A Tikhonov *G*-space not admitting a compact Hausdorff *G*-extension or *G*-linearization

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A Tikhonov G-space not admitting a compact Hausdorff G-extension or G-linearization

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We preserve the terminology of [1]-[5].

***Proposition* 1.** *There exists a Tikhonov G-space not admitting equivariant compact extensions.*

Proof. Let $I = [0;\ 1]$ be a numerical interval, let Homeo(I) be the group of homeomorphisms of I in the topology of uniform convergence, let $S = \{1/n\colon n \in \mathbf{N}\}$, and let

$$G_1 = \{g \in \mathrm{Homeo}\,(I)\colon \text{ for all } s \in S,\ gs = s\}.$$

Then G_1 is a closed subgroup of Homeo(I). The natural action $\alpha_1 : G_1 \times I \to I$ is continuous. It has countably many orbits. All points of S and the point o are fixed. All the remaining orbits are the intervals of the form $(1/(n+1);\ 1/n)$, where $n \in \mathbf{N}$. The point o has a base consisting of invariant neighbourhoods $[0;\ 1/n)$, $n \in \mathbf{N}$.

(1) If $\{O_k\colon k \in \mathbf{N}\}$ is a system of open neighbourhoods of o in the space I such that $[GO_k] \subseteq O_{k+1}$ for all $k \in \mathbf{N}$, then $O_{k_0+1} = I$, where k_0 is a natural number such that $[o;\ 1/k_0] \subseteq O_1$.

Let $\{(G_n,\ I_n,\ \alpha_n)\colon n \in \mathbf{N}\}$ be a countable system of continuous transformation groups, where each (G_n, I_n, α_n) is a copy of the action (G_1, I, α_1). We form a special sum (G, X, α), where $X = \oplus\{I_n\colon n \in \mathbf{N}\}$ is the topological sum with the natural embeddings $i_n : I_n \to X$ and the group $G = \Pi\ \{G_n\colon n \in \mathbf{N}\}$ is a product of topological groups with the natural projections $j_n : G \to G_n$. The action of α is defined in the following natural manner:

$$\alpha\,(g,\ x) = i_{n_0}\,(\alpha_{n_0}\,(j_{n_0}\,(g),\ x_0))$$

where $g \in G$, $x \in i_{n_0}(I_{n_0})$, and $i_{n_0}(x_0) = x$.

We now form a set Y by identifying, in X, all "null" points, that is, we define an equivalence relation whose only non-trivial equivalence class is the set $\{i_n(o)\colon n \in \mathbf{N}\}$. This "singular" point of Y will be denoted by w. Let $p : X \to Y$ be the canonical projection. We define a topology on Y, where a neighbourhood base of W is the system $\{A_k(w)\colon k \in \mathbf{N}\}$ with

$$A_k\,(w) = \cup\{p\,(i_n\,[o;\ 1/k))\colon n \in \mathbf{N}\}.$$

At all other points we take the usual neighbourhoods. It is easy to verify that Y is homeomorphic to the so-called metrizable hedgehog $J(\aleph_0)$ with countably many thorns (see [4]).

We define an action $\bar{\alpha}$ of G on Y. Every point of the set $p^{-1}(w)$ is fixed. Therefore, there exists a unique action $\bar{\alpha}$ on Y under which p is equivariant. Formally,

$$\bar{\alpha}\,(g,\ a) = p\,(\alpha\,(g,\ p^{-1}\,(a))),\ \text{where}\ (g,\ a) \in G \times Y.$$

Every $A_k(w)$ is an invariant set under $\bar{\alpha}$. Therefore, $\bar{\alpha}$ is continuous at points of the form (g, w), where $g \in G$. The continuity of $\bar{\alpha}$ at the remaining points is obvious. Hence Y is a Tikhonov G-space and $w(G) = w(Y) = \aleph_0$, dim $Y = 1$, and G is complete in the sense of Raikov. We prove that the G-space Y is of the desired kind. We first recall the following definition.

