \chi(L) = \kappa$ and $t(L) = t(X \oplus \mathbb{T})$; in particular, L is metrizable if and only if X is metrizable, i.e., $\kappa = \omega$;

(iii) L is locally countably compact, whenever X is countably compact;

(iv) L is locally pseudocompact, whenever X is pseudocompact and

$$\dim L = \dim X + 1;$$

(v) $c(L) = \mathbb{T} \oplus X$, whenever X is connected (in particular, L is locally connected if and only if X is locally connected);

(vi) *L is NSS if and only if X is NSS (Definition 7.6).*

(B) *Assume that X, in addition, is non-archimedean. Then* $L := \mathbb{Q}/\mathbb{Z} \oplus X \leftthreetimes \widetilde{X}^\wedge$ *is a locally precompact zero-dimensional minimal group and* $\mathbb{Q}/\mathbb{Z} \oplus X$ *is an open subgroup of L;*

(C) *Assume that X, in addition, has finite exponent m. Then* $L := \mathbb{Z}_m \oplus X \leftthreetimes \widetilde{X}^\wedge$ *is a locally precompact minimal group with the following properties:*

(i) $\mathbb{Z}_m \oplus X$ *is an open subgroup of L;*

(ii) *L has finite exponent* m^2*;*

(iii) *L has no convergent sequences whenever X has no convergent sequences;*

(iv) *L is locally countably compact, whenever X is countably compact;*

(v) *L is locally pseudocompact, whenever X is pseudocompact.*

Proof. (A) $\widetilde{X}^\wedge$ is discrete because $\widetilde{X}$ is compact. Hence, $\mathbb{T} \oplus X$ is open in L being the kernel of the retraction $L \to \widetilde{X}^\wedge$. This proves (i) which easily implies (ii) – (vi). The equality $\dim L = \dim X + 1$ from (iv) easily follows from the main result of [201].

(B) Since X is non-archimedean and precompact, for every character $\chi : X \to \mathbb{T}$ the image $\chi(X)$ is a finite subgroup of $\mathbb{T}$. Indeed, choose a neighborhood U of $\{0\}$ in $\mathbb{T}$ which contains no nontrivial subgroup of $\mathbb{T}$. For the character $\chi : X \to \mathbb{T}$ choose an open subgroup V in X such that $\chi(X) \subseteq U$. Then $\chi(V) = 0$ by the choice of U. This means that χ vanishes on V, so factorizes through the factor $X \to X/V$, where X/V is finite, since X is precompact. Hence, $\chi(X) \subseteq \mathbb{Q}/\mathbb{Z}$. It follows that the group L is well-defined. The minimality of L follows from Remark 5.13.3.

(C) Use Lemma 5.7.2 to see that $L := \mathbb{Z}_m \oplus X \leftthreetimes \widetilde{X}^\wedge$ is a well-defined group and it has exponent m^2. The minimality of L follows from Theorem 3.5. The proof of (i) is similar to that of (A)(i). Other properties follow from (i). □

5.3 Representations on Heisenberg groups

The aim of this brief subsection is to explain how representations of groups can help to build minimal groups. In particular, in Section 5.5 we represent any topological group G as a retract of a (minimal) group $M = H(w) \leftthreetimes G$, where $H(w)$ is an appropriate Heisenberg group and the minimality of M is ensured by Theorem 5.20.

Definition 5.17 *Let G be a topological group and let* $w : E \times F \to A$ *be a continuous biadditive mapping. A (continuous) birepresentation of G in w is a pair* $\Psi = (\alpha_1, \alpha_2)$ *of (continuous) actions by group automorphisms* $\alpha_1 : G \times E \to E$ *and* $\alpha_2 : G \times F \to F$ *such that w is G-invariant, i.e.,* $w(gx, gf) = w(x, f)$*.*

Here comes the leading pair of examples that triggered the above notion of birepresentations.

Example 5.18 *Let* α *be a representation of G on a Banach space V (or on a LCA group X). The* adjoint (dual) representation α^* *of G on the dual Banach space* V^* *is defined by*

$$\alpha^* : G \to \mathrm{Iso}_{lin}(V^*), \ \alpha^*(g)(\psi) = g\psi, \ where \ (g\psi)(v) = \psi(g^{-1}v), \ v \in V, \psi \in V^*.$$

Similarly, the adjoint (dual) representation $\alpha^\wedge : G \times X^\wedge \to X^\wedge$ *on the Pontryagin dual* $X^\wedge$ *is defined by* $(g, f) \mapsto gf$ *with* $(gf)(x) := f(g^{-1}x)$*, where* $g \in G$ *and* $f \in X^\wedge$*.*

(a) The pair $\Psi := (\alpha, \alpha^\wedge)$ *is a birepresentation of G in* $w : X \times X^\wedge \to \mathbb{T}$*. If* α *is continuous, then* $\alpha^\wedge$ *is also continuous and hence* Ψ *is continuous too.*

(b) The pair $\Psi := (\alpha, \alpha^*)$ *is a birepresentation of G in* $w : V \times V^* \to \mathbb{R}$*. The adjoint representation* α^* *of G need not be continuous even if* α *is continuous and G is compact (see the discussion before Definition 5.34).*

More global view on birepresentations comes from the following observations. We have a naturally defined action of $Aut(E) \times Aut(F)$ on the set $BD(E, F, A)$ of all biadditive mappings $E \times F \to A$. Denote by $Aut(w)$ the stabilizer subgroup of $w \in BD(E, F, A)$. Then the homomorphisms $G \to Aut(w)$ completely describe the possible birepresentations of G on w.

Definition 5.19 *Let* $\Psi = (\alpha_1, \alpha_2)$ *be a continuous birepresentation of G in* $w : E \times F \to A$*. Similar to Definition 4.8(2), we say that* Ψ *is* t-exact *if for every strictly coarser,* not necessarily Hausdorff*, group topology on G the birepresentation does not remain continuous.*

It is equivalent to say that the natural coordinatewise action $\alpha : G \times (E \times F) \to (E \times F)$ of G on the group $E \times F$ is t-exact in the sense of Definition 4.8(2). The kernel of the latter action

$$ker(\alpha) = \{g \in G : gx = x, gy = y \ \ \forall (x, y) \in X \times Y\}$$

is exactly $ker(\alpha_1) \cap ker(\alpha_2)$. If $ker(\alpha_1) \cap ker(\alpha_2) = \{e\}$, we say that Ψ is an *algebraically exact birepresentation*.

These observations together with Remark 4.9 show that a birepresentation Ψ is t-exact if and only if Ψ is algebraically exact and for every strictly coarser *Hausdorff* group topology on G the birepresentation does not remain continuous.

If one of the actions α_1 or α_2 is t-exact (algebraically exact) then clearly Ψ is t-exact (resp., algebraically exact).

Let Ψ be a continuous G-birepresentation

$$\Psi = (w : E \times F \to A,\ \alpha_1 : G \times E \to E,\ \alpha_2 : G \times F \to F).$$

Then it induces a continuous representation of G on the Heisenberg group $H(w)$ defined by the following action

$$\pi : G \times H(w) \to H(w), \quad \pi(g, (a, x, f)) = (a, gx, gf).$$

The corresponding topological semidirect product $M(\Psi) := H(w) \leftthreetimes_\pi G$ is said to be the *induced group* of the birepresentation Ψ. The action π can be viewed also as a restriction of the natural action $Aut(w) \times H(w) \to H(w)$.

Theorem 4.13 and the central retraction concept, Definition 4.12 (along with Lemma 5.1.4), play major role in the proof of the following result.

Theorem 5.20 *Let $w : E \times F \to A$ be a minimal biadditive mapping with minimal group A. Suppose that Ψ is a continuous t-exact birepresentation of a topological group G in w. Then the induced group $M(\Psi) = H(w) \leftthreetimes_\pi G$ is minimal.*

Proof. We have to show that $M(\Psi)$ is minimal. Let $\gamma_1 \subseteq \gamma$ be a coarser Hausdorff group topology on $M(\Psi) = X \leftthreetimes G$, where $X := H(w)$. By Theorem 5.8 the group X is minimal. Therefore, $\gamma_1|_X = \gamma|_X$, while γ/X is the original topology on G. According to Merson's Lemma, to prove that $\gamma_1 = \gamma$, we have to check that $\gamma_1/X = \gamma/X$.

The G-retractions $q_E : H(w) \to E, q_F : H(w) \to F$ are central by Lemma 5.1.4. The subgroups E, F are also G-invariant as it follows by the definition of $M(\Psi)$. Now we may apply Theorem 4.13 twice for the retractions q_E and q_F. Then we obtain that each of the following actions are continuous:

$$(G, \gamma_1/X) \times (E, \gamma_1|_E) \to (E, \gamma_1|_E)\,, \qquad (G, \gamma_1/X) \times (F, \gamma_1|_F) \to (F, \gamma_1|_F)$$

Since, $\gamma_1|_E = \gamma|_E$, $\gamma_1|_F = \gamma|_F$ and $\gamma_1/X \subseteq \gamma/X$, the t-exactness of the birepresentation Ψ entails $\gamma_1/X = \gamma/X$. □

5.4 Some applications

Representations on locally compact groups

Let X be a locally compact group and $Aut(X)$ be the group of all automorphisms endowed with the *Birkhoff topology* (see [68, p. 260]). The latter is a Hausdorff group topology on $Aut(X)$ having as a local base at id_X the sets

$$\mathcal{B}(K, O) := \{f \in Aut(X) : \ f(x) \in Ox \text{ and } f^{-1}(x) \in Ox \ \ \forall\, x \in K\}$$

where K runs over compact subsets and O runs over neighborhoods of the identity in X. In the sequel $Aut(X)$ is always equipped with the Birkhoff topology.

For every subgroup $G \leq Aut(X)$ the Birkhoff topology is the coarsest group topology on G which makes the natural action of G on X continuous (for details see [137, Remark 1.9]). So in particular the natural continuous (and algebraically exact) action $\alpha : G \times X \to X$ is t-exact for every topological subgroup $G \leq Aut(X)$.

As already mentioned in Example 5.18(a), the dual action $\alpha^\wedge : G \times X^\wedge \to X^\wedge$ is also continuous for every LCA group X and the pair $\Psi := (\alpha, \alpha^\wedge)$ is a continuous birepresentation of G in $w : X \times X^\wedge \to \mathbb{T}$ in the sense of Definition 5.17. Moreover, Ψ is t-exact because already the original action α is t-exact. As in Section 5.3 we have the induced group $M(\Psi) = H(X) \leftthreetimes_\pi G$. This fact will be used below in Theorem 5.21 for instance.

When X is a compact group then $Aut(X)$ is a *topological* subgroup of Homeo$\,(X)$. This fact can be derived from the t-exactness of the action $Aut(X) \times X \to X$ mentioned above.

Recall that the canonical biadditive mapping $G \times G^\wedge \to \mathbb{T}$ is strongly minimal (Lemma 5.7.1). Taking into account again the t-exactness of $G \times X \to X$ for any LCA X and $G \leq Aut(X)$ as mentioned above, we obtain the following application of Theorem 5.20.

Theorem 5.21 *$M(\Psi) = H(X) \leftthreetimes_\pi G$ is a minimal group for every LCA group X and $G \leq Aut(X)$.*

Remark 5.22 ([137, Prop. 2.3]) *Every LCA group G naturally embeds $G \hookrightarrow Aut(\mathbb{T} \oplus G^\wedge)$ using the action $g(t, \chi) = (t + \chi(g), \chi)$.*

Theorem 5.23 *1. Every closed subgroup G of $GL(n, \mathbb{R})$ is a group retract of the minimal Lie group $H(\mathbb{R}^n) = ((\mathbb{T} \oplus \mathbb{R}^n) \leftthreetimes \mathbb{R}^n) \leftthreetimes G$.*

2. *If X is an LCA group containing $\mathbb{T}$, as a topological subgroup, then $X \leftthreetimes Aut(X)$ is minimal.*

Proof. (1) Apply Theorem 5.21 to $X := \mathbb{R}^n$.

(2) The subgroup $\mathbb{T}$ splits in X by [119, Section 25.31]. So, $X = \mathbb{T} \oplus G$ for some LCA group G. The dual group $G^\wedge$ is embedded into $Aut(\mathbb{T} \oplus G)$ (dual version of Remark 5.22). Moreover, the corresponding action of $G^\wedge$ on $X = \mathbb{T} \oplus G$ is exactly the action $w^\nabla : G^\wedge \times (\mathbb{T} \oplus G) \to (\mathbb{T} \oplus G)$ which determines the Heisenberg group $H(G)$. By Theorem 5.10, $H(G) = X \leftthreetimes G^\wedge$ is minimal. Then X is a (strongly) $G^\wedge$-minimal group (Definition 4.8). The action of $G^\wedge$ on X is a restriction of the action of $Aut(X)$ on X. Hence, X is $Aut(X)$-minimal, as well. Now $X \leftthreetimes Aut(X)$ is minimal by Theorem 4.14. □

Let $\alpha : G \times X \to X$ be a t-exact action in the sense of Definition 4.8. Since this action is by automorphisms, algebraically G can be identified with a subgroup of $Aut(X)$. If X is locally compact, then by the property of the Birkhoff topology we mentioned above, G can topologically be identified with its natural image in the topological group $Aut(X)$. This observation is used in Lemma 5.24.

Recall that by Definition 4.19 a topological group X is said to be *Aut-minimal* if $X \leftthreetimes_\alpha G$ is minimal for some topological group G and some continuous algebraically exact action $\alpha : G \times X \to X$.

Lemma 5.24 1. *If X is an LC group then X is Aut-minimal if and only if $X \leftthreetimes_\alpha G$ is minimal for some topological subgroup $G \leq Aut(X)$.*

2. *If X is LCA then the following conditions are equivalent:*
   - *X is Aut-minimal;*
   - *there exists a topological subgroup G of $Aut(X)$ such that X is G-minimal;*
   - *$X \leftthreetimes Aut(X)$ is minimal.*

Proposition 5.25 *We list here some non-trivial examples of Aut-minimal groups:*

1. *Every LCA group X which contains $\mathbb{T}$ as a topological subgroup.*
2. *The group F^n (in particular, $\mathbb{R}^n$) for any non-discrete locally compact field F and $n \in \mathbb{N}$.*
3. *The non-minimal group $X := (\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$ (see Example 1.1) is Aut-minimal.*

Proof. (1) In the case when $\mathbb{T}$ is embedded into X we apply the assertion (2) of Theorem 5.23 and Lemma 5.24.

(2) Combine Theorem 4.39 and Lemma 5.24.

(3) By Example 4.7 we know that $X \leftthreetimes_\alpha \mathbb{Z}_2$ is minimal for a natural action α. □

The additive group of any normed space E, as well as the Heisenberg group $H(w)$ for every normed space E and the canonical bilinear form $E \times E^* \to \mathbb{R}$, are also examples of Aut-minimal groups (see Proposition 5.37).

Besides Aut-minimality of locally compact groups X it would be interesting to study also when $X \leftthreetimes Aut(X)$ is totally minimal. In this case $Aut(X)$ should be totally minimal (see , e.g., Problem 5.29).

We recall the following general questions.

Question 5.26 [137] (see also [28])

(a) *Is every locally compact group a retract of a minimal locally compact group ?*

(b) *Is every Lie group a retract of a minimal Lie (or, at least, locally compact) group ?*

(c) *For which classes $\mathscr{K}$ of topological groups every $G \in \mathscr{K}$ is a group retract of a minimal group $M \in \mathscr{K}$?*

Every closed subgroup of $GL(n, \mathbb{R})$ is a Lie group. On the other hand many Lie groups are non-matrix groups, i.e., non-embeddable into $Aut(\mathbb{R}^n) = GL(n, \mathbb{R})$. For example, the 3-dimensional Weyl-Heisenberg minimal group $H_0(\mathbb{R}) = (\mathbb{T} \oplus \mathbb{R}) \leftthreetimes \mathbb{R}$ is one of such examples, [116, p. 314]. The following question is related to several issues raised in this section, but has undoubtedly its own interest. It may lead to interesting new classes of topological groups.

