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COMPACTIFICATIONS OF SEMIGROUPS AND SEMIGROUP ACTIONS

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Abstract. An action of a topological semigroup S on X is *compactifiable* if this action is a restriction of a jointly continuous action of S on a Hausdorff compact space Y. A topological semigroup S is compactifiable if the left action of S on itself is compactifiable. It is well known that every Hausdorff topological group is compactifiable. This result cannot be extended to the class of Tychonoff topological monoids. At the same time, several natural constructions lead to compactifiable semigroups and actions.

We prove that the semigroup $C(K,K)$ of all continuous selfmaps on the Hilbert cube $K = [0,1]^\omega$ is a universal second countable compactifiable semigroup (*semigroup version of Uspenskij's theorem*). Moreover, the Hilbert cube K under the action of $C(K,K)$ is universal in the realm of all compactifiable S-flows X with compactifiable S where both X and S are second countable.

We strengthen some related results of Kocak & Strauss [19] and Ferry & Strauss [13] about Samuel compactifications of semigroups. Some results concern compactifications with separately continuous actions, LMC-compactifications and LMC-functions introduced by Mitchell.

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1. INTRODUCTION

A major role of semigroup actions and semigroup compactifications is now well understood. See for example the books [4, 5, 37, 1]. Very little is known however about sufficient conditions which ensure the existence of *proper* compactifications (for definitions see Section 3) in the case of monoidal actions. This contrasts the case of topological group actions (see for example [44, 46, 45, 48, 2, 30, 25, 24, 26, 29, 35, 31]).

A semigroup action $S \times X \to X$, or, a *flow* (S, X), is *compactifiable* if there exists a *proper* S-compactification $X \hookrightarrow Y$. That is, if the original action is a restriction of a jointly continuous action on a Hausdorff compact S-flow Y. In this article we require that S is a *topological semigroup* (the multiplication is jointly continuous). We say that a topological semigroup S is *compactifiable* if the flow (S, S), the regular left action, is compactifiable. Passing to the *Ellis semigroup* $E(Y)$ of an S-compactification Y of a monoid S we see that S is compactifiable iff S has a proper *dynamical compactification* in the sense of Ruppert [37] (see also the *monoidal compactification* in the sense of Lawson [22]). Recall that a dynamical compactification of S is a right topological semigroup compactification $S \to T$ such that the associated action $S \times T \to T$ is jointly continuous (see Definition 3.3 and Proposition 3.5).

If a topological semigroup S algebraically is a group we say that S is a *paratopological group*. As usual, *topological group* means that in addition we require the continuity of the inverse operation. Due to Teleman [39] every Hausdorff (equivalently: Tychonoff) topological group is compactifiable. This classical result cannot be extended to the class of Tychonoff topological semigroups. For instance, the multiplicative monoid $([0, \infty), \cdot)$ of all nonnegative reals is not compactifiable (see Example 6.3.2 below) and not even *LMC-compactifiable* as it follows by a result of Hindman and Milnes [17]. The latter means in fact that there is no proper S-compactification $S \to Y$ with a separately continuous action on Y. LMC is an abbreviation of *Left Multiplicatively Continuous*. LMC-compactifications and LMC-functions for semigroups were introduced by Mitchell, [32, 17, 4]. The case of separately continuous compactifications is parallel to the theory of right topological semigroup compactifications and *generalized LMC-functions* (see Definition 3.13). This direction is linked to Banach representations of semigroups and actions (in the sense of [28]) and to corresponding generalized matrix coefficients.

One of our aims in the present paper is to study the similarities and differences in the theory of flow compactifications when we pass from groups to semigroups. We emphasize the limitations providing several noncompactifiable semigroups and actions with "good topological properties" (contrasting the case of topological groups).