Definition [1], [3]. A continuous function $f : X \to \mathbf{R}$ defined on a G-space X is said to be *α-uniform* if for any $\varepsilon > 0$ there exists a neighbourhood $U(e)$ of the identity such that

$$|\,f(x) - f(gx)\,| < \varepsilon \ \text{ for all } g \in U(e) \text{ and all } x \in X.$$

Theorem [1], [3]. *A G-space X admits equivariant compact extensions if and only if the α-uniform functions on X separate points and closed sets.*

It turns out that in the G-space Y the closed set $F = \{p(i_n(1))\colon n \in \mathbf{N}\}$ of all "end-points" cannot be separated from the point w by any α-uniform function. We assume the contrary. Let $f : Y \to [0;\ 1]$ be an α-uniform function such that $f(w) = 0$ and $f(F) = 1$. The definition of an α-uniform function implies the existence of a sequence $\{O_n(w)\}$ of open neighbourhoods of w and of a sequence of neighbourhoods of the identity such that

(2) $\{V_n(e)\}$ $O_n \cap F = \varnothing$ and $[V_nO_n] \subseteq O_{n+1}$ for all $n \in \mathbf{N}$.

By the definition of the local base at w, for any $k_0 \in \mathbf{N}$ we have $A_{k_0}(w) \subseteq O_1(w)$. Since $G = \Pi\{G_n : n \in \mathbf{N}\}$, there exists $n_0 \in \mathbf{N}$ such that if $1 \leqslant k \leqslant k_0 + 1$, $n \geqslant n_0$, then $j_n(V_k) = G_n$. Using property (1) of the action α_1 and the fact that the restriction of $\bar{\alpha}$ to each "needle" $p(I_m)$ is equivalent to α_1, we see that $O_{k_0+1}(w)$ contains $\bigcup \{p(i_m(I_m)) : m \geqslant n_0\}$ and, in particular, $O_{k_0+1}(w)$ intersects F, which contradicts (2).

***Proposition* 2.** *There exists a Tikhonov G-space admitting no G-linearization.*

This assertion follows immediately from Proposition 1 and the following theorem.

Theorem. *Every linear G-space has compact G-extensions.*

Proof. We showed in [5] that a G-space X has compact G-extensions exactly when there exists a quasibounded [5] uniformity on X. It remains to verify that the natural uniformity is quasibounded on every linear G-space.

***Proposition* 3.** *There exists a continuous action of a zero-dimensional compact metrizable group G on a 1-dimensional G-space X with a countable base such that* $\dim \beta_G X = 2$, *where* $\beta_G X$ *is a maximal compact G-extension.*

There exists a continuous action α_1 of a zero-dimensional compact metrizable group G_1 on a 1-dimensional compact space X_1 such that the dimension of the orbit space is equal to two [3]. Let $\{(G_n, X_n, \alpha_n)\}$ be a countable system of copies of the action α_1. It turns out that the special sum $(\Pi G_n, \oplus X_n, \alpha)$ (see the proof of Proposition 1) is the desired action.

Remark. Propositions 1 and 2 answer questions of de Vries [2], and Proposition 3 answers a question of Yu.M. Smirnov. For the special sum, see [2].

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References

[1] J. de Vries, On the existence of G-compactifications, Bull. Acad. Polon. Sci. Ser. Sci. Math. **26** (1978), 275–280. MR **58** # 31002.

[2] ————, Topological transformation groups. I, Math. Centre Tracts 65, Math. Centrum, Amsterdam 1975. MR **54** # 3671.

[3] Yu. Smirnov, Compactifications, dimension and absolutes of topological transformation groups, Proc. Conf. on Topology and Measure III, Greifswald 1982, 259–266.

[4] R. Enkelking, General topology, PWN, Warsaw 1977. MR **58** # 18316b.
Translation: *Obshchaya topologiya*, Mir, Moscow 1986. MR **87k**:54002.

[5] M.G. Megrelishvili, Equivariant completions and compact extensions, Soobshch. Akad. Nauk Gruzin. SSR **115**:1 (1984), 21–24. MR **86m**:54054.

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