Question 5.27 *Which topological groups G embed into $Aut(X)$ for some locally compact (compact, locally compact abelian, locally compact nilpotent, etc.) group X ?*

A very recent result from [148] completely resolves the case of compact abelian X as follows,

Theorem 5.28 [148, Theorem 5.1] *For a topological group G the following are equivalent:*

(a) *G is non-archimedean;*

(b) *G is a topological subgroup of $Aut(X)$ for some compact abelian group X;*

(c) *G is a topological subgroup of $Aut(\mathbb{Z}_2^\kappa)$ for the Cantor additive group $\mathbb{Z}_2^\kappa$ of weight $\kappa = w(G)$.*

In this result it is essential that the compact group X is *abelian.* For every connected non-abelian compact group X the group $Aut(X)$ cannot be non-archimedean containing a nontrivial continuous image of X (namely, the non-trivial subgroup of internal automorphisms). All non-archimedean groups are totally disconnected and for every locally compact totally disconnected group G both G and $Aut(G)$ are non-archimedean (see Theorems 7.7 and 26.8 in [119]).

In the spirit of questions on Homeo(K) for a compact space K, one can ask the following question (closely related also to the (total) minimality of $G \leftthreetimes Aut(G)$ by Theorem 4.3):

Problem 5.29 *Describe the compact groups G such that $Aut(G)$ is minimal. When $Aut(G)$ is totally minimal ?*

Using Pontryagin duality and the groups from Example 4.20 one can get arbitrarily large compact connected abelian groups G with $|Aut(G)| = 2$. In general, $Aut(G)$ need not be minimal even for compact abelian groups. For example, let G be the Pontryagin dual of the discrete group $\mathbb{Q}$. Then $Aut(G)$ is topologically isomorphic to the discrete multiplicative group $\mathbb{Q}\setminus\{0\}$. The latter of course is not minimal (being abelian and non-precompact). As to an argument why $Aut(G)$ here is discrete note that $Aut(G)$ is Polish for every compact metrizable group G. Hence $Aut(G)$, being also countable, is discrete.

Problem 5.29 may have an interesting flavor for compact groups G with *countable* $Aut(G)$, since in this case the latter group is discrete. The countable discrete minimal groups are not easy to come by, as we mentioned in §2.1. As far as we know, automorphism groups have never been used in the framework of Markov's problem. So finding a compact group G such that $Aut(G)$ is both minimal and countable will be of considerable interest.

For more information about $Aut(X)$ see for example [119, 128, 199].

Representations on Stone spaces

To every *Stone space* (i.e., zero-dimensional compact space) X one may naturally associate the biadditive mapping $w : B \times B^\wedge \to \mathbb{Z}_2$, where $B := C(X, \mathbb{Z}_2)$ is a discrete Boolean ring which can be identified with the ring of all clopen subsets of X (with group operation the symmetric difference and multiplication the intersection). In the sequel we consider the underlying discrete Boolean group of B disregarding the ring structure, then the Pontryagin dual $B^\wedge$ can be identified with the compact group $Hom(B, \mathbb{Z}_2)$. Consider the corresponding Heisenberg group

$$H(w) = (\mathbb{Z}_2 \oplus B) \leftthreetimes B^\wedge.$$

It is curious to note, that $Z(H(w)) = \mathbb{Z}_2$, so $H(w)$ has exponent 4, as $H(w)/Z(H(w))$ is a Boolean group. Even if $Z(H(w))$ is a compact two-element group, the quotient $H(w)/Z(H(w))$ of the minimal group $H(w)$ is not minimal if B is infinite.

Non-archimedean groups are exactly subgroups $G \leq$ Homeo(X), where X runs over the class of Stone spaces, [147]. Every such subgroup naturally acts on X and defines a t-exact birepresentation on $w : B \times B^\wedge \to \mathbb{Z}_2$, where $B := C(X, \mathbb{Z}_2)$. Then Theorem 5.20 (taking into account Lemma 5.7.2) leads to the following result.

Theorem 5.30 [148] *For every Stone space X and every subgroup $G \leq$ Homeo(X) the groups $H(w)$ and $H(w) \leftthreetimes G$ are minimal and non-archimedean.*

This leads to the following result: for every non archimedean (and locally compact) G there exists a non archimedean (and locally compact, resp.) minimal group M such that G is a group retract of M. For discrete groups G the locally compact version was proved by S. Dierolf and U. Schwanengel [39].

Representations on normed spaces and in bilinear maps

The classical 3-dimensional Heisenberg group is not minimal. Moreover, $H(w)$ is not minimal for every separated bilinear mapping $w : E \times F \to \mathbb{R}$, where E and F are normed spaces (over the field of reals). Indeed, by Proposition 3.3, the closed central subgroup of a minimal group must be minimal, and the center of $H(w)$ is the subgroup $\mathbb{R}$ (Lemma 5.1.3) which is not minimal. There are two ways to produce a minimal group out of the non-minimal $H(w)$. The first one is suggested by the minimality of $\mathbb{R} \leftthreetimes \mathbb{R}_+$ (see Example 4.15). Following [137], consider the action

$$\alpha : \mathbb{R}_+ \times H(w) \to H(w), \quad \alpha(t, (a, x, f)) = (ta, tx, f).$$

and semidirect product $H_+(w) := H(w) \leftthreetimes_\alpha R_+$. (where the subgroup $\mathbb{R}$ of $H(w)$ becomes *relatively minimal* (in terms of Section 6)). Another possibility is to use the Weyl-Heisenberg groups (Theorem 5.11) which is simpler and have several advantages. For example, being a particular case of Heisenberg groups, every Weyl-Heisenberg group is 2-step nilpotent. For the sake of completeness we give the details also for the first case.

Theorem 5.31 *For every strong duality $w : E \times F \to \mathbb{R}$ with normed spaces E and F the corresponding (metrizable) group $H_+(w)$ is minimal. In particular, for every normed space V and the canonical bilinaer mapping $w : V \times V^* \to \mathbb{R}$ the corresponding group $H_+(w)$ is minimal.*

Proof. Let $\gamma_1 \subseteq \gamma$ be a coarser Hausdorff group topology on $H_+(w)$. Dierolf-Schwanengel's minimal group $\mathbb{R} \leftthreetimes \mathbb{R}_+$ is naturally embedded into $H_+(w)$. Therefore, $\gamma_1|_{\mathbb{R}} = \gamma|_{\mathbb{R}}$. From Lemma 5.9 we get $\gamma_1|_{H(w)} = \gamma|_{H(w)}$. The natural retraction $H(w) \to \mathbb{R}$ is central (Lemma 5.1.4) and the action of $\mathbb{R}_+$ on $\mathbb{R}$ is t-exact. Making use of Theorem 4.13, we obtain that $\gamma_1/H(w) = \gamma/H(w)$. So, by Merson's Lemma we conclude $\gamma_1 = \gamma$. □

Assume that the pair $\alpha_1 : G \times E \to E$, $\alpha_2 : G \times F \to F$ of continuous linear actions determines a birepresentation Ψ of G in $w : E \times F \to \mathbb{R}$. By the *induced group* $M_+(\Psi)$ of the given continuous birepresentation Ψ we mean the topological semidirect product $H_+(w) \leftthreetimes_\pi G$, where the action $\pi : G \times H_+(w) \to H_+(w)$ is defined by $\pi(g, (a, x, f, t)) := (a, gx, gf, t)$.

Analogously, as in Section 5.3, Ψ defines on the Weyl-Heisenberg group $H(w_0)$ the following natural action $\pi : G \times H(w_0) \to H(w_0)$ by the rule $\pi(g, (a, x, f)) := (a, gx, gf)$. The following *induced group* $H(w_0) \leftthreetimes_\pi G$ will be denoted by $M_0(\Psi)$.

Theorem 5.32 *Let $\Psi = (\alpha_1, \alpha_2)$ be a t-exact G-birepresentation of a topological group G into a strongly minimal $w : E \times F \to \mathbb{R}$ with normed spaces E and F. Then the corresponding induced groups $M_+(\Psi) = H_+(w) \leftthreetimes_\pi G$ and $M_0(\Psi) = H(w_0) \leftthreetimes_\pi G$ are both minimal.*

Proof. $M_0(\Psi) = H(w_0) \leftthreetimes_\pi G$ is minimal by Theorem 5.20. The minimality of $M_+(\Psi)$ follows using arguments similar to the arguments in the proofs of Theorems 5.31 and 5.20. □

Recall that every topological group is topologically isomorphic to a subgroup of $\mathrm{Iso}_{lin}(V)$ for a suitable Banach space V (see Section 4.3). So one may try to build minimal groups using induced groups of birepresentations from Section 5.4 (for canonical bilinear maps $V \times V^* \to \mathbb{R}$) keeping in mind Theorem 5.32. A principal difficulty here is the continuity of the adjoint representation. To emphasize this point, call a continuous representation $h : G \to \mathrm{Iso}_{lin}(V)$ on a Banach space V *adjoint continuous* if the adjoint representation $h^* : G \to \mathrm{Iso}_{lin}(V^*)$ is also continuous. It is a well known phenomenon in Functional Analysis that continuous representations on *general* Banach spaces need not be adjoint continuous (even for compact groups).

Example 5.33 *The regular representation of the circle group $\mathbb{T}$ on $V := C(\mathbb{T})$ is continuous but not adjoint continuous. Indeed, the continuity of the adjoint representation $\mathbb{T} \to \mathrm{Iso}\,(C(\mathbb{T})^*)$ is equivalent to the norm continuity of all orbit maps $\tilde{v} : \mathbb{T} \to C(\mathbb{T})^*, t \mapsto tv$ for every functional $v \in C(\mathbb{T})^*$. Now observe that the map $\mathbb{T} \to C(\mathbb{T})^*, t \mapsto t\delta_{x_0}$ is discontinuous for every point measure δ_{x_0}, where $\delta_{x_0}(f) := f(x_0)$.*

See some other examples and sufficient conditions in [140].

Definition 5.34 *A topological group G is said to be* adjoint continuously represented *on a Banach space V if there exists an adjoint continuous topological embedding $h : G \hookrightarrow \mathrm{Iso}_{lin}(V)$. Denote by $\mathcal{A}dj$ the class of all adjoint continuously representable topological groups.*

The class $\mathcal{A}dj$ will be involved in relevant applications in the sequel (see Corollary 5.36). This leads to the question which has its own interest: how large is $\mathcal{A}dj$? We briefly discuss this issue in the sequel.

It is well known (and easy to see) that if $V = H$ is a Hilbert space then the canonical adjoint representation for $U(H)$ (and a fortiori for every its subgroup) is adjoint continuous. Therefore, locally compact groups (being embedded into the unitary group $U(H)$ for Hilbert spaces H) are in $\mathcal{A}dj$.

Example 5.35 *A Banach space V is* Asplund *if for every separable linear subspace $W \subseteq V$ the dual Banach space W^* is separable. Every reflexive space is Asplund. All continuous representations of G on Asplund Banach spaces are adjoint continuous (this is one of the main results from [140]). So it follows that for any Asplund space V any topological subgroup $G \leq \mathrm{Iso}_{lin}(V)$ belongs to $\mathcal{A}dj$.*

Examples of groups $G \notin \mathcal{A}dj$ are not easy to come by. Such a group was found only recently in [107] (and also by Uspenskij, unpublished), namely $\mathrm{Homeo}_+[0,1] \notin \mathcal{A}dj$ (where, $\mathrm{Homeo}_+[0,1]$ is the group of all end-point preserving homeomorphisms of $[0,1]$).

The next result follows from Theorem 5.32.

Corollary 5.36 *Let $h : G \hookrightarrow \mathrm{Iso}_{lin}(V)$ be an adjoint continuous embedding and $\Psi = (h, h^*)$ be the corresponding* continuous *birepresentation. Then the corresponding induced groups $M_+(\Psi) = H_+(w) \leftthreetimes_\pi G$ and $M_0(\Psi) = H(w_0) \leftthreetimes_\pi G$ are both minimal.*

The case of M_+ is covered by results of [137], while the case of $M_0(\Psi)$ is new and has not appeared before.

Theorem 5.31 allows us to obtain two more examples of Aut-minimal groups (see Proposition 5.25):

Proposition 5.37 *The following are examples of Aut-minimal groups:*

1. *the Heisenberg group $H(w)$ for every normed space E and the canonical bilinear form $E \times E^* \to \mathbb{R}$;*
2. *the additive group of any normed space V.*

Proof. (1) By Theorem 5.31, $H_+(w) := H(w) \leftthreetimes \mathbb{R}_+$ is minimal. Clearly, the action of $\mathbb{R}_+$ on $H(w)$ is algebraically exact (it is already true for the restricted action of $\mathbb{R}_+$ on $\mathbb{R} \leq H(w)$ (see Example 4.15)).

(2) Represent V as a topological direct sum $\mathbb{R} \oplus E$, where E is a closed linear subspace of E of codimension 1. Consider the bilinear mapping $w : E \times E^* \to \mathbb{R}$. By Theorem 5.31 there exists an action $\alpha : \mathbb{R}_+ \times H(w) \to H(w)$ such that $H_+(w) := H(w) \leftthreetimes \mathbb{R}_+$ is minimal. It is easy to see that the minimal group $H_+(w)$ is naturally topologically isomorphic to the semidirect product $(\mathbb{R} \oplus E) \leftthreetimes (E^* \times \mathbb{R}_+)$ and the action of $E^* \times \mathbb{R}_+$ on $\mathbb{R} \oplus E$ is algebraically exact. Hence $V = \mathbb{R} \oplus E$ is Aut-minimal. □

5.5 Every topological group is a group retract of a minimal group

In the present section we discuss the following theorem and give various applications.

Theorem 5.38 [145] *For every topological group G there exists a G-group X such that the topological semidirect product $M = X \leftthreetimes G$ is a minimal group.*

In particular, it follows that every topological group G is a group retract of a minimal group M.

As we already know the Heisenberg type groups are minimal under certain mild conditions (Theorems 5.10, 5.11, 5.20 and 5.32). Moreover, many naturally defined minimal groups arise out of representations of groups via appropriate group automorphisms. So looking for "sufficiently many" representations of topological groups on Heisenberg groups it is a reasonable (and seemingly, also powerful) idea for constructing "sufficiently many" minimal groups. The general idea in the proof of Theorem 5.38 is to explore the induced group of a birepresentation on a Heisenberg group, Theorem 5.32.

We saw in Corollary 5.36 how our results on minimality of Heisenberg type groups can be useful proving Theorem 5.38 in the class $\mathcal{A}dj$ (see Definition 5.34). This class already contains a great part of all topological groups (but not all of them) and at the same time emphasizes the difficulty of the general case.

An immediate consequence of Corollary 5.36 is that every topological group G from $\mathcal{A}dj$ is a retract of a minimal group. As we saw in §5.4, locally compact groups are in $\mathcal{A}dj$, as well as topological subgroups $G \leq \mathrm{Iso}_{\,lin}(V)$ for any Asplund space V (see Example 5.35), yet $\mathcal{A}dj$ does not contain all topological groups. So Corollary 5.36 does not cover all topological groups (for more information see [145, Section 7]).

The recent development, allowing to complete the proof of Theorem 5.38, is based on some methods of dynamical systems and Banach spaces. The idea is to find sufficiently many t-exact representations of G on some bilinear mappings $w : E \times F \to \mathbb{R}$ that are not necessarily canonical. One of the crucial parts is to represent any bounded left and right uniformly continuous function on a topological group G as a matrix coefficient of some suitable continuous birepresentation of G. More precisely we show that for every $f \in \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$ there exist: a strongly minimal (Definition 5.3) bilinear mapping $w : E \times F \to \mathbb{R}$ with Banach spaces E and F, a continuous birepresentation (h_1, h_2) of G in w and a pair $(u, v) \in E \times F$, such that $f(g) = w(gu, v)$ for every $g \in G$. The proof of this fact, as well as that of item (a) in the next lemma, uses a dynamical modification of a Davis-Figiel-Johnson-Pelczyński factorization method from Banach space theory.