The classical Gelfand-Naimark 1-1 correspondence between Banach subalgebras of $C(X)$ and the compactifications of X can be extended to the category of S-flows describing jointly continuous S-compactifications by subalgebras of the algebra $\mathrm{RUC}_S(X)$ of all *right uniformly continuous functions* on X (see Definition 3.9). This theory is well known for topological group actions (see, for example, J. de Vries [45]). One can easily extend it to the case of topological semigroup actions. Some results in this direction can be found in the work of Ball and Hagler [6]. For instance the authors gave an example of a second countable noncompactifiable monoid S.

We establish some sufficient and necessary conditions in terms of uniform structures. In particular, we strengthen two results of Kocak and Strauss [19] and also a result of Ferry and Strauss [13] (see Corollary 4.12 and Remark 4.16.1). These results provide an additional information about points of joint continuity of semigroup actions. Note that this direction is an object of a quite intensive investigation (see for example [11, 16, 15, 21, 20, 37, 40] and the references therein).

The topological monoid $C(K,K)$ of all continuous self-maps endowed with the compact open topology is compactifiable. If E is a normed space then the monoid $(\Theta(E), norm)$ of all contractive linear self-operators $E \to E$ is compactifiable endowed with the norm topology. It is not true with respect to the strong operator topology τ_s on $\Theta(E)$. However, its topological opposite semigroup $(\Theta(E)^{op}, \tau_s)$ is always compactifiable (Corollary 4.15).

A paratopological group G is compactifiable iff G is a topological group. It follows in particular, that the Sorgenfrey Line, as an additive monoid, is not compactifiable.

One of our main results states that the semigroup $U := C(I^\omega, I^\omega)$ is a universal second countable compactifiable semigroup. It is a semigroup version of Uspenskij's theorem [41] about the universality of the group Homeo (I^ω). Moreover, strengthening a result of [26], we establish that the action of U on I^ω is universal in the realm of compactifiable S-flows X (with compactifiable S) where X and S both are separable and metrizable.

The present paper influenced especially by [13, 19, 34, 41].

2. Semigroup actions: natural examples and representations

Let $\pi : P \times X \to Z$ be a map. For $p_0 \in P$ and $x_0 \in X$ define *left and right translations* by

$$\lambda_{p_0} : X \to Z, \quad x \mapsto \pi(p_0, x)$$

and

$$\rho_{x_0} : P \to Z, \quad p \mapsto \pi(p, x_0)$$

respectively. The map π is *left (right) continuous* if every left (right) translation is continuous. Note that some authors define 'left continuity' as our 'right continuity' and vice versa.

Lemma 2.1. *Let $\pi : P \times X \to Z$ be a right continuous map, P' and X' be dense subsets of P and X respectively. Assume that the map $\lambda_{p'} : X \to Z$ is continuous for every $p' \in P'$ and Z is a regular space. Then if $P' \times X' \to Z$ is continuous at (p', x') then $\pi : P \times X \to Z$ is continuous at (p', x').*

Proof. Let O be a neighborhood of $\pi(p'x')$ in Z. Since Z is regular one can choose a neighborhood U of $\pi(p'x')$ such that $cl(U) \subset O$. Now by continuity of π' at (p', x') choose the neighborhoods V of p'

in P' and W of x' in X' s.t. $\pi'(t,y) \in U$ for every $(t,y) \in V \times W$. Now $\pi(p,x) \in cl(U) \subset O$ for every $p \in cl(V)$ and $x \in cl(W)$. Indeed choose two nets $a_i \in V$ and $b_j \in W$ s.t. $\lim_i a_i = p$ in P and $\lim_j b_j = x$ in X.

Since $a_i \in P'$ the map λ_{a_i} is continuous for every i. We have $\lim_j \pi(a_i, b_j) = \pi(a_i, x) \in cl(U)$ for every i. Now by right continuity of π we obtain $\lim_i \pi(a_i, x) = \pi(p,x) \in cl(U)$. This implies the continuity of π at (p', x') because $cl(V)$ and $cl(W)$ are neighborhoods of p' and x' in P and X respectively. $\square$

A topologized semigroup S is: (a) *left* (*right*) topological; (b) *semitopological*; (c) *topological* if the multiplication function $S \times S \to S$ is left (right) continuous, separately continuous, or jointly continuous, respectively.