Remark 5.39 *Theorem 5.38 can be proved in two different ways that we describe now. In (a) we offer some more details on the proof given in [145, Theorem 7.2]). In (b) we discuss an alternative way, using the Weyl-Heisenberg groups.*

(a) *There exists a strongly minimal bilinear mapping $w : E \times F \to \mathbb{R}$ with Banach spaces E, F and a t-exact birepresentation $\Psi = (\alpha_1, \alpha_2)$ of G into w such that: α_1 and α_2 are actions by linear isometries, $max\{d(E), d(F)\} \leq w(G)$ and the desired minimal group M can be obtained as the induced group $M_+(\Psi)$ of Ψ (according to Theorem 5.32). That is*

$$M := M_+(\Psi) = H_+(w) \leftthreetimes G.$$

The desired group retraction is the canonical projection $p : M \to G$ and the kernel of this retraction $ker(p) = H_+(w)$ is minimal, Weil complete and solvable.

(b) *Theorem 5.38 can be proved also using Weyl-Heisenberg groups, which is a new approach based on Theorems 5.11 and 5.32. This case allows one to present M in a simpler form. More precisely, the desired minimal group is*

$$M := M_0(\Psi) = H(w_0) \leftthreetimes G$$

where w and Ψ are the same as in item (a). The kernel $ker(p) = H(w_0)$ of the retraction $p : M \to G$ is minimal, Weil complete and 2-step nilpotent.

(c) *In both cases (a) and (b) the group M is perfectly minimal and the following properties can be preserved from G to M: (1) weight; (2) character; (3) metrizability; (4) pseudocharacter; (5) density character; (6) (Weil) completeness; (7) solvability; (8) (local) connectedness; (9) NSS and NSnS (Definition 7.6).*

Some comment on item (c) are in order. Concerning perfect minimality of M, recall that a topological group is perfectly minimal if and only if its center is perfectly minimal, by Theorem 3.19. So it remains to observe that $Z(M_+(\Psi)) = \{e\}$ and $Z(M_0(\Psi)) = \mathbb{T}$. Item (6) can be derived from the fact that semidirect products preserve (Weil) completeness (Lemma 4.2.2). Regarding item (7) observe that M is an extension of a solvable group G by the solvable group $H_+(w)$ (respectively, by the 2-step nilpotent group $H(w_0)$). For (9) see Lemma 7.7.

Theorem 5.38 answers Pestov's Conjecture 4.18 and Arhangel′skiĭ's Questions 4.16, 4.17 about quotients and (closed) subgroups of minimal groups. Recall that earlier Uspenskij 4.34 proved that every topological group is a topological subgroup of a totally minimal topologically simple group which is complete and Roelcke-precompact. Theorem 5.38 also answers the following two questions:

Question 5.40 *1.* (Arhangel′skiĭ; see [28, Section 3.3D]) *Let M be a complete minimal group. Must $\chi(G) = \psi(G)$? What if M is Weil-complete?*

2. (Arhangel′skiĭ [6, Problem VI.4]; see also [28, Section 3.3G]) *Suppose that a minimal group acts continuously and transitively on a compact space X. Must X be a dyadic space ? Must X be a Dugundji space?*

Theorem 5.38, combined with Remark 5.39(c), imply that for every (Weil) complete group G with $\chi(G) \neq \psi(G)$ the minimal group M has the same properties concerning both completeness and the gap between these two cardinal invariants. In particular, one can achieve the same gap between $\chi(M) > \psi(M)$ for a (Weil) complete minimal group M, as one can achieve *without asking minimality.* This answers negatively Question 5.40.1. (see also the discussion of Arhangel′skiĭ's question 1.3 in Section 2).

As another corollary of Theorem 5.38 we get

Corollary 5.41 *Every compact Hausdorff homogeneous space admits a transitive continuous action of a minimal group.*

Proof. Let X be a compact Hausdorff homogeneous space. Then the group $G := \mathrm{Homeo}\,(X)$ of all homeomorphisms of X is a topological group with respect to the usual compact open topology. The natural action $\alpha : G \times X \to X$ is continuous. This action is transitive because X is homogeneous. By Theorem 5.38 there exists a minimal group M and a continuous group retraction $p : M \to G$. Then the action $M \times X \to X$, $mx := \alpha(p(m), x)$ is also continuous and transitive. □

Applying this corollary to a compact homogeneous space X which is not dyadic we get an immediate negative answer to Question 5.40.2.

Remark 5.42 *Uspenskij [208] derives from Theorem 5.38 that every topological group G is a quotient of a* Weil-complete *minimal group. Indeed, G is a quotient of a Weil-complete group G_c by a result of Uspenskij. Now by Theorem 5.38, G_c is a retract of a Weil-complete minimal group M.*

6 Relative minimality and co-minimality of subgroups

Definition 6.1 *1.* [143, 64] *A subset X of a topological group (G, τ) is* relatively minimal *in G if $\sigma|_X = \tau|_X$ for every Hausdorff group topology $\sigma \subseteq \tau$ of G.*

2. [64] *A subgroup X of (G, τ) is* co-minimal *in G if every coarser Hausdorff group topology $\sigma \subseteq \tau$ of G induces on the coset space G/X the original topology. That is, $\sigma/X = \tau/X$.*

Every subgroup H of a minimal group G is obviously relatively minimal in G, while any minimal group is relatively minimal in a larger topological group.

Less obvious are the following properties (see [64]). A G-group X is G-minimal (Definition 4.8) if and only if X is relatively minimal in $X \leftthreetimes G$. A subgroup H of a topological group G is relatively minimal in G if and only if its closure is relatively minimal in G. Merson's Lemma implies that G is minimal if and only if there exists $H \leq G$ such that H is relatively minimal and co-minimal in G. Every G is a closed relatively minimal (or/and co-minimal) subgroup in some group by Theorem 5.38.

Theorem 6.2 *Let X be a central subgroup of a topological group G and let H denote the closure of X in G. Then the following are equivalent:*

(a) X is relatively minimal in G;

(b) X is relatively minimal in H;

(c) H is a minimal topological group.

As a corollary one gets: a closed central (one cannot omit "central" by Theorem 6.3 below) subgroup X of a topological group G is relatively minimal in G if and only if X is minimal. In particular, the center $Z(G)$ of G is relatively minimal if and only if $Z(G)$ is minimal. Non-trivial examples of relative minimality we may find in non-abelian groups. For instance, in Heisenberg groups. The proof of the following result is similar to the proof of Lemma 5.9.

Theorem 6.3 [64] *Let $w : E \times F \to A$ be a strongly minimal biadditive mapping. Then:*

1. *the subgroups E and F are relatively minimal in the Heisenberg group $H(w)$;*
2. *the subgroups $A, A \oplus E$ and $A \oplus F$ are co-minimal subgroups of $H(w)$.*

The first assertion in Theorem 6.3 is reversible. For every separated biadditive mapping $w : E \times F \to A$ if the subgroups E and F are relatively minimal in $H(w)$ then w is a strongly minimal mapping. This observation easily follows from the fact (mentioned in Section 5.1) that every compatible triple $(\sigma_1, \tau_1, \nu_1)$ of Hausdorff group topologies gives rise to a coarser Hausdorff group topology on $H(w)$.

Let R be a topological associative unital ring. Denote by $M_n(R)$ the matrix ring equipped with the Tychonov topology. In the sequel we consider topological subgroups of the linear group $GL_{n+2}(R)$. More specifically, the subgroup $U_{n+2}(R)$ of all $(n+2) \times (n+2)$ upper unitriangular matrices and for pairs of indexes $(i, j) \neq (1, n+2)$, $i < j$ the subgroup $G_{ij}^{n+2}(R)$ of $U_{n+2}(R)$ of the form

$$G_{ij}^{n+2}(R) := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & a_{ij} & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \middle| \, a_{ij} \in R \right\}.$$

Theorem 6.4 (Shlossberg [187]) *Let F be a topological division ring furnished with an archimedean absolute value. Then every subgroup $G_{ij}^{n+2}(F)$ (with $(i, j) \neq (1, n+2)$, $i < j$) is relatively minimal in the group $U_{n+2}(F)$.*

In the particular case of $F = \mathbb{R}$, $n = 1$ and $1 = i < j = 2$ this theorem gives exactly the case of 3-dimensional Heisenberg group. This case can be derived also from Theorem 6.3 for $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

For every normed space V consider the corresponding *affine group* $V \leftthreetimes GL(V)$, where $GL(V)$ carries the uniform topology. The proof of following theorem is a variant of the proof in Proposition 5.37.2.

Theorem 6.5 [139] *Every normed space (as an additive topological group) V is* ***relatively*** *minimal in the affine group $V \leftthreetimes GL(V)$.*

Question 6.6 *For what infinite-dimensional normed spaces V is the affine group $V \leftthreetimes GL(V)$ minimal? This question seems to be unclear even for the separable Hilbert space $V = l_2$.*

By Theorem 4.14 it is equivalent to ask: is the action of $GL(V)$ on a normed space V t-exact? The answer is "Yes" for finite dimensional V because by Remus-Stoyanov theorem the affine group $\mathbb{R}^n \leftthreetimes GL(n, \mathbb{R})$ is minimal, [174]. Compare also with Theorem 4.39, where $K^n \leftthreetimes GL(n, K)$ is minimal for every non-discrete locally retrobounded complete field K.

Definition 6.7 *A subgroup $H \leq G$ of a topological group G is* strongly closed (strongly open) *in G if H is closed (open) for every coarser Hausdorff group topology.*

These concepts and co-minimality are strongly related as the following properties confirm.

- If H is closed (open) and co-minimal in G, then H is strongly closed (open) in G.
- If H is a strongly closed normal subgroup of G and G/H is minimal, then H is co-minimal in G.
- If H is a closed normal subgroup of finite index of a topological group G, then H is co-minimal if and only if H is strongly closed.

6.1 Some applications of (relative) minimality to group representations

First we give some useful facts about Banach representations of topological groups. For more details we refer to [16, 158, 144, 108]. A *representation* (by linear isometries) of a topological group G on a Banach space V is a continuous homomorphism $h \colon G \to \mathrm{Iso}_{lin}(V)$. As we already mentioned in Section 4.3 by Teleman's result any topological group G admits an embedding (topologically faithful representation) into $\mathrm{Iso}_{lin}(V)$ at least for the Banach space $V := \mathrm{RUC}(G)$. So any G is *Banach representable.* While every locally compact group is Hilbert (say also, *unitary*) representable according to the celebrated Gelfand–Raĭkov theorem, there are many known examples of Polish groups which are not unitary representable (see Banasczyk [16]). A topological group G is reflexively representable (i.e., G is embedded into $\mathrm{Iso}_{lin}(V)$ endowed with the strong operator topology for some reflexive V) if and only if the algebra $\mathrm{WAP}(G)$ of all weakly almost periodic functions determines the topology of G.

The first example distinguishing Hilbert and reflexive cases appears in [141]; the additive group of the Banach space $L_4 := L_4[0, 1]$ is reflexively representable, but not Hilbert representable. A typical application of relative minimality is transforming "non-embedding" examples into "non-injective". This idea works in Theorems 6.8 and 6.11. The following result answers a question of A. Shtern [189].

Theorem 6.8 [143] *There exists a (Polish) group G which is reflexively representable but its Hilbert representations do not separate the points.*

Proof. Consider the pairing $L_4 \times L_{\frac{4}{3}} \to \mathbb{R}$ and its Heisenberg group $G = H(L_4)$. As we already noticed L_4 is reflexively representable, [141]. Hence, WAP(G) generates the topology of L_4. One may show that the same is true for G making use Grothendieck's double limit test of WAP (see details in [143]). Therefore, G is reflexively representable, too. At the same time G is not Hilbert representable. Indeed, L_4 is a relatively minimal subgroup in G (Theorem 6.3). Hence every continuous injective homomorphism $G \to U(H)$ into the unitary group induces an embedding of L_4. But L_4 is not Hilbert representable, a contradiction. □

The question if WAP(G) determines the topology of a topological group G was raised by Ruppert [178]. This question was negatively answered in [142] by showing that the topological group $G := \mathrm{Homeo}_+[0,1]$ of all end-point preserving homeomorphisms of the closed interval has only constant WAP functions (and every representation on a reflexive Banach space is trivial). Glasner and Megrelishvili asked (see [144]) if there is an *abelian* group which is not reflexively representable ? The following important result of Ferri and Galindo solves this and also some related questions posed in [144].

Theorem 6.9 (Ferri-Galindo [100]) *The additive group $G := c_0$ is not reflexively representable.*

However there exists a continuous injective representation of the group c_0 on a reflexive space. This fact led to the following natural

Question 6.10 (Ferri-Galindo [100]) *Is it true that every* abelian *topological group has a continuous injective representation on a reflexive Banach space?*

We have a partial answer at least for 2-step nilpotent groups.

Theorem 6.11 *There exists a 2-step nilpotent (Polish) group G which does not admit any injective continuous reflexive representation.*

Proof. The group c_0 is a relatively minimal subgroup in the Heisenberg group $H(c_0) = (\mathbb{R} \oplus c_0) \leftthreetimes l_1$. Hence every continuous injective homomorphism of $H(c_0)$ induces an embedding of c_0 which is ruled out by Ferri-Galindo's Theorem 6.9. □

7 Local minimality

The next notion, proposed by Pestov and Morris [152] (see also T. Banakh [12]) is a common generalization of minimal groups, locally compact groups and normed spaces:

Definition 7.1 *A topological group (G,τ) is* locally minimal *if there exists a neighborhood V of e such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of e, then $\sigma = \tau$.*

To underline that the neighborhood V witnesses local minimality for (G,τ), we say sometimes (G,τ) is *V-locally minimal.*

A first supply of examples, beyond locally compact groups, groups having an open minimal subgroup (see also Proposition 7.4(b)) and normed spaces, is given by the following:

Theorem 7.2 [152, Corollary 2.8] *All subgroups of Banach-Lie groups are locally minimal. In particular, all subgroups of a finite-dimensional Lie groups are locally minimal.*

Another supply of examples comes in the next example:

Example 7.3 *According to Example 3.15, $\mathbb{Q}^n$ admits no minimal group topology for every positive $n \in \mathbb{N}$, G_π, admits no minimal group topology if $\emptyset \neq \pi \neq \mathbb{P}$ and $G = \mathbb{Q}^{(\kappa)}$ does not admit minimal group topologies for a non-exponential cardinal $\kappa > \mathfrak{c}$.*

- *(a) For every $n \in \mathbb{N}$, the group $\mathbb{Q}^n$, being isomorphic to a subgroup of $\mathbb{T}$, admits a locally minimal precompact group topology.*
- *(b) For every $\pi \subseteq \mathbb{P}$, the group G_π, being isomorphic to a subgroup of $\mathbb{T}$, admits a locally minimal precompact group topology.*
- *(c) Using (a), one can see that for any cardinal $\kappa > 0$, the group $G = \mathbb{Q}^{(\kappa)}$ admits a non-discrete locally minimal locally precompact groups topologies.*

7.1 General properties of locally minimal groups

According to Theorem 5.38, closed subgroups (even group retracts) of minimal groups can be arbitrarily chosen Hausdorff groups. Hence, the class of locally minimal groups is not stable under taking group retracts (hence not stable under taking closed subgroups or quotients). However, in some natural situations, a closed subgroup of a locally minimal group may still be locally minimal:

Proposition 7.4 *Let G be a topological group and let H be a closed subgroup of G.*

(a) [10, Proposition 2.5] *If G is locally minimal and H is central, then H is locally minimal.*

(b) [10, Proposition 2.4] *If H is open and locally minimal, then G is locally minimal.*

(c) [65] *If G is U-locally minimal, H is central and $HV \subseteq U$ for some $U, V \in \mathscr{V}(e_G)$, then H is minimal.*

Here we see that the Heisenberg groups frequently are locally minimal.

Corollary 7.5 *Let $w : E \times F \to A$ be a minimal biadditive mapping and $H(w)$ be the corresponding Heisenberg group. Then $H(w)$ is a locally minimal group if and only if A is a locally minimal group.*

Proof. Indeed, if $H(w)$ is locally minimal then its center A is also locally minimal (Proposition 7.4). Conversely, if A is locally minimal then by Lemma 5.9 it follows that $H(w)$ is a locally minimal group. □

Note that $H(w)$ in Corollary 7.5 is minimal if and only if A is minimal by Theorem 5.8. The similarity in the formulations in 7.5 and 5.8 makes it natural to suggest examining "locally minimal analog" of biadditive mappings in the spirit of Definition 5.3.

The next condition became popular after the solution of Hilbert's fifth problem:

Definition 7.6 *A topological group (G, τ) is called* NSS group *(resp.,* NSnS group*) if a suitable neighborhood $V \in \mathscr{V}(e_G)$ contains only the trivial (normal) subgroup.*

The abbreviation NSS (resp., NSnS) stays for No Small (normal) Subgroups. The locally compact NSS groups are precisely the Lie groups.

Lemma 7.7 *NSS property is stable under taking subgroups, finer group topologies and group extensions (e.g., semidirect products). NSnS property is stable under taking finer group topologies and group extensions (e.g., semidirect products).*

Proof. The first two properties for NSS and the first one for NSnS are obvious. Let K is a topological group with a closed normal subgroup X such that K/X and X both are in NSS. We have to show the same for K. Choose $U \in \mathscr{V}(e_G)$ and $V \in \mathscr{V}(e_X)$ which are free from nontrivial subgroups. There exist neighborhoods $U_1 \in \mathscr{V}(e_G)$ and $U_2 \in \mathscr{V}(e_G)$ such that $U_1 \cap X \subseteq V$ and $p(U_2) \subseteq U$ where, $p : K \to K/X = G$ is the natural map. We claim that the neighborhood $O := U_1 \cap U_2$ of unity in K is also free from nontrivial subgroups. Let $H \subseteq O$ be a subgroup of K. Then $p(H) \subseteq U_2$. By our assumption on U_2 we get $p(H) = \{e\}$ in G. Therefore, $H \subseteq X = ker(p)$. We obtain $H \subseteq O \cap X \subseteq U_1$. Now by our assumption on U_1 we get that H is the trivial subgroup of X.

Similar proof works for NSnS groups. □

A topological group containing a dense NSS subgroup need not be NSS itself, as the following example shows:

Example 7.8 *Let τ denote the Bohr topology of $\mathbb{Z}$, so that the completion of $G = (\mathbb{Z}, \tau)$ is the Bohr compactification $b\mathbb{Z}$ of $\mathbb{Z}$. It is known that $b\mathbb{Z}$ is non-metrizable, so $b\mathbb{Z}$ is not NSS and G is non-metrizable. On the other hand, if $\chi : \mathbb{Z} \to \mathbb{T}$ is an injective homomorphism, then χ is τ-continuous, hence G is NSS, as the subgroup $\chi(\mathbb{Z})$ of $\mathbb{T}$ is NSS, by Lemma 7.7.*

It is easy to see that if K is an abelian group containing a dense and *essential* NSS subgroup G, then K is NSS itself. In particular, if G is a minimal abelian NSS group with completion K, then K is compact by Theorem 3.9 and G is essential in K by Theorem 3.1, so K is NSS. As the compact NSS groups are metrizable (actually, Lie groups), we conclude that *every minimal abelian NSS group is metrizable.* This can be reinforced as follows. Recall, that a SIN group is a topological group having a local base of invariant (under conjugation) neighborhoods of e (abelian groups, as well as precompact groups are SIN).

Proposition 7.9 [10, Proposition 2.13] *Every locally minimal SIN and NSnS group G is metrizable.*

In particular, every locally minimal precompact NSnS group G is metrizable (as precompact groups are SIN). One can be tempted to remove completely SIN (in particular, precompact). As mentioned in [10], in Proposition 7.9 one cannot trade SIN for the property "totally minimal", since for uncountable set X the group $S(X)$ is totally minimal and NSnS, yet non-metrizable.

However, it remained unclear whether the conjunction "SIN and NSnS" in Proposition 7.9 can be traded for the stronger property NSS alone, i.e., the question of whether every locally minimal NSS group is metrizable remained open ([10, Question 6.7]). Now we show that the technique from §5.2 can be used to provide a negative answer to this question. In fact, the group providing the counterexample is *minimal* rather than (only) *locally minimal.*

Example 7.10 *To the non-metrizable NSS group $X = \mathbb{Z}$ from Example 7.8 apply Lemma 5.16(A) to get a minimal (nilpotent class 2 and locally precompact) NSS group L that is not metrizable (actually, $\chi(L) = \mathfrak{c}$).*

Our next aim are the dense subgroups of locally minimal groups. The following "local" version of essentiality was proposed in [10]:

Definition 7.11 *Let H be a subgroup of a topological group G. We say that H is* locally essential *in G if there exists a neighborhood V of e in G such that $H \cap N = \{e\}$ implies $N = \{e\}$ for all closed normal subgroups N of G contained in V.*

When necessary, we shall say H is locally essential with respect to V *to indicate also V. Note that if V witnesses local essentiality, then any smaller neighborhood of the neutral element does, too.*

Remark 7.12 *(a) Clearly, essential subgroups are also locally essential (simply take $V = G$).*

(b) The essentiality coincides in the case of discrete groups with the known notion of essentiality in algebra. In contrast with this, every subgroup of a discrete group is locally essential, i.e., local essentiality becomes vacuous in the discrete case.

We now give a criterion for local minimality of dense subgroups.

Theorem 7.13 [10] *Let H be a dense subgroup of a topological group G. Then H is locally minimal if and only if G is locally minimal and H is locally essential in G.*

Remark 7.14 *From Theorem 7.13 one obtains:*

(a) Completions of locally minimal groups are locally minimal;

(b) A dense subgroup of a locally compact NSnS group is locally minimal.

It is easy to see that if K is a torsion-free pro-finite abelian group, then a dense locally essential subgroup G of K is already essential in K, since for some open subgroup V of K the subgroup $G \cap V$ is essential in V. But since $[K : V] < \infty$ and K is torsion-free, we deduce that $G \cap V$ is essential in K. This implies that the group $G = (\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p)$ (see Example 1.1(b)) is not even locally minimal.

7.2 Cardinal invariants of locally minimal groups

The following two cardinal invariants of topological group G were introduced in [20]: $TD(G)$ (resp., $ED(G)$) is the minimum cardinality of a totally dense (resp., dense essential) subgroup of G. The cardinal invariants $ED(G)$ and $TD(G)$ in the case of an abelian group G were studied in [26, 19]. Their connection to (total) minimality is very simple: if G is a (totally) minimal group, then $ED(G)$ (resp., $TD(G)$) coincides with the minimum cardinality of a dense (totally) minimal subgroup of G. Obviously,

$$w(G) \leq ED(G) \leq TD(G) \leq |G|,$$

as every minimal group H satisfies $w(H) \leq |H|$ (see Theorem 7.18) so $w(G) \leq |H| \leq |G|$ for every dense minimal subgroup of G. If G is topologically simple, then $ED(G) = TD(G) = d(G)$.

The equality $ED(G) = TD(G)$ for compact abelian groups was established by Stoyanov [192] (see also [19]). According to [26, Theorem B], there exists a minimal abelian group G with $ED(G) < TD(G)$, moreover, in certain models of ZFC this group can be chosen to be also pseudocompact.

The class $\mathcal{P}$ of the compact abelian groups G with countable $ED(G)$ was described in §3.3. If $G \notin \mathcal{P}$, then $ED(G) \geq \mathfrak{c}$. More precisely, $ED(G)$ can be computed as follows:

(a) [19, Corollary 3.8] $ED(G) = |G| = 2^{w(G)}$, if G is either connected or a pro-p-group for some prime p;

(b) [19, Theorem 3.12] $G = \sup_p ED(G_p) = \sup_p 2^{w(G_p)}$, when G is a pro-finite group.

For an arbitrary compact abelian group $K \notin \mathcal{P}$ one has (combining (a), (b) and [19, Theorem 3.14]):

$$ED(K) = 2^{w(c(G))} \cdot \sup\{2^{w((K/c(K))_p)} \ : \ p \in \mathbb{P}\} \tag{8}$$

and

$$w(K) = w(c(G)) \cdot \sup\{w((K/c(K))_p) \ : \ p \in \mathbb{P}\}. \tag{9}$$

Problem 7.15 *Describe the cardinal invariants $TD(K)$ and $ED(K)$ for arbitrary compact groups K. When they coincide?*

For infinite cardinals τ and λ the symbol $\mathrm{Min}(\tau, \lambda)$ denotes the following statement: There exists a sequence of cardinals $\{\lambda_n : n \in \mathbb{N}\}$ such that

$$\lambda = \sup_{n\in\mathbb{N}} \lambda_n \quad \text{and} \quad \sup_{n\in\mathbb{N}} 2^{\lambda_n} \leq \tau \leq 2^{\lambda}. \tag{10}$$

We say that the sequence $\{\lambda_n : n \in \mathbb{N}\}$ as above *witnesses* $\mathrm{Min}(\tau, \lambda)$. Following [60], call a cardinal number κ a *Stoyanov cardinal*, provided that κ is either finite, or infinite and satisfies $\mathrm{Min}(\kappa, \sigma)$ for some infinite cardinal σ. It was proved by Stoyanov [194] that these cardinals are exactly the cardinalities of the minimal abelian groups. More precisely, $|G|$ is a Stoyanov cardinal for every minimal abelian group G. On the other hand, a free abelian group of size κ admits minimal group topologies if (and only if) κ is a Stoyanov cardinal. Extending this theorem of Stoyanov, one can prove that $\mathrm{Min}(|G|, w(G))$ holds for every infinite minimal abelian group G (applying (8) and (9) to the compact completion K of G and noting that $ED(K) \leq |G| \leq |K| = 2^{w(K)}$). For an alternative proof see also [60]. This result was extended in two directions, namely to locally minimal groups G with $\chi(Z(G)) = \chi(G)$ (in particular, for every locally minimal abelian group), in [77]:

Theorem 7.16 [77] *If G is an infinite locally minimal precompact group with $\chi(Z(G)) = \chi(G)$, then* $\mathrm{Min}(|G|, w(G))$ *holds.*

Note that "precompact" cannot be relaxed to locally precompact, since every discrete group is both locally minimal and locally precompact. So every discrete group G such that $|G|$ is not a Stoyanov cardinal will fail to satisfy the conclusion of the above theorem.

It follows from Theorem 7.16 that the cardinalities of the locally minimal precompact abelian groups coincide with those of the minimal ones. More precisely:

Corollary 7.17 *For a free abelian group F the following are equivalent:*

(a) F admits a locally minimal precompact group topology;

(b) $|F|$ is a Stoyanov cardinal;

(c) F admits a minimal group topology.

Indeed, the implication (a) $\to$ (b) follows from Theorem 7.16, while the implication (b) $\to$ (c) is due to Stoyanov [194]. The remaining implication (c) $\to$ (a) is trivial.

It is important to note that the equivalence between (a) and (c) concerns only *free groups*. Indeed there are many abelian groups admitting locally minimal precompact group topologies, that admit no minimal ones (see Example 7.3).

The next theorem is the counterpart for locally minimal groups of a well-known theorem of Arhangel′skiĭ [4, Corollary 2] about minimal groups (for minimal group, the second assertion of the theorem is [4, Corollary 3]).

Theorem 7.18 [10] *For a locally minimal group (G, τ) one has $w(G) = nw(G)$. In particular, every countable locally minimal group is metrizable.*

The next theorem relates the tightness and the weight of a locally minimal precompact abelian group:

Theorem 7.19 [77, Corollary 2.8] *$t(G) = w(G)$ for every locally minimal precompact abelian group G.*

The equality $t(G) = w(G)$ was established for every compact group G by Arhangel′skiĭ [5].

Since sequential spaces have countable tightness, we obtain:

Corollary 7.20 [77, Corollary 2.9] *A sequential locally minimal and locally precompact abelian group is metrizable.*

This is related to Malykhin's problem: *are countable Fréchet-Urysohn groups metrizable*? Here countable can be replaced by separable. Replacing separability by the conjunction of local minimality and local precompactness allows us to obtain a positive answer to a stronger version of Malykhin's question (where the Fréchet-Urysohn property is relaxed to sequentiality).

Now we give a series of corollaries about minimal abelian groups. Since every minimal abelian group is precompact by Prodanov-Stoyanov's precompactness theorem 3.9 one obtains $t(G) = w(G)$ for every minimal abelian group G. The next corollary answers a question of O. Okunev (2007):

Corollary 7.21 [77, Corollary 2.12] *Every minimal abelian group having countable tightness is metrizable.*

It is natural to ask whether these corollaries or Theorem 7.19 can be extended to the case of nilpotent groups. The next example shows that commutativity plays a crucial role in the above corollaries.

Example 7.22 *For an arbitrary uncountable cardinal κ let X be the Σ product of κ copies of $\mathbb{T}$. Then X is a connected ω-bounded (hence, countably compact, so precompact, as well) abelian Fréchet-Urysohn group with $\chi(X) = \kappa$ (so non-metrizable). Applying Lemma 5.16(A) we can obtain a minimal (nilpotent of class 2) locally countably compact group L of countable tightness and weight κ, such that $c(G)$ is an open normal ω-bounded subgroup of L. Actually, L is Fréchet-Urysohn, as $X \oplus \mathbb{T} \cong X$.*

From Corollary 7.21 one can deduce also:

Theorem 7.23 [77, Theorem 3.4] *Every totally minimal nilpotent groups of countable tightness is metrizable.*

The next theorem concerns another generalization of Fréchet-Urysohn groups, namely radial groups (i.e., the topological groups which are radial as topological spaces):

Theorem 7.24 [77, Theorem 2.16] *If K is a compact abelian group containing a locally essential radial subgroup, then K is metrizable.*

Corollary 7.25 *Every radial locally minimal and locally precompact abelian group is metrizable. In particular, every radial minimal abelian group is metrizable.*

The next example shows that local minimality of G cannot be traded for some other strong compactness properties.

Example 7.26 *Let κ be an uncountable cardinal. The Σ-product G of κ-many copies of the circle group $\mathbb{T}$ is abelian, ω-bounded and Fréchet-Urysohn, hence of countable tightness and radial, while G is non-metrizable (as $\chi(G) = w(G) = \kappa$). The group G is connected, to obtain an example of a zero-dimensional non-metrizable Fréchet-Urysohn ω-bounded abelian group replace $\mathbb{T}$ by any finite abelian group F.*

7.3 Locally GTG groups and UFSS groups

For a symmetric subset U of a group G with $e_G \in U$, and $n \in \mathbb{N}$ let

$$(1/n)U := \{x \in G : x^k \in U \ \forall k \in \{1, 2, \cdots, n\}\}$$

and

$$U_\infty := \{x \in G : x^n \in U \ \forall n \in \mathbb{N}\}.$$

The following group analog of a normed space was introduced by Enflo [99]:

Definition 7.27 *A topological group (G, τ) is* uniformly free from small subgroups (UFSS *for short*) *if for some neighborhood U of e_G, the family $\{(1/n)U\}_{n=1}^\infty$ is a neighborhood basis at e_G for τ.*

When we need to emphasize the neighborhood U from the definition, we say U-UFSS group.

A topological vector space is UFSS as a topological group if and only if it is locally bounded (in particular, normed spaces are UFSS). Every Banach-Lie group is UFSS [152, Theorem 2.7].

The class of UFSS groups is stable under taking subgroups, completions, finite direct products and local isomorphisms. It has also the three space property.

If a topological group (G, τ) is U-UFSS and $\sigma \subseteq \tau$ is a Hausdorff group topology such that $U \in \sigma$, then obviously $(1/n)U \in \sigma$ for all $n \in \mathbb{N}$, so $\sigma = \tau$. Hence every UFSS group is locally minimal ([10, Facts 3.3(a)]). The converse of this implication holds if the group is precompact and NSnS:

Proposition 7.28 [10, Proposition 3.8] *For a locally minimal precompact group G the following are equivalent:*

(a) G is NSnS;

(b) G is NSS;

(c) G is UFSS;

(d) G is isomorphic to a dense subgroup of a compact Lie group.

This proposition implies that a continuous homomorphic image of a precompact UFSS group is still UFSS (this fails without the assumption of precompactness, see [10, Example 3.14]).