A *topological (left) S-flow* (or an *S-space*) is a triple (S, X, π) where $\pi : S \times X \to X$ is a jointly continuous left action of a topological semigroup S on a topological space X; we write it also as a pair (S, X), or simply, X (when π and S are understood). As usual we write sx instead of $\pi(s,x) = \breve{s}(x) = \tilde{x}(s)$. "Action" means that always $s_1(s_2x) = (s_1s_2)x$. Every $x \in X$ defines the orbit map $\tilde{x} : S \to X,\ s \mapsto sx$. Every $s \in S$ gives rise to the s-translation $\breve{s} : X \to X,\ x \mapsto sx$. The action is *monoidal* If S is a monoid and the identity e of S acts as the identity transformation of X.

If the action $S \times X \to X$ is separately continuous (that is, all orbit maps $\tilde{x}$ and all translations $\breve{s} : X \to X$ are continuous) then we say that X (or, (S, X)) is a *semitopological S-flow.*

A *right flow* (X, S) can be defined analogously. If S^{op} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left flow (S^{op}, X) (and vice versa).

Let $h : S_1 \to S_2$ be a semigroup homomorphism, S_1 act on X_1 and S_2 on X_2. A map $\alpha : X_1 \to X_2$ is said to be *h-equivariant* if $\alpha(sx) = h(s)\alpha(x)$ for every $(s,x) \in S_1 \times X_1$. Sometimes we say that the pair (h, α) is *equivariant.* For $S_1 = S_2$ with $h = id_S$, we say: *S-map.* The map $h : S_1 \to S_2$ is a *co-homomorphism* iff $S_1 \to S_2^{op}, s \mapsto h(s)$ is a homomorphism. We say that (h, α) is *proper* if α is a topological embedding.

Let μ be a uniform structure on a set X. We assume that it is separated. Then the induced topology $top(\mu)$ on X is Tychonoff. A uniformity μ on a topological space (X, τ) is said to be *compatible* if $top(\mu) = \tau$. "Compact" will mean compact and Hausdorff.

Recall some natural ways getting topological monoids and monoidal actions.

Let V be a normed space. The closed unit ball of V we denote by B_V. The weak star compact unit ball B_{V^*} in the dual space V^* will be sometimes denoted by B^*.

Examples 2.2. (1) *Let Unif(Y,Y) be the set of all uniform self-maps of a uniform space (Y, μ). Denote by $\mu_{\sup}$ the uniformity of uniform convergence on Unif(Y,Y). Then under the corresponding topology top$(\mu_{\sup})$ on Unif(Y,Y) and the usual composition we get a topological monoid. For every subsemigroup $S \subset$ Unif(Y,Y) the induced action $S \times Y \to Y$ defines a topological flow.*

(2) *For every compact space Y the semigroup $C(Y,Y)$ endowed with the compact open topology is a topological monoid. It is a particular case of (1). Note also that the subset* Homeo (Y) *in $C(Y,Y)$ of all homeomorphisms $Y \to Y$ is a topological group.*

(3) *For every metric space (M,d) the semigroup $\Theta(M,d)$ of all d-contractive maps $f : X \to X$ (that is, $d(f(x), f(y)) \leq d(x,y)$) is a topological monoid with respect to the topology of pointwise convergence. Furthermore, the evaluation map $\Theta(M,d) \times M \to M$ is a jointly continuous monoidal action.*

(4) *For every normed space $(V, ||\cdot||)$ the semigroup $\Theta(V)$ of all contractive linear operators $V \to V$ endowed with the* strong operator topology *(being a topological submonoid of $\Theta(V,d)$ where $d(x,y) := ||x-y||$) is a topological monoid. The subspace $Is(V)$ of all linear onto isometries is a topological group.*

(5) *For every normed space V and a subsemigroup $S \subset \Theta(V)^{op}$ the induced action $S \times B^* \to B^*$ on the compact space B^* is jointly continuous.*

(6) *Every normed algebra A treated as a multiplicative monoid is a topological monoid. The subset B_A is a topological submonoid. In particular, for every normed space V the monoids $L(V)$ and $B_{L(V)}$ of all bounded and, respectively,*

of all contractive linear operators $V \to V$ are topological monoids endowed with the norm topology. Observe that $B_{L(V)}$ and $\Theta(V)$ algebraically are the same monoids.