A subset A of a topological abelian group G is called *quasi-convex*, if for every $x \in G \setminus A$ there exists $\chi \in G^\wedge$ such that $\chi(A) \subseteq \mathbb{T}_+$, but $\chi(x) \notin \mathbb{T}_+$, where $\mathbb{T}_+$ is the image of the segment $[-\frac{1}{4}, \frac{1}{4}]$ with respect to the natural quotient map $\mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. A topological abelian group G is *locally quasi-convex* if $\mathscr{V}(e_G)$ has a base of quasi-convex sets. Quasi-convexity was introduced by Vilenkin to replace convexity from TVS in the framework of abelian topological groups. Locally precompact abelian groups are locally quasi-convex.

The following notions belongs to V. Tarieladze:

Definition 7.29 [10, Definition 4.2] *Let G be a topological abelian group.*

(a) For a symmetric subset of G, with $0 \in U$, we say that U is a group topology generating subset *of G* (GTG subset of G *for short*) *if the family $\{(1/n)U : n \in N\}$ is a basis of neighborhoods of zero for a not necessarily Hausdorff group topology $\mathcal{T}_U$ on G.*

(b) G is said to be locally GTG *if G admits a basis of neighborhoods of the identity formed by GTG subsets of G.*

Obviously, U is a GTG set precisely when there exists $m \in \mathbb{N}$ such that $(1/m)U + (1/m)U \subseteq U$. Hence U_∞ is a subgroup of G, whenever U is a GTG set (as it coincides with the $\mathcal{T}_U$-closure of 0).

A subgroup of an abelian group G is always a GTG set. Same applies to any quasi-convex subset E of a Hausdorff topological abelian group G, as $(1/2)E + (1/2)E \subseteq E$. In particular, abelian non-archimedean groups, as well as locally quasi-convex groups are locally GTG.

Clearly, every UFSS group is locally GTG. The converse of this implication holds true in the presence of local minimality and the property NSS:

Theorem 7.30 [10, Theorem 5.10] *A topological abelian group (G, τ) is a UFSS group if and only if (G, τ) is locally minimal, locally GTG and NSS.*

We shall see below that when G is of finite square-free exponent, then the conjunction of local minimality and local GTG turns out to be equivalent to local compactness (see Remark 7.32 (c)).

Theorem 7.31 [10, Theorem 5.12] *If G is a U-locally minimal abelian group where U is a GTG set, then U_∞ is a minimal subgroup.*

One can deduce from Theorem 7.31 that every locally minimal abelian group contains a minimal, hence precompact, G_δ-subgroup ([10, Proposition 5.13]). In particular, complete locally minimal abelian groups are almost metrizable, i.e., contain a compact G_δ-subgroup [10, Corollary 5.14].

In the next remark we discuss the properties of the locally GTG groups of bounded exponent. We see, among others, that such a group is locally minimal if and only if it contains an open minimal subgroup.

Remark 7.32 *Let G be a bounded topological abelian group of exponent m. From the above results (see also [10]) one can derive the properties.*

- *(a) A subset U of G is a GTG set precisely when U_∞ is a subgroup (note that $U_\infty = (1/m)U$). If U is open, then U_∞ is an open subgroup of G. Consequently, G is locally GTG if and only if G is non-archimedean.*
- *(b) If $U \in \mathscr{V}(e_G)$ is a GTG set, then G is locally U-minimal if and only if U_∞ is an open minimal subgroup of G.*
- *(c) If G is locally GTG and locally minimal, then G is locally precompact and $soc(G)$ is locally compact.*
- *(d) If $exp(G)$ is square-free, then G is locally compact if and only if G is locally GTG and locally minimal.*

The next theorem can be deduced from the above remark. Item (b) can be weakened to: G admits a non-discrete U-locally minimal group topology for some GTG set U.

Theorem 7.33 [10, Theorem 5.18] *Let G be a bounded abelian group. Then the following are equivalent:*

- *(a) $|G| \geq \mathfrak{c}$;*
- *(b) G admits a non-discrete locally minimal and locally GTG group topology;*
- *(c) G admits a non-discrete locally compact metrizable group topology.*

Question 7.34 *(a)* [10, Question 6.2] *Is every locally minimal abelian group necessarily locally GTG?*

- *(b)* [10, Question 6.5] *Is every locally minimal NSS abelian group necessarily locally GTG?*

A positive answer to the next question implies a negative answer to Question 7.34(a).

Question 7.35 *(a)* [10, Question 6.3] *Does the infinite Boolean group $\mathbb{Z}_2^{(\mathbb{N})}$ admit a non-discrete locally minimal group topology?*

- *(b) More generally, does the group $\mathbb{Z}_p^{(\mathbb{N})}$ admit a non-discrete locally minimal group topology for every prime p?*

The interest in item (b) of the above questions is explained by the fact that a positive answer to it, along with Example 7.3, will prove that *every infinite abelian group admits a non-discrete locally minimal group topology.* As we saw in Section 3.3, many infinite abelian groups, say ($\mathbb{Q}$ or $\mathbb{Z}_p^{(\mathbb{N})}$, for a prime p), do not admit a non-discrete minimal group topology.

7.4 Almost minimal groups

Here we discuss the notion of almost minimal group. Our aim is to replace local minimality by a stronger property that still covers local compactness (in the abelian case), UFSS and minimality, but goes closer to them in the following natural sense:

Definition 7.36 [10, Definition 6.1] *A topological group G is called* almost minimal *if it has a closed, minimal normal subgroup N such that the quotient group G/N is UFSS.*

In the definition, N is a G_δ-set in G (since G/N is metrizable). Hence, if N is also abelian then Theorem 3.9 implies that N is precompact.

Minimal groups, as well as UFSS groups, are almost minimal. We shall see below that all LCA groups are almost minimal (Corollary 7.40).

Theorem 7.37 *For a topological abelian group G the following conditions are equivalent:*

- *(a) G is almost minimal.*
- *(b) There exists a GTG neighborhood of zero U such that G is U-locally minimal and G/U_∞ is UFSS.*

It is not difficult to show that if G is almost minimal, a convenient $U \in \mathcal{V}(0)$ can be chosen which satisfies (b). However, in order to avoid ambiguities, in what follows we use the expression *G is U-almost minimal* as substitute for condition (b).

From Theorem 7.37 one deduces that the completion of an almost minimal group is almost minimal ([10, Proposition 6.6]).

Remark 7.38 *Complete almost minimal abelian groups have some interesting properties that are worth mentioning. Indeed, such a group G has a compact subgroup N such that G/N is UFSS, in particular metrizable. Hence G is almost metrizable so Čech complete (being complete). This implies that G is a k-space and also a Baire space.*

In the rest of this subsection we discuss the fact that every locally quasi-convex locally minimal group can be embedded into an almost minimal group. This embedding allows us to show that every complete locally quasi-convex locally minimal groups is already almost minimal.

Theorem 7.39 [11] *Let (G, τ) be a locally quasi-convex locally minimal topological group. Then (G, τ) can be embedded into a complete locally quasi–convex topological group A having a compact subgroup B such that:*

(a) A/B is UFSS and $N := G \cap B$ is a minimal subgroup of G.

In particular, A is an almost minimal group.

Corollary 7.40 [11] *Every complete locally quasi-convex locally minimal abelian group is almost minimal. In particular, every LCA group is almost minimal.*

Taking into account that both local quasi-convexity (Corollary 6.17 in [8]) and local minimality (Remark 7.14(a)) are stable under taking completions, Corollary 7.40 admits the following generalization:

Corollary 7.41 [11] *The completion of a locally quasi-convex locally minimal abelian group is almost minimal.*

However, a locally quasi–convex (even a precompact) locally minimal abelian group need not be almost minimal. There exists a locally minimal, metrizable, precompact abelian group which is not almost minimal. More precisely, in [10, Example 6.17] one can find a metrizable precompact locally minimal abelian group G which has a closed subgroup N such that G/N is not UFSS and any minimal subgroup of G is contained in N. Hence, if N_1 is any closed minimal subgroup of G, there exists a continuous epimorphism $G/N_1 \to G/N$ Since every continuous homomorphic image of a precompact UFSS group is UFSS, this implies that G/N_1 cannot be UFSS.

Finally, we give a positive result which pushes the criterion of local minimality to a higher level, thus identifying the almost minimal groups among all precompact locally minimal groups.

Proposition 7.42 [11] *If G is an abelian precompact locally minimal group, then G has a minimal closed subgroup N such that G/N admits a continuous injective homomorphism $G/N \hookrightarrow \mathbb{T}^n$ for some $n \in \mathbb{N}$.*

As we have already said, this "smallness" of the quotient G/N with respect to the closed minimal subgroup N does not imply that all precompact locally minimal groups are almost minimal, as the above mentioned example from [11] shows. Indeed, the continuous injective homomorphism $G/N \hookrightarrow \mathbb{T}^n$ only shows that the quotient group G/H admits a coarser UFSS topology (the one induced from $\mathbb{T}^n$).

7.5 Products and quotients of locally minimal groups

Dikranjan and Morris [65] proposed the following notion that goes closer to local minimality:

Definition 7.43 *A topological group G is said to be* locally q-minimal *if there exists a neighbourhood V of the identity of G such that whenever H is a Hausdorff group and $f : G \to H$ is a continuous surjective homomorphism with $f(V) \in \mathscr{V}(e_H)$, then f is open.*

Locally compact groups are locally q-minimal. It is shown in [65] that under certain circumstances complete locally q-minimal groups are locally compact. This occurs for subgroups of products of locally compact groups in two cases: (a) for products of locally compact abelian groups; (b) for connected subgroups of products of locally compact MAP groups. It is also shown in [65] that MAP cannot be removed.

Let us briefly discuss here products of locally minimal groups.

We start by an example:

Example 7.44 *$G = \mathbb{R}^\omega$ is not locally minimal. This follows from the following more general fact: if $\{G_i\}_{i \in I}$ is a family of topological groups such that infinitely many groups G_i are not minimal, then the product $\prod_{i \in I} G_i$ is not locally minimal.*

The following result is a natural modification of the similar fact for minimal groups (see Theorem 4.3).

Theorem 7.45 *Let P be a topological group and let X be a complete normal subgroup of P. If X and P/X are both locally minimal, then P is locally minimal, too. In particular, if X is complete then the semidirect product $X \rtimes G$ is locally minimal for locally minimal groups X and G, whenever X is a G-group.*

Proof. Assume that $G := P/X$ is V-minimal and X is U-minimal, where $V \in \mathscr{V}(e_G)$ and $U \in \mathscr{V}(e_X)$. Choose $O \in \mathscr{V}(e_P)$ such that $O \cap X \subseteq U$. Let us show that P is W-minimal, where $W := O \cap q^{-1}(V)$ and $q : P \to G = P/X$ is the natural projection. Let $\sigma \subseteq \tau$ be a coarser Hausdorff group topology on P such that W is a σ-neighborhood of e. We claim that then $\sigma = \tau$. Indeed, on the subgroup $X \leq P$ we get a coarser Hausdorff group topology $\sigma|_X \subseteq \tau|_X$ such that $U \in \sigma|_X$, because, $W \cap X \subseteq U$. By U-minimality of X we have $\sigma|_X = \tau|_X$. By Merson's Lemma 4.4 it suffices to show that $\sigma/X = \tau/X$. The normal subgroup X clearly remains σ-closed because X is complete. So the quotient topology σ/X is a Hausdorff group topology on G. By our assumption, W is a σ-neighborhood of e_P. Then $q(W) = q(O) \cap V$ is a σ/X-neighborhood of e_G and $q(W) \subseteq V$. Hence, V is a σ/X-neighborhood of e_G too. By V-minimality of G we obtain that σ/X is the original topology on G. Hence, $\sigma/X = \tau/X$, as desired. □

The proof of the following theorem will appear elsewhere:

Theorem 7.46 *A product of minimal abelian groups is locally minimal if and only if it is minimal.*

The following nice characterization of the groups of p-adic integers was found by Prodanov [162]: for an infinite compact abelian group G all subgroups of G are minimal if and only if $G \cong \mathbb{J}_p$ for some prime p.

Problem 7.47 *Describe the locally compact groups G such that every subgroup of G is locally minimal.*

The following theorem (its proof will appear elsewhere) gives a solution in the abelian case.

Theorem 7.48 *For a locally compact abelian group G the following are equivalent:*

(a) *every subgroup of G is locally minimal;*

(b) *G is either a Lie group or has an open subgroup isomorphic to $\mathbb{J}_p$ for some prime p.*

We conjecture that the answer of the following question, suggested by the above theorem, is positive:

Problem 7.49 *If every subgroup of a locally compact connected group G is locally minimal, is G necessarily a Lie group?*

If one removes the restraint "locally compact", then every normed space V has again the property to have all its subgroups locally minimal. This leaves open the following general problem:

Problem 7.50 *Describe the topological (abelian) groups G such that every subgroup of G is locally minimal.*

The next problem should be compared to Question 4.22.

Problem 7.51 *Study the compact spaces K for which the group* Homeo(K) *is* locally *minimal.*

8 Minimality and (dis)connectedness

The dimension of a topological space is usually considered as a measure of its connectedness. This section will study the connection of minimality to dimension and (dis)connectedness.

8.1 Zero-dimensionality vs total (hereditary) disconnectedness

A topological group G is *hereditarily (locally) pseudocompact* if every closed subgroup of G is (locally) pseudocompact. Obviously, every (locally) countably compact group is hereditarily (locally) pseudocompact. Natural source of hereditarily locally pseudocompact groups is provided in the theorem below.

Call a group G *h-sequentially complete* if all continuous homomorphic images of G are sequentially complete. Countably compact groups are h-sequentially complete.

Theorem 8.1 [79] *A h-sequentially complete group G is pseudocompact if and only if G is precompact. In particular, nilpotent h-sequentially complete groups are pseudocompact.*

The above theorem implies that abelian h-sequentially complete groups are hereditarily pseudocompact [79]. The properties of being sequentially complete, pseudocompact and minimal are independent, i.e., the conjunction of any two of them does not imply the third even in the class of abelian groups [80, Example 3.1].

In the sequel we denote by $q(G)$ the quasi-component of a topological group. As $c(G)$, also $q(G)$ is a closed normal subgroup of G containing $c(G)$. We say that G is *hereditarily (totally) disconnected*, if $c(G) = \{e_G\}$ (resp., $q(G) = \{e_G\}$). Always, $c(G) \subseteq q(G)$. Hence totally disconnected groups are hereditarily disconnected. It is known that some compact-like properties of G (e.g., countable compactness or local compactness) imply $c(G) = q(G)$ [49, 50]. On the other hand, for other classes of groups, that are less close to compactness (e.g., pseudocompact groups), this fails to be true.

To describe better the connectedness properties of a topological group G we need one more functorial subgroup, introduced in [52, 1.1.1]: $z(G)$ denotes the intersection of all kernels of continuous homomorphisms from G into zero-dimensional groups. This subgroup is closed and normal, with $c(G) \subseteq q(G) \subseteq z(G) \subseteq o(G)$. A group G with trivial $z(G)$ need not be zero-dimensional, but admits a coarser zero-dimensional group topology.

The following classical result of van Dantzig [119] describes the hereditarily disconnected locally compact groups:

Theorem 8.2 *Every hereditarily disconnected locally compact group G has a local base at identity consisting of open compact subgroups. In particular, G is non-archimedean.*

This theorem gives

$$c(G) = q(G) = z(G) = o(G) \tag{V}$$

for every locally compact group.

The following unpublished result of D. Shakhmatov nicely connects total disconnectedness and zero-dimensionality for pseudocompact groups.

Theorem 8.3 *If G is a totally disconnected pseudocompact group then G admits a coarser zero-dimensional group topology. Therefore, every minimal, pseudocompact, totally disconnected group is zero-dimensional.*

For the extension of this theorem in the locally pseudocompact case see Theorem 8.6. One can give this theorem in a stronger form: for pseudocompact groups $q(G) = z(G) = o(G)$ [50]. The validity of this equality was extended to locally pseudocompact groups in [29].

The first assertion of the next theorem can be deduced from Theorem 8.3. The proof of the remaining part makes use of Fact 3.8.

Theorem 8.4 [50, Theorem 7.1],[49, Theorem 3] *Let G be a pseudocompact group. Then $G/q(G)$ admits a coarser zero-dimensional group topology. Moreover, the implications (c) $\Rightarrow$(b)$\Leftrightarrow$(a) hold for the following conditions:*

(a) *$G/q(G)$ is zero-dimensional;*

(b) *$q(G)$ is dense in $c(\widetilde{G})$;*

(c) *$G/q(G)$ is minimal.*

If G is minimal and abelian, then they all are equivalent.

8.2 Vedenissov groups and locally pseudocompact groups

According to Vedenissov's classical theorem, hereditarily disconnected locally compact (Hausdorff) spaces are zero-dimensional. Since for every topological group G the quotient $G/c(G)$ is hereditarily disconnected, Vedenissov's theorem justifies the following

Definition 8.5 [61] A topological group G is *Vedenissov* if the quotient $G/c(G)$ is zero-dimensional; if in addition $z(G) = o(G)$, then we say that G is *strongly Vedenissov.*

A group G is Vedenissov precisely when $c(G) = q(G) = z(G)$ and $G/z(G)$ is zero-dimensional (and G is strongly Vedenissov if (V) holds and $G/z(G)$ is zero-dimensional).

Clearly, a group G is (strongly) Vedenissov if and only if $G/c(G)$ is (strongly) Vedenissov. So the study of the (strongly) Vedenissov property can be reduced to the case of hereditarily disconnected groups. A hereditarily disconnected group G is Vedenissov if and only if it is zero-dimensional (and G is strongly Vedenissov if and only if G is zero-dimensional and $o(G)$ is trivial).

So Vedenissov's theorem implies that the locally compact groups are Vedenissov. Actually, by Theorem 8.2, the hereditarily disconnected locally compact group are non-archimedean. This means that the locally compact groups are strongly Vedenissov.

On the other hand, zero-dimensional pseudocompact groups are strongly Vedenissov [50], this result was extended to locally pseudocompact groups in [29].

It is possible to extend Theorem 8.3 to locally pseudocompact groups:

Theorem 8.6 [61, Theorem B] *Every locally pseudocompact totally disconnected group admits a coarser zero-dimensional group topology. Therefore, every minimal, locally pseudocompact, totally disconnected group is zero-dimensional and thus strongly Vedenissov.*

Local pseudocompactness cannot be omitted neither in the first, nor in the second assertion of the theorem (see Question 8.29 and Example 8.30).

Corollary 8.7 [61, Theorem C] *Let G be a totally minimal locally pseudocompact group. Then $c(G) = q(G)$ if and only if $G/c(G)$ is zero-dimensional, in which case G is strongly Vedenissov.*

The next theorem is contained in [50] for pseudocompact groups (can be easily deduced from Theorem 8.4) and extended to locally pseudocompact groups in [29].

Theorem 8.8 *For a locally pseudocompact group G the following are equivalent:*

(a) *G is Vedenissov;*

(b) *G is strongly Vedenissov;*

(c) *$c(G)$ is dense in $c(\widetilde{G})$.*

Remark 8.9 *For a minimal pseudocompact abelian group G there is a subtle difference between being (strongly) Vedenissov and having $\dim G/q(G) = 0$ which need not imply Vedenissov's property (see Corollary 8.13 for examples to this effect). In particular, from Theorem 8.4 and the above theorem we deduce that a minimal pseudocompact abelian group G is Vedenissov if and only if $c(G) = q(G)$.*

The next theorem is a simultaneous generalization of the classical result in the locally compact case and the case of hereditary pseudocompactness established in [50].

Theorem 8.10 [61, Theorem A] *Let G be a hereditarily locally pseudocompact group. Then $G/c(G)$ is zero-dimensional and $c(G) = q(G) = z(G) = o(G)$; that is, G is strongly Vedenissov.*

Corollary 8.11 *Every h-sequentially complete abelian group is strongly Vedenissov.*

A hereditarily disconnected pseudocompact group may fail to be Vedenissov in two ways. It may be (even) totally disconnected but not zero-dimensional (such groups, of arbitrary positive dimension were built by Comfort and van Mill [30]). The alternative is to have a hereditarily disconnected pseudocompact group that is not totally disconnected. In the sequel we recall a construction of totally minimal pseudocompact, hereditarily disconnected groups of arbitrary positive dimension. So they are a *counterpart* of Comfort and van Mill's groups with the extra property of being totally minimal (but they will be non-totally disconnected, according to Theorem 8.3). The next theorem provides a large class of examples of groups with these properties, with some extra additional properties (assigned connected component of the completion, assigned compact space $G/q(G)$ of the quasi-components, etc.):

Theorem 8.12 [52, Theorem 4.1] *For every connected compact metrizable abelian group K and for every totally disconnected compact abelian group N admitting a dense pseudocompact totally minimal subgroup H with $r(N/H) \geq 2^\omega$ there exists a hereditarily disconnected, totally minimal, pseudocompact abelian group $G = G_{K,N}$ such that:*

(i) $\dim G = \dim K$, $\dim\ q(G) = \dim\ G/q(G) = 0$*;*

(ii) the subgroup $q(G)$ is metrizable and splits algebraically in G;

(iii) $G/q(G) \cong N$ is a compact group;

(iv) there exists a closed subgroup B of G such that $G/B \cong c(\tilde{G}) \cong K$.

For a complete proof of the existence of a group N as above (and actually, with stronger properties) see [61, Lemma 4.2].

The next corollary produces an n-dimensional hereditarily disconnected, totally minimal, pseudocompact abelian group H_n for every natural number n or $n = \omega$.

Corollary 8.13 [52, Corollary 4.2] *For every natural number n or $n = \omega$ there exists a hereditarily disconnected, totally minimal, pseudocompact abelian group H_n with:*

(i) $\dim\ H_n = n$, $\dim\ q(H_n) = \dim\ H_n/q(H_n) = 0$*;*

(ii) $H_n/q(H_n)$ is a compact totally disconnected group;

(iii) H_n has a compact connected quotient group of dimension n (isomorphic to $c(\tilde{G})$);

(iv) the subgroup $q(H_n)$ is metrizable and splits algebraically in H_n.

A similar result (groups with a stronger property in the first assertion, but without the extra properties) can be found in [61, Theorem D].

Some compact-like properties (as local or countable compactness) impose preservation of total disconnectedness or zero-dimensionality under taking quotients. The next theorem shows that total minimality fails to have this property in a strong way:

Theorem 8.14 [53, Theorem 1.2] *Every abelian precompact group G is a quotient of a zero-dimensional precompact torsion-free abelian group H_G of the same weight. Moreover, if G is (totally) minimal, then H_G has the same property.*

8.3 Sequentially complete minimal groups

The combination of sequential completeness and minimality gives rise to surprisingly nice properties, as we saw in Theorem 3.20. We are going to describe some further properties now, starting with the totally minimal case. Then we expose the general structure results about sequentially complete minimal groups and finally comes the case of hereditary disconnected ones.

Theorem 8.15 [80, Theorem 3.8] *The sequentially complete totally minimal abelian groups are compact.*

For countably compact groups this was proved in [71]. For a different point of view see Theorem 9.16.

It is not clear if Theorem 8.15 can be extended to non-abelian groups:

Question 8.16 [80, Question 6.9] *Can Theorem 8.15 be extended to nilpotent groups?*

With "h-sequentially complete" in place of "sequentially complete", the affirmative answer follows immediately after iterated applications of Theorem 8.15.

Theorem 8.15 leaves open the following question from [54, Question 7.3].

Question 8.17 *(i) Are G' and $c(G)$ totally minimal for a countably compact totally minimal group G?*

(ii) Is a solvable (in particular, metabelian) countably compact totally minimal group necessarily compact?

(iii) Can "countably compact" be weakened to "sequentially complete" also in the nilpotent (solvable, metabelian) case?

"Yes" to the first part of (i) implies "Yes" to (ii) after iterated applications of Theorem 8.15.

The following structure theorem for sequentially complete minimal abelian groups G shows, among others, that the connected component $c(G)$ of G is always "as big as" the connected component of the completion $\widetilde{G}$ of G.

Theorem 8.18 [80, Theorems 3.2, 3.6, 4.8] *For a sequentially complete minimal abelian group G one has $w(c(G)) = w(c(\widetilde{G}))$. In addition, if $|c(G)|$ is not measurable, then $c(G) = c(\widetilde{G})$ and $G/c(G)$ is minimal and zero-dimensional.*

In [80, Theorem 4.8] it is claimed only that the quotient $G/c(G)$ is zero-dimensional, which only implies that G is Vedenissov. But a more careful look at Theorem 8.18 shows that under the current hypothesis $c(G)$ coincides with $c(\widetilde{G}) = q(\widetilde{G}) = z(\widetilde{G}) = o(\widetilde{G})$, hence G is also strongly Vedenissov. For reader's convenience we formulate this conclusion explicitly in the next corollary.

Corollary 8.19 [80, Theorem 4.8] *Every minimal sequentially complete abelian group G, such that $|c(G)|$ is not measurable, is strongly Vedenissov.*

Theorem 8.18 gives the following bold compactness criterion: each connected sequentially complete minimal abelian group of non-measurable size is compact [80, Corollary 3.3]. Therefore, under the ZFC-consistent assumption that no measurable cardinals exist, one has:

(i) minimal sequentially complete abelian groups are strongly Vedenissov;

(ii) connected sequentially complete minimal abelian groups are compact.

Here is another consequence of Theorem 8.18:

Corollary 8.20 *Monothetic h-sequentially complete minimal groups are compact. Consequently, the center of a h-sequentially complete minimal group is covered by compact subgroups.*

Indeed, a monothetic group G is abelian and has size $\leq 2^{\mathfrak{c}}$ (so non-measurable), hence $c(G) = c(\widetilde{G})$ is compact, by Theorem 8.18. So it remains to show that $G/c(G) = G/c(\widetilde{G})$ is compact. Since this quotient group is isomorphic to a dense subgroup of the compact quotient $\widetilde{G}/c(\widetilde{G})$, it suffices to note that $G/c(G)$ is sequentially complete and to recall that $\widetilde{G}/c(\widetilde{G})$, as a monothetic totally disconnected compact group, is metrizable. The second assertion immediately follows from the first one, since both properties in the hypothesis are inherited by closed central subgroups.

Remark 8.21 *(a) Actually, by using Remark 8.9, one can relax the hypothesis "h-sequentially complete minimal group" in Corollary 8.20 to "pseudocompact sequentially complete minimal group".*

(b) Since the minimal abelian groups covered by compact subgroups are perfectly minimal by a theorem of Stephenson [191], Corollary 8.20 provides a new proof of Theorem 3.20 (since a minimal group G is perfectly minimal precisely when $Z(G)$ is perfectly minimal, Theorem 3.19.2).

Item (1) in the next question from [54, Question 9.6, (1)–(3)] is suggested by Corollary 8.20. Item (3) is a counterpart of Uspenskij's Theorem 4.35 (or 5.38, for the "retract" option) for countably compact groups.

Question 8.22 *(1) Is every countably compact minimal group G covered by compact subgroups?*

(2) Let G be a sequentially complete minimal group. Is the center of the subgroup $c(G)$ is covered by compact subgroups?

(3) Characterize the (closed) subgroups of countably compact minimal groups. Is every countably compact group a closed subgroup (a retract) of a countably compact minimal group ? What about the totally minimal case?

One may ask if Theorem 8.18 remains true for nilpotent groups (of class 2). We formulate the question in the stronger form with countable compactness:

Question 8.23 *Let G be a countably compact minimal nilpotent group G with non-measurable $c(G)$. Is $c(G)$ compact?*

The next example shows that this is not the case if one weakens "countably compact" to "locally countably compact". (Note that locally countably compact groups are still strongly Vedenissov by Theorem 8.10.)

Example 8.24 *Let X be any countably compact non-compact connected abelian group. According to Lemma 5.16.(A) the group $L := (\mathbb{T} \oplus X) \leftthreetimes \widetilde{X}^\wedge$ is minimal, $\widetilde{X}^\wedge$ is discrete and $\mathbb{T} \oplus X$ is an open connected subgroup of L, so $\mathbb{T} \oplus X = c(L)$. Since $c(L)$ is an open countably compact connected subgroup of L, the group L is also locally countably compact. To conclude, L* is a minimal locally countably compact nilpotent (of class 2) group with non-compact $c(L)$. *The example can be chosen of every uncountable weight; in particular, of weight ω_1.*

The next proposition provides further evidence for the close relation between sequentially complete minimal abelian groups and compact abelian groups, by showing that the connection between the size and the weight of the group remain the same.

Proposition 8.25 [54, Corollary 4.15] *Let G be an infinite sequentially complete minimal abelian group. Then $|G| = 2^{w(G)}$ in each of the following cases:*

- *G is connected;*
- *G is totally disconnected;*
- *G is h-sequentially complete.*

In particular, every infinite sequentially complete minimal abelian group has cardinality $\geq \mathfrak{c}$.

Here we compare the various levels of disconnectedness of the sequentially complete minimal groups, in particular, when a hereditarily disconnected sequentially complete minimal group is Vedenissov.

Corollary 8.26 *Let G be a minimal sequentially complete hereditarily disconnected abelian group. Then:*

(a) [80, Corollary 4.9] $\dim \widetilde{G} = 0$*, so G is strongly Vedenissov;*

(b) [54, Corollary 4.12] *for every prime number p the subgroup G_p is closed (hence, minimal and sequentially complete) and $G = \prod_p G_p$ topologically.*

The converse of item (b) above holds true: if $G = \prod_p G_p$ topologically for an abelian topological group G, and each G_p is minimal (and sequentially complete), then also G is minimal (and sequentially complete).

According to Corollary 8.13 sequential completeness of the group G in Corollary 8.26 (a) cannot be replaced by pseudocompactness even in the presence of total minimality.

It is not clear whether one can replace "abelian" with "nilpotent" in Corollary 8.26 (a). It is shown in [80, Corollary 4.11] that this is possible for nilpotent groups of class 2: if G is a hereditarily disconnected sequentially complete minimal nilpotent group of class 2, then $\dim G = 0$.

Theorem 8.18 leaves open the following

Question 8.27 [80, Question 6.4] *Let G be a sequentially complete minimal abelian group. Is then $G/c(G)$ sequentially complete?*

A positive answer to this question, along with Theorem 8.18 will give: *For an abelian topological group G with $c(G)$ of non-measurable size the following are equivalent: (a) G is sequentially complete and minimal; (b) $c(G)$ is compact and $G/c(G)$ is sequentially complete and minimal.* Hence, this will reduce the study of sequentially complete minimal abelian groups with small connected component to the study of the same property in totally disconnected groups. Furthermore, under a positive answer to Question 8.27 and according to Theorem 3.9, the group $G/c(G)$ will be a dense essential sequentially closed subgroup of a compact zero-dimensional group (isomorphic to $\widetilde{G}/c(G)$).

Corollary 8.26 (a) shows that the stronger version of Question 8.28(b) is true for abelian minimal sequentially complete groups in a strong way (indeed, since $\widetilde{G}$ is compact, $\dim \widetilde{G} = 0$ implies that $\widetilde{G}$ is non-archimedean, so G is non-archimedean as well).

According to Theorem 8.18, the answer of the following question is in the positive in the abelian case:

Question 8.28 *(a)* [80, Question 6.7] *Does every totally disconnected sequentially complete group admit a coarser zero-dimensional group topology?*

(b) [54, Question 7.1(5)] *Is a minimal totally (hereditarily) disconnected sequentially complete group necessarily zero-dimensional?*

In view of Corollary 8.26, (b) is true in abelian case.

Clearly, a positive answer of (a) yields a positive answer to (b) (in the option "totally disconnected") as well, yet it is not clear if (b) may hold true in case (a) fails to be true. In Example 8.30 the totally disconnected minimal non-zero-dimensional group is metric and not complete, so it is not sequentially complete, hence cannot answer (b).

8.4 On Arhangel′skiĭ's question

We see here that Theorems 8.3 and 8.6 fail without imposing any additional compactness-like property on the group G.