We omit the straightforward arguments.

An action $S \times X \to X$ on a metric space (X, d) is *contractive* if every s-translation $\tilde{s} : X \to X$ lies in $\Theta(X, d)$. It defines a natural homomorphism $h : S \to \Theta(X, d)$.

Remark 2.3.
1. If an action of S on (X, d) is contractive then it is easy to show that the following conditions are equivalent:
   - (i) The action is jointly continuous.
   - (ii) The action is separately continuous.
   - (iii) The restriction $S \times Y \to X$ to some dense subspace Y of X is separately continuous.
   - (iv) The natural homomorphism $h : S \to \Theta(X, d)$ is continuous.
2. If $j : V \hookrightarrow \widehat{V}$ is the completion of a normed space V then we have the following canonical equivariant inclusion of monoidal actions
$$(\Theta(V), V) \rightrightarrows (\Theta(\widehat{V}), \widehat{V}).$$

The Banach algebra of all continuous real valued bounded functions on a topological space X will be denoted by $C(X)$. Every left action $\pi : S \times X \to X$ induces the co-homomorphism $h_\pi : S \to C(X)$ and the right action $C(X) \times S \to C(X)$ where $(fs)(x) = f(sx)$. While the translations are continuous, the orbit maps $\tilde{f} : S \to C(X)$ are not necessarily norm (even weakly) continuous and require additional assumptions for their continuity (see Definition 3.9).

For every normed space V the usual adjoint map

$$adj : L(V) \to L(V^*), \quad \phi \mapsto \phi^*$$

is an injective co-homomorphism of monoids.

The following two simple lemmas are very useful. For some closely related results see [41], [1, Chapter 5] and [28, Fact 2.2].

Lemma 2.4. *For every normed space V the injective map*

$$\gamma : \Theta(V)^{op} \hookrightarrow C(B^*, B^*)$$

induced by the adjoint map adj $: L(V) \to L(V^*)$, *is a topological (even uniform) monoid embedding. In particular,*

$$\Theta(V)^{op} \times B^* \to B^*$$

is a jointly continuous monoidal action of $\Theta(V)^{op}$ *on the compact space* B^*.

Proof. The strong uniformity on $\Theta(V)$ is generated by the family of pseudometrics $\{p_v : v \in V\}$,where $p_v(s,t) = ||sv - tv||$. On the other hand the family of pseudometrics $\{q_v : v \in V\}$,where $q_v(s,t) = \sup\{|(fs)(v) - (ft)(v)| : f \in B^*\}$ generates the natural uniformity inherited from $C(B^*, B^*)$. Now observe that $p_v(s,t) = q_v(s,t)$ by the Hahn-Banach theorem. This proves that γ is a uniform (and hence, also, topological) embedding. □

Lemma 2.5. *Let* V *be a Banach space. Suppose that* $\pi : V \times S \to V$ *is a right action of a topologized semigroup* S *by linear contractive operators. The following are equivalent:*

(i) *The co-homomorphism* $h : S \to \Theta(V)$, $h(s)(v) := vs$ *is strongly continuous.*

(ii) *The induced affine action* $S \times B^* \to B^*$, $(s\psi)(v) := \psi(vs)$ *on the weak star compact ball* B^* *is jointly continuous.*

Proof. (i) $\Rightarrow$ (ii): Let $h : S \to \Theta(V)$ be a strongly continuous co-homomorphism. Then by Lemma 2.4 the composition $\gamma \circ h : S \to C(B^*, B^*)$ is a continuous homomorphism. This yields (ii) (see Example 2.2.2).