Question 8.29 *(a)* (Arhangel′skiĭ) *Does a totally disconnected group always admit a coarser zero-dimensional group topology ?*

(b) ([51, Question 7.9]) *Is a minimal totally disconnected group always zero-dimensional ?*

In terms of the radicals $q(G)$ and $z(G)$, (a) can be formulated also as follows: does $q(G) = \{1\}$ imply $z(G) = \{1\}$? Theorems 8.3 and 8.6 show that in the presence of some degree of compactness, Arhangel′skiĭ's question has a positive answer. Obviously, a negative answer to (b), implies a negative answer to (a) as well. Now we see that the answer to (b) is negative in the general case.

Example 8.30 *There exists a totally disconnected second countable 2-step nilpotent minimal group which is not zero-dimensional. The Weyl-Heisenberg group $H_0(l_2)$ is minimal (Remark 5.13.3). Consider its dense subgroup $G := (\mathbb{Q}/\mathbb{Z} \oplus E) \leftthreetimes F$, where E is the Erdös space $E := \{(a_i) \in l_2| \quad a_i \in \mathbb{Q}\}$ and $F := \{(b_i) \in l_2| \quad b_i \in \mathbb{Q} \ \ b_i = 0$ for almost all i's$\}$. Then G is minimal being a dense essential subgroup in $H_0(l_2)$ (Remark 5.13.4). Moreover, G contains a subgroup isomorphic to the Erdös space E, which is not zero-dimensional, hence G is not zero-dimensional either. Finally, being topologically isomorphic to the product $\mathbb{Q}/\mathbb{Z} \times E \times F$ of totally disconnected spaces, G is totally disconnected.*

The first counterexample to Question 8.29 was constructed in [139], where the target group G was the minimal group $(\mathbb{Q} \oplus E) \leftthreetimes F \leftthreetimes \mathbb{Q}_+$, which is solvable whereas the group in the above example is 2-step nilpotent.

9 Related results

9.1 Convergent sequences in minimal groups

While infinite compact groups, being dyadic spaces, have always non-trivial (from now on) convergent sequences, even a countably compact group may consistently fail to have convergent sequences (an example under the assumption of MA was given by van Douwen [92]). Many examples (in ZFC) of pseudocompact groups without convergent sequences are known [41]. Every minimal abelian group has convergent sequences [54, Theorem 2.11] (see also [184] for a different and more detailed proof), while Shakhmatov [184] proved that the non-abelian ones may fail to have them.

Question 9.1 [54, Question 7.2] *Which of the following properties of an infinite minimal group guarantee the existence of converging sequences: i) countably compact and minimal; ii) countably compact and totally minimal; iii) totally minimal?*

Item (ii) was proposed also in [74, Problem 910].

The following example shows that the answer is consistently negative even for nilpotent groups of class 2 if we weaken "countably compact" to "locally countably compact":

Example 9.2 *Let B be a countably compact Boolean group without converging sequences. The existence of such a group was established by van Douwen [92] under MA. Apply Lemma 5.16.(C) to the exponent 2 group B. Then we obtain the group $L := \mathbb{Z}_2 \oplus B \leftthreetimes \widetilde{B}^\wedge$ such that* G is a minimal, locally countably compact, nilpotent (of class 2) group without non-trivial convergent sequences. *This example is very delicate, since the group G has a two-element center and $G/Z(G)$ is abelian.*

A slight variation on this topic is the study of those minimal (or more generally, compact-like) groups that have convergent sequences with some additional property, e.g., of being quasi-convex. Note that a quasi-convex convergent sequence $x_n \to x$ in a topological abelian group G must be a *null sequence* (i.e., $x = 0$) and must be *symmetric* (i.e., for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $-x_n = x_m$).

Theorem 9.3 [63] *For every minimal abelian group G, the following statements are equivalent:*

1. *G admits no non-trivial quasi-convex null sequences;*
2. *$G \cong P \times F$, where P is a minimal bounded abelian p-group ($p \leq 3$) admitting no non-trivial quasi-convex null sequences, and F is a finite abelian group;*
3. *one of the subgroups $G[2]$ or $G[3]$ is sequentially open in G;*
4. *G contains a sequentially open compact subgroup of the form $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$ for some cardinal κ.*

Furthermore, if G is totally minimal, then these conditions are also equivalent to:

5. *$G \cong \mathbb{Z}_2^\kappa \times F$ or $G \cong \mathbb{Z}_3^\kappa \times F$, where κ is some cardinal and F is a finite abelian group (so G is compact);*
6. *one of the subgroups $G[2]$ or $G[3]$ is open in G;*

7. *one of the subgroups 2G and 3G is finite.*

Moreover, for an ω-bounded abelian group G it was proved in [63] that items 1, 6 and 7 of Theorem 9.3 are equivalent. Obviously, all items of this theorem are equivalent whenever G is compact, this result was established in [62, Theorem A]. For a locally compact abelian group the equivalence of 1, 6 and the following statement was proved in [62, Theorem A]:

8. *G contains an open compact subgroup of the form $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$ for some cardinal κ.*

It is not clear, whether minimality can be relaxed to precompactness in the first part of Theorem 9.3:

Question 9.4 [63, Problem I] *Let G be a (locally) precompact abelian group, and suppose that G admits no non-trivial quasi-convex null sequences. Is one of the subgroups $G[2]$ or $G[3]$ sequentially open in G?*

We recall also another similar problem.

Problem 9.5 [62, Problem II] *Describe the countably compact abelian groups, admitting no non-trivial quasi-convex null sequences.*

9.2 Minimality combined with other compact-like properties

Many examples of non-compact locally compact minimal groups were already discussed (e.g., those given in [96], [174]).

Here we deal mainly with pseudocompactness and its stronger forms. Note that these properties are "transversal", in a certain sense, to local compactness, since locally compact pseudocompact groups are compact.

According to a well-known criterion of Comfort and Ross [36], a topological group G is pseudocompact precisely when G is precompact and meets every non-empty G_δ set in its (compact) completion K (i.e., G is G_δ-dense in K). Therefore, a (totally) minimal pseudocompact abelian group is precisely a G_δ-dense essential (totally dense) subgroup of a compact abelian group.

The topic in this subsection comes from two problems set and intensively studied by W. Comfort and coauthors.

The following general problem was studied first in [37]:

Problem 9.6 *Describe the compact groups K admitting a proper totally dense pseudocompact subgroup.*

Clearly, groups K as above give rise to totally minimal pseudocompact non-compact groups.

Comfort and Robertson showed that a radical approach to Problem 9.6, providing a totally dense pseudocompact subgroup of smaller cardinality cannot work in general:

Theorem 9.7 [34, Theorem 6.2] *ZFC cannot decide whether there exists a compact abelian group K with a totally dense pseudocompact subgroup of size $< |K|$.*

This theorem suggests a more careful analysis, based on the algebraic and topological properties of the compact groups in question.

Following [59], we call a compact abelian group G *singular*, if the subgroup mG of G is metrizable for some $m > 0$. Since, $mG \cong G/G[m]$, this means that G is singular if and only if $G[m]$ is a G_δ-subgroup of G. Hence, for a singular G, the torsion subgroup $tor(G)$ contains a closed G_δ-subgroup. The latter condition appeared much earlier in [71] as a *necessary condition*, to ensure the existence of a proper totally dense pseudocompact subgroup of G. One can easily see that this condition is *equivalent* to singularity of G. Indeed, if some closed G_δ-subgroup N of G is torsion, we deduce that N is bounded torsion (as N is compact). If $mN = 0$ for some $m > 0$, then the subgroup $mG \cong G/G[m]$ of G is metrizable, so G is singular.

Obviously, all metrizable compact abelian groups are singular. Let us see now that *a singular compact abelian group cannot have proper totally dense pseudocompact subgroups.* Indeed, assume that $G[m]$ is a G_δ-subgroup of G. Then every dense pseudocompact subgroup H of G must meet every coset $x + G[m]$, so $H + G[m] = G$. On the other hand, H contains $G[m]$, being totally dense. Therefore, $H = G$. It turned out to be quite hard to prove that every non-singular compact abelian group G admits a proper totally dense pseudocompact subgroup. This was done in [71] under the assumption of Lusin's hypothesis $2^{\omega_1} = 2^\omega$. The final solution of Problem 9.6 in the abelian case came much later in [56]:

Theorem 9.8 [56, Theorem 1.5] *For a compact abelian group K the following are equivalent:*

(a) K admits a proper dense totally minimal pseudocompact group;

(b) $tor(K)$ contains no G_δ-subgroup (i.e., K is not singular);

(c) there exists a continuous surjective homomorphism of K onto S^{ω_1}, where S is a compact non-torsion abelian group;

(d) K admits a proper totally dense subgroup containing a dense ω-bounded subgroup.

The implication (a) → (b) was discussed above, while (d) → (a) since ω-bounded groups are pseudocompact. The implication (c) → (d) follows from the fact that the Σ-product in S^{ω_1} is ω-bounded and one can produce a proper totally dense subgroup H of S^{ω_1} containing the Σ-product, so that the inverse image of H under the continuous surjective homomorphism $K \to S^{\omega_1}$ works. The major difficulty is hidden in the proof of the implication (b) → (c).

A more general version of this theorem was proved in [58], depending on the stronger notion of *λ-pseudocompact group*, introduced by Kennison [124] ($f(G)$ is compact for every continuous function $f : G \to \mathbb{R}^\lambda$). Let us also recall that a group is called *λ-bounded*, if every subset of size $\leq \lambda$ is contained in a compact subset.

Theorem 9.9 [58] *For a compact abelian group K the following are equivalent:*

(a) *K admits a proper dense totally minimal λ-pseudocompact group;*

(b) *$tor(K)$ contains no G_λ-subgroups;*

(c) *there exists a continuous surjective homomorphism of K onto S^λ, where S is a compact non-torsion abelian group;*

(d) *K admits a proper totally dense subgroup containing a dense λ-bounded subgroup.*

The natural counterpart of Theorem 9.8 about dense minimal subgroups was obtained by Giordano Bruno in Theorem 9.12. It is related also to another long standing problem we are going to recall now.

Following [33], call a pseudocompact group *s-extremal* if it contains no proper dense pseudocompact subgroup. Metrizable pseudocompact (hence, compact) groups are s-extremal. The problem to invert this implication (i.e., show that every s-extremal pseudocompact group is metrizable) was posed by Comfort and Robertson [33] and attacked by many authors in a long series of papers and resolved in the positive by Comfort and van Mill [31] about 25 years later:

Theorem 9.10 [31] *Every non-metrizable pseudocompact abelian group G admits a proper dense pseudocompact subgroup.*

In the next cluster of problems proposed in [105], various weaker levels of extremal pseudocompactness are considered imposing on the proper dense pseudocompact subgroup some additional properties related to *minimality*:

Problem 9.11 *For $i = 1, 2, 3, 4$ describe the class $\mathcal{K}_i$ of pseudocompact abelian groups that admit proper dense subgroups with the following property: (1) minimal and pseudocompact; (2) essential and pseudocompact; (3) totally minimal and pseudocompact; (4) totally dense and pseudocompact.*

Let us note that a pseudocompact group G from $\mathcal{K}_1$ (resp., $\mathcal{K}_3$) is necessarily (totally) minimal. On the other hand, $G \in \mathcal{K}_1$ and $G \in \mathcal{K}_2$ (resp., $G \in \mathcal{K}_3$ and $G \in \mathcal{K}_4$) are equivalent for (totally) minimal pseudocompact abelian groups G. Clearly, none of the classes $\mathcal{K}_i$ contains metrizable groups as metrizable groups do not admit any proper dense pseudocompact subgroup. By Theorem 9.10, a non-metrizable pseudocompact abelian group G admits a proper dense pseudocompact subgroup, but it is not clear if such a subgroup can be chosen also to be essential. A class $\mathcal{K}$ containing all minimal groups from $\mathcal{K}_1$ is discussed below (see Problem 9.20).

Here comes a description of the *compact* groups from $\mathcal{K}_1$ (and from $\mathcal{K}_2$), showing that a non-metrizable compact abelian group (necessarily containing a proper dense pseudocompact subgroup by Theorem 9.10) need not contain a proper dense minimal pseudocompact subgroup.

Theorem 9.12 [105] *For a compact abelian group K the following are equivalent:*

(a) *K admits no proper dense minimal pseudocompact subgroup;*

(b) *$soc(K)$ contains a G_δ-subgroup;*

(c) *K is singular and $p \cdot \mathrm{cl}(K_p)$ is metrizable for every prime p.*

A compact abelian group may have a proper dense essential pseudocompact subgroup even if it has no proper totally dense subgroup (see [105, Example 3.1]). The groups with the latter property are described in item (a) of the next theorem.

Theorem 9.13 [105, Corollary 4.2] *Let K be a compact abelian group. Then*

(a) *K has no proper totally dense subgroup if and only if K is torsion.*

(b) *K has no proper dense minimal subgroups if and only if $soc(K)$ is open (so K is torsion) and $p \cdot \mathrm{cl}(K_p)$ is finite for every $p \in \mathbb{P}$.*

Theorem 9.13(a) suggests the following counterpart of Problem 9.11.

Problem 9.14 [105, Problem 1.10]

(a) *Describe the pseudocompact abelian groups that admit proper dense minimal subgroups.*

(b) *Describe the pseudocompact abelian groups that admit proper dense essential subgroups.*

The next two problems are inspired by Theorem 9.13, where the problem was resolved for compact abelian G:

Problem 9.15 *Describe the minimal (abelian) groups G such that:*

(a) G has proper dense minimal subgroups;

(b) G has proper dense totally minimal subgroups.

According to Theorem 3.6, G satisfying (b) must necessarily be totally minimal.

D. Dikranjan, A. Giordano Bruno and D. Shakhmatov [60] described the free abelian groups admitting pseudocompact minimal group topologies. These are precisely the free groups that admit a minimal group topology and also a pseudocompact group topology.

In the rest of this subsection we concentrate on dense (minimal) countably compact subgroups. Theorem 8.15 can be announced in the following equivalent form: *no compact abelian group contains proper, totally dense, sequentially complete subgroups.* Inspired by [71, Theorem 1.4], we announce the following result that simultaneously generalizes [71, Theorem 1.4] and Theorem 8.15 (the proof can be obtained by a combination of both proofs):

Theorem 9.16 *If the group G is covered by its compact subgroups, then G contains no proper sequentially complete subgroup H such that $H \cap N$ is dense in N for every closed subgroup N of G. In particular, no ω-bounded abelian group contains proper sequentially complete totally dense subgroups.*

The condition "$H \cap N$ is dense in N for every closed subgroup N of G", satisfied by G is stronger than total density. This explains the strong conclusion of the theorem. In the sequel we replace it by total density.

Let us recall also the following open problem from [71].

Problem 9.17 *Characterize the class $\mathcal{T}$ of pseudocompact groups which do not admit proper totally dense countably compact subgroups.*

Since every group containing a dense pseudocompact (in particular, countably compact) subgroup is pseudocompact itself, it is necessary to impose pseudocompactness in the definition of $\mathcal{T}$. Obviously, $\mathcal{T}$ contains the class of all compact metrizable groups.

Remark 9.18 Using the fact that central subgroups are normal and arguing by induction on the nilpotency class, it can be deduced from Theorem 9.16, that all ω-bounded nilpotent groups belong to the class $\mathcal{T}$. We do not know if $\mathcal{T}$ contains also all nilpotent countably compact groups as well. An example of a pseudocompact abelian group that does not belong to this class can be found in [54, Example 7.5].

Example 9.19 (a) To obtain an example of a compact group that does not belong to the class $\mathcal{T}$ take an uncountable family $\{K_i : i \in I\}$ of compact simple groups, then $K = \prod_{i \in I} K_i \notin \mathcal{T}$, as the Σ-product is a proper totally dense ω-bounded subgroup of K (see Example 3.23).

(b) Let $\{F_i : i \in I\}$ be an uncountable family of finite simple groups and consider the Σ-product G in the product $K = \prod_{i \in I} F_i$. Then G is an ω-bounded group in $\mathcal{T}$. Indeed, one can prove even a stronger property. Assume that H is a totally dense sequentially complete subgroup of G. Then H contains the direct sum $D = \bigoplus_{i \in I} F_i$, as each F_i is a finite closed normal subgroup of K (hence of G as well) and H is totally dense. Since D is sequentially dense in G, while H must be sequentially closed in G, we deduce that $H = G$. Thus, $G \in \mathcal{T}$.

Now we propose a variant of Problem 9.11, by defining a class $\mathcal{K}$ of minimal pseudocompact groups that contains all minimal groups from the class $\mathcal{K}_1$ defined in Problem 9.11.

Problem 9.20 *Characterize the class $\mathcal{K}$ of minimal pseudocompact abelian groups which do not admit proper dense minimal countably compact subgroups.*

Imposing minimality and pseudocompactness on the groups of $\mathcal{K}$ makes sense, since a group having a dense minimal pseudocompact subgroup, is minimal and pseudocompact itself.

To see that $\mathbb{T}^\kappa$ belongs to $\mathcal{K}$ for every cardinal κ it suffices to note that $soc(\mathbb{T}^\kappa)$ is sequentially dense in $\mathbb{T}^\kappa$. Hence, $\mathbb{T}^\kappa$ has an even stronger property: it has no proper dense *sequentially complete* minimal groups. For non-measurable κ this follows from the following much more general fact. According to Theorem 8.18, a connected abelian group K of non-measurable size contains no proper dense minimal sequentially complete abelian groups G (indeed, $c(G) = c(\widetilde{G}) = c(\widetilde{K}) = \widetilde{K}$ yields $G = K = \widetilde{L}$). On the other hand, there exists a plenty of non-compact minimal countably compact groups (see [72, 85]), their completions are compact abelian groups that do not belong to $\mathcal{K}$.

9.3 Generators of minimal topological groups

A discrete subset S of a topological group G is called *suitable* for G, if S generates a dense subgroup of G and $S \cup \{e\}$ is closed in G. The notion of suitable set was invented by Hofmann and Morris [120], who proved that every locally compact group has a suitable set. Metrizable groups have suitable sets [32, Theorem 6.6]. For further examples the reader may consult also [81].

Here are two leading examples of a suitable set: (a) a finite set S; (b) a sequence $S = (s_n)$ of distinct elements $s_n \neq e$ of G converging to e. The second instance has a more general version: a *supersequence* S converging to e, i.e., a discrete subset $S \subseteq G \setminus \{e\}$ such that $S \cup \{e\} = \alpha S$ is the one-point compactification of the discrete space S. Hence every supersequence S converging to e that generates a dense subset is a suitable set. On the other hand, the symmetric group $S(X)$ has a suitable set (the set of all transpositions) that is not a supersequence converging to e.

The following properties of suitable sets should be taken into account.

Fact 9.21 *(a) A suitable set S in a group G has size $\leq w(G)$.*

(b) A suitable set S in a countably compact group is either finite or a supersequence converging to e. Therefore, countably compact groups containing an infinite suitable set contain a non-trivial converging sequence.

(c) If $f\colon G \to H$ is a continuous homomorphism and G is countably compact, then the existence of a suitable set of G yields the existence of a suitable set of H whenever f is either surjective or a dense embedding.

Since locally compact groups admit a suitable set, it seems natural to investigate under this point of view also the (totally) minimal groups.

Theorem 9.22 [81] *Let G be a totally minimal group.*

(a) ([81, Theorem 4.4]) *If G is abelian, then G has a suitable set.*

(b) ([81, Theorem 4.5]) *If G is connected and precompact, then G has a suitable set.*

A minimal countably compact abelian group need not have a suitable set (see Remark 9.24(b) for a stronger assertion). Things change under the additional assumption of connectedness:

Theorem 9.23 [81, Theorem 4.8] *Every minimal countably compact connected abelian group G has a suitable set.*

Remark 9.24 *Let us discuss the necessity of some of the hypothesis in Theorem 9.23.*

(a) It is conjectured in [81] that "abelian" can be eliminated.

(b) Connectedness cannot be removed in the above theorem. An example of a totally disconnected ω-bounded (and hence countably compact) minimal abelian group without suitable sets can be found in [81, Example 4.10].

The following question was left open in [81, Problem 4.6]:

Question 9.25 *Does every totally minimal group contain a suitable set ? What about precompact totally minimal groups?*

Question 9.26 [54, Question 7.7] *Prove or disprove the following statements for a countably compact group G.*

(1) G admits a suitable set if G has a compact normal subgroup N such that G/N has a suitable set;

(2) G admits a suitable set if G is totally minimal and $G/Z(G)$ has a suitable set;

(3) G admits a suitable set if G is totally minimal.

Clearly, (3) implies (2) (by Fact 9.21(c)), while (1) and (2) together imply (3) (since $Z(G)$ is compact by Theorem 8.15).

The next question is motivated by Theorems 9.22 and 9.23.

Question 9.27 [54, Question 7.8] *Does every connected minimal abelian group admit a suitable set?*

A new line in the study of groups with suitable sets of topological generators was proposed in [86] by imposing additional restraint not on the (topological) nature of the set of generators of a topological group G, but on the level of density in G of the subgroup they generate. A suitable subset S of a topological group G is called *totally suitable* for G if S generates a totally dense subgroup of G. In a topologically simple group every suitable set is totally suitable, so the symmetric topological groups $S(X)$ have totally suitable sets.

The class $\mathcal{S}_t$ of groups with totally suitable sets was studied in [19], [86]. It is closed with respect to taking closed continuous homomorphic images ([86, Proposition 4.2], for countably compact groups this follows also from Fact 9.21 (b)).

The compact groups G with a totally suitable set are subject to some restraints. The first comes from Fact 9.21 (a), which implies the inequality

$$TD(G) \leq w(G). \tag{11}$$

Example 9.28 *(a)* $\mathbb{T}^{\mathbb{N}} \notin \mathcal{S}_t$, *as* $\mathbb{T}^{\mathbb{N}}$ *does not satisfy (11).*

(b) For every prime p, $\mathbb{Z}_p^{\mathbb{N}} \notin \mathcal{S}_t$, *since again (11) fails.*

Theorem 9.29 [86, Theorem 4.6] *Let* G *be a compact abelian group. Then* $G \in \mathcal{S}_t$ *if and only if* $G \in \mathcal{P}$.

Proof. We shall only give a brief sketch of the proof of the implication

$$G \in \mathcal{S}_t \Rightarrow G \in \mathcal{P}.$$

The proof of the missing implication can be found in [86, Theorem 4.6].

Let us see first that $G \in \mathcal{S}_t$ implies that G is metrizable. Indeed, it suffices to show that both $c(G)$ and $G^0 := G/c(G)$ are metrizable. To this end we exploit the fact that the class $\mathcal{S}_t$ is closed under taking quotients, $\mathbb{T}^{\mathbb{N}} \notin \mathcal{S}_t$, and $\mathbb{Z}_p^{\mathbb{N}} \notin \mathcal{S}_t$ for every prime p, according to Example 9.28. The first and the second properties imply $\dim G = \dim c(G) < \infty$ (so $c(G)$ is metrizable), while the latter one yields that G/pG is finite for every prime p. As G^0 is pro-finite, one has

$$G^0 \cong \prod_{p \in \mathbb{P}} G_p^0 \quad \text{and} \quad G/pG \overset{(1)}{\cong} G^0/pG^0 \cong G_p^0/pG_p^0, \tag{12}$$

where the isomorphism (1) can be deduced from $pG \supseteq c(G)$ for all primes p. To see that G^0 is metrizable it suffices to check that each G_p^0 is metrizable for all primes p. This follows from the fact that the group G_p^0 is a pro-p-group with finite G_p^0/pG_p^0 (due to (12)).

Since $G \in \mathcal{S}_t$ is metrizable, (11) implies that G has a countable totally dense subgroup. Hence, G belongs to the class $\mathcal{P}$ of Prodanov groups (see Example 3.14). □

These results can be extended to countably compact abelian groups G by noting that if $G \in \mathcal{S}_t$ is countably compact, then G is compact (hence $G \in \mathcal{P}$, so metrizable) [86, Theorem 4.12]. This left open the following question:

Question 9.30 *Is a pseudocompact abelian group* $G \in \mathcal{S}_t$ *necessarily metrizable (hence, compact)?*

In the sequel we recall the solution of the counterpart of Question 9.30 ([86, Problem 4.14]) for LCA groups.

For an LCA group G, we denote by $B(G)$ the union of all its compact subgroups and by G^+ the group G when endowed with its Bohr topology, that is the topology induced on G by the Bohr compactification $b : G \to bG$ (it is injective in this case). It is well known that G is topologically isomorphic to a product $\mathbb{R}^n \times G_0$, where $n \in \mathbb{N}$ and G_0 contains a compact open subgroup K. In case G_0 itself is compact, one takes $K = G_0$. Hence, the index $\varrho(G) = [G : K]$ is either 1 (precisely when G is compact), or infinite. If K_1 is another compact open subgroup of G_0, then both $[K : (K \cap K_1)]$ and $[K_1 : (K \cap K_1)]$ are finite, so $[G : K] = [G : K_1]$ in case at least one of these indexes is infinite. In other words, $\varrho(G)$ is uniquely determined by the group G, i.e., does not depend on the choice of K. Moreover, $TD(K) = TD(K_1) = TD(K \cap K_1)$ when G is not compact, i.e., $TD(K)$ does not depend on the choice of the compact open subgroup K.

Theorem 9.31 [20] *Let* G *be a locally compact abelian group with compact open subgroup* K.

(A) If $G \neq B(G)$ *and* G *is non-discrete, then the following conditions are equivalent:*

- *(A*$_1$*)* $G \in \mathcal{S}_t$;
- *(A*$_2$*)* G *admits a closed suitable set;*
- *(A*$_3$*)* $\varrho(G) \geq \mathfrak{c} \cdot |K|$.

(B) If $G = B(G)$ *and* G *is non-compact, then the following conditions are equivalent:*

- *(B*$_1$*)* $G \in \mathcal{S}_t$;
- *(B*$_2$*)* G *admits a closed totally suitable set;*
- *(B*$_3$*)* $\varrho(G) \geq TD(K)$.

Let $\mathscr{T}$ be the class of all abelian groups admitting a suitable set and in which *every* suitable set is also totally suitable. It was proved in [86, Theorem 4.17] that the groups $\mathbb{Z}_p, p \in \mathbb{P}$ are the only infinite countably compact groups in $\mathscr{T}$. This suggests the conjecture that the groups $\mathbb{Q}_p, p \in \mathbb{P}$ are the only non-compact and non-discrete groups locally compact abelian groups in $\mathscr{T}$. We see below that this is true for separable groups, although it fails in the general case. Indeed, an example of a non-discrete and non-compact LCA group $G \in \mathscr{T}$ with $d(G) = \mathfrak{c}$ (so $G \not\cong \mathbb{Q}_p$ for any prime p) can be found in [20, Example 2.16].

Theorem 9.32 [20] *Let* G *be a locally compact abelian group that is neither compact nor discrete. If* $d(G) < \mathfrak{c}$, *then the following are equivalent for* G:

(a) $G \in \mathscr{T}$;

(b) $G \cong \mathbb{Q}_p$ *for some prime* p.

9.4 Miscellanea

Here we list only briefly several topics that we did not cover in the survey.

Concerning the basic properties of minimal groups, we skipped a great deal of the wealth of known results in the abelian case (especially concerning products and the structure theory of minimal groups). The reader is advised to consult the surveys [51, 54], the books [68] and [131, Chap. 3] and the references given there.

Categorical compactness in an abstract category was introduced by Manes [133]. It was studied for the first time in the category of topological groups (under the name *c-compactness*) in [87], where (among others) its nice connection to minimality was pointed out. A topological group G is *c-compact*, if for every topological group H the second projection $G \times H \to H$ sends closed subgroups of $G \times H$ to closed subgroups of H. Compact groups are c-compact by Kuratowski closed projection theorem, while c-compact groups are h-complete [87]. Every totally minimal locally compact group is obviously h-complete, so h-completeness does not imply compactness even for locally compact groups. The following bold question from [87] still remains open:

Question 9.33 ([87]) *Are c-compact groups compact?*

The answer was shown to be positive for solvable groups, while nilpotent h-complete groups are compact [87] (see also Theorem 8.1 for a stronger result). On the other hand, it is still open even in the discrete case. More precisely, it was shown in [87] that a countable discrete group is c-compact if and only if it is hereditarily non-topologizable (see §2.1). We formulate the question in the form given in [87, Question 5.2].

Question 9.34 *Is a discrete c-compact group necessarily: (a) finite; (b) finitely generated; (c) of finite exponent; (d) countable?*

Since every separable c-compact group is minimal ([87, Corollary 3.6]), one obtains another weaker and natural form of Question 9.33

Question 9.35 ([87, Question 1.2]) *Are c-compact groups minimal?*

Further information can be found in the surveys [51, 52, 54, 74], as well as the work of Lukács (the paper [130], the book [131, Chap. 4] and his PhD thesis).

A *paratopological group* is a pair (G, τ) of a group G and a topology τ on G such that the group operation $G \times G \to G$ is continuous. Recently, paratopological groups are attracting more attention, see for example the contribution of Tkachenko in the present volume and also [7]. A Hausdorff paratopological group G is said to be minimal if the topology of G is a minimal element of the set of all Hausdorff group paratopologies on G. The following two questions of Guran aim to clarify whether the concept of minimality, defined in the framework of paratopological groups, gives something new:

Question 9.36 (Guran [115]) *(a) Is there a minimal Hausdorff paratopological group which is not a topological group?*

(b) Does every Hausdorff paratopological group G admit a weaker Hausdorff group topology?

Clearly, a positive answer to (b) gives a negative answer to (a). Ravsky [171] answered negatively (b), while Banakh and Ravsky [14] showed (b) has a positive answer in a rather wide class of groups, containing all nilpotent groups. Therefore, this result provides a negative answer, in this class of groups, to (a). This item still remains open in the general case. Lin [129] treats a paratopological version of *local minimality*. Here the situation is different, the Sorgenfrey line and its finite powers are locally minimal paratopological groups, but not topological groups.

Let $\mathcal{L}(G)$ denote the lattice of all group topologies on a group G and let $\mathcal{H}(G) \subseteq \mathcal{L}(G)$ be the poset of Hausdorff group topologies on G. So far special emphasis was given to the minimal elements of $\mathcal{H}(G)$, namely the minimal topologies on G. The remarkable idea of Prodanov [167] was to study the minimal topologies by means of the maximal ones (these are the maximal elements in the set of non-discrete Hausdorff group topologies, see §3.2). The advantage of this point of view is the abundance of maximal topologies (as Zorn's lemma can be applied to produce maximal topologies on the group). Moreover, every minimal group topology on an abelian group G is contained in every maximal topology on G. This made obvious the important role of the *submaximal topology* $\mathscr{M}_G$ on an abelian group G (that is, the infimum of all maximal topologies on the group G), described explicitly by Prodanov [167]. See also [55, 82] for more details on the submaximal topology. Therefore, on abelian groups G having $\mathscr{M}_G$ precompact (e.g., divisible groups, or finitely generated groups) every minimal topology will automatically be precompact.

Minimal algebraic structures (rings, modules and fields) were not discussed here, the reader may address the books [9, 210] the survey [51, §7.4], or [42, 43, 138]. For minimality in topological vector spaces see [93, 94].

Note added in proofs: Recently Klyachko, Olshanskii and Osin [126] resolved Question 9.33 and all items of Question 9.34 in the negative. Very recently, I. Ben Yaacov and T. Tsankov [17] announced the following result: Every Polish Roelcke precompact group G satisfying the condition $\mathrm{RUC}(G) \cap \mathrm{LUC}(G) = \mathrm{WAP}(G)$ (i.e., a *wap group* in terms of [108, Def. 6.5]), is totally minimal. Various large groups are wap, e.g., the unitary groups $U(H)$ and $Aut(X, \mu)$. This gives short proofs of Theorems 4.40 (Stoyanov) and 4.42 (Glasner) and provides a positive answer to Question 4.41 in the case of wap groups. In general the answer to Question 4.41 is negative as it follows from the same work of I. Ben Yaacov and T. Tsankov.

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