(i) $\Leftarrow$ (ii): Since the action $S \times B^* \to B^*$ is continuous and B^* is compact the homomorphism $S \to C(B^*, B^*)$, $s \mapsto \breve{s} = \gamma(h(s))$ is continuous. Again by Lemma 2.4 we get that the co-homomorphism $h : S \to \Theta(V)$ is strongly continuous. □

Definition 2.6. (1) [28, Definition 3.1] A (continuous) *representation of a flow* (S, X) *on a normed space* V is an equivariant pair

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$$

where $\alpha : X \to B^*$ is weak* continuous and $h : S \to \Theta(V)^{op}$ is a (resp.: strongly continuous) homomorphism.

(2) A *representation of* (S, X) *on a uniform space* (Y, μ) is an equivariant pair

$$(h, \alpha) : (S, X) \rightrightarrows (\mathrm{Unif}(Y, Y), Y)$$

where $h : S \to \mathrm{Unif}(Y, Y)$ is a continuous homomorphism and $\alpha : X \to (Y, top(\mu))$ is a continuous map (cf. Definition 4.2.3).

Definition 2.7. (1) Let $S \times X \to X$ be a semigroup action. A uniformity μ on X is *equicontinuous* if for every $\varepsilon \in \mu$ and any $x_0 \in X$ there exists a neighborhood O of x_0 such that $(sx, sx_0) \in \varepsilon$ for every $x \in O$ and every $s \in S$. If there exists $\delta \in \mu$ such that $(sx, sy) \in \varepsilon$ holds for every pair $x, y \in \delta$ then as usual we say that μ is *uniformly equicontinuous.* In the case of right actions the definitions are similar.

(2) A pseudometric d on a semigroup S is *right contractive* if $d(xs, ys) \leq d(x, y)$ for every $x, y, s \in S$.

(3) A uniform structure μ on a semigroup S is *right invariant* (see also [13, p. 98] and Lemma 2.8) if for every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that $\delta \subset \varepsilon$ and $(sx, tx) \in \delta$ for every $(s, t) \in \delta$, $x \in S$.

Lemma 2.8. *Let μ be a compatible uniform structure on a topological semigroup S. The following conditions are equivalent:*

(1) *μ can be generated by a family of right contractive pseudometrics.*

(2) *μ is right invariant on S.*

(3) *The right action of S on itself is μ-uniformly equicontinuous (that is, for every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that $(sx, tx) \in \varepsilon$ for every $(s, t) \in \delta$, $x \in S$).*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (1): Assume that the right action of S on itself is μ-uniformly equicontinuous. Choose a family $\{d_i\}_{i \in I}$ of pseudometrics on S which generates the uniformity μ. For every $i \in I$ define

$$d_i^*(x, y) := \max\{\sup_{s \in S} d_i(xs, ys), d_i(x, y)\}$$

Then the new system $\{d_i^*\}_{i \in I}$ consists by right contractive pseudometrics and still generates the same uniformity μ. □

Example 2.9.

1. *For every topological group G the right uniformity $\mathcal{R}(G)$ of G is the* unique *right invariant compatible uniformity on G,* [36, Lemma 2.2.1].
2. *Let (X,μ) be a uniform space and $\mu_{\sup}$ be the corresponding natural uniformity on Unif(X,X). Assume that S is a subsemigroup of Unif(X,X). Then the subspace uniformity $\mu_{\sup}|_S$ on S is right invariant.*

The following proposition is an equivariant version of the well known Arens-Eells embedding construction [3].

Proposition 2.10. *Let $S \times X \to X$ be a continuous contractive action of a semigroup S on a bounded metric space (X,d). Then there exists a normed (equivalently: Banach) space E and an equivariant pair*

$$(h,\alpha) : (S,X) \rightrightarrows (\Theta(E),E)$$

such that $h : S \to \Theta(E)$ is a strongly continuous homomorphism and $\alpha : X \to E$ is an isometric embedding.

Proof. By Remark 2.3 it suffices to give a proof for normed E. Since the metric is bounded we can suppose that X contains a fixed point z (adjoining if necessary a fixed point z and defining $d(x,z) = diam(X,d) < \infty$ for every $x \in X$). We can use the *Arens-Eells isometric embedding*

$$i : X \to A(X),\ x \mapsto x - z$$

(see [3]) of a pointed metric space (X,z,d) into a normed space $(A(X), ||\cdot||)$. The elements of $A(X)$ are the formal sums of the form $\sum_{i=1}^n c_i(x_i - y_i)$, where $x_i, y_i \in X$ and $c_i \in \mathbb{R}$. Define the natural left action

$$S \times A(X) \to A(X), \quad s\sum_{i=1}^n c_i(x_i - y_i) := \sum_{i=1}^n c_i(sx_i - sy_i).$$

The desired norm on $A(X)$ is defined by setting

$$||u|| := \inf \sum_{i=1}^n |c_i| d(x_i, y_i)$$

where we compute the infimum with respect to the all presentations of $u \in A(X)$ as the sums $u = \sum_{i=1}^n c_i(x_i - y_i)$ with $x_i, y_i \in X$. This explicit description shows that $||su|| \leq ||u||$ for every $s \in S$

because $d(sx_i, sy_i) \leq d(x_i, y_i)$. Therefore the action $S \times X \to X$ can be extended to the canonically defined action $S \times A(X) \to A(X)$ by contractive linear operators. Moreover it is clear that every orbit mapping $S \to A(X)$, $s \mapsto su$ is continuous for every $u \in A(X)$. Thus we get a continuous homomorphism $h : S \to \Theta(A(X))$. Moreover, since $i : X \to A(X)$ is an isometric embedding it follows that $E := A(X)$ is the desired normed space. $\square$

Remark 2.11. (1) This result in fact is known; (at least for group actions) it can be derived from results of Pestov [33]. In the construction Arens-Eells space can be replaced by *Free Banach spaces*, as in above mentioned work of Pestov.

(2) Proposition 2.10 provides only a sufficient condition for linearizability of contractive actions because we assume that the metric space (X, d) is bounded (which certainly is not a necessary condition). The same restriction, as to our knowledge, appears in each previous form of equivariant Arens-Eells embedding (see e.g. [33]). An elegant necessary and sufficient condition has been recently found by Schröder [38]. Specifically, he shows that the contractive (*nonexpansive*, in other terminology) S-action on (X, d) is linearizable if and only if all orbits Sx $(x \in X)$ are bounded.

3. S-Compactifications and functions

Here we discuss how the classical Gelfand-Naimark 1-1 correspondence between Banach subalgebras of $C(X)$ and the compactifications of X can be extended to the category of S-flows. This theory is well known for topological group actions (see, for example, J. de Vries [44, 45]). One can easily extend it to the case of semigroup actions (Ball and Hagler [6]). Separately continuous compactifications are closely related to the theory of right topological compactifications and LMC-functions (see Definition 3.13).

First we briefly recall some classical facts about compactifications. A compactification of X is a pair (Y, ν) where Y is a compact (Hausdorff) space and ν is a continuous map with a dense range. If ν is a topological embedding then the compactification is said to be *proper*.

Due to the Gelfand-Naimark theory there is a 1-1 correspondence (up to the equivalence classes of compactifications) between Banach *unital* (that is, the containing the constants) subalgebras $\mathcal{A} \subset C(X)$ and the compactifications $\nu : X \to Y$ of X. Any Banach unital S-subalgebra $\mathcal{A}$ of $C(X)$, induces the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$, where $X^{\mathcal{A}}$ is the Gelfand space (or, the *spectrum* – the set $MM(\mathcal{A})$ of all multiplicative means [5]) of the algebra $\mathcal{A}$ (see also Definition 2.6.1). The map $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ is defined by the *Gelfand transform*, the evaluation at x multiplicative functional, $\alpha_{\mathcal{A}}(x)(f) := f(x)$. Conversely, every compactification $\nu : X \to \ Y$ is equivalent to the *canonical $\mathcal{A}_\nu$-compactification* $\alpha_{\mathcal{A}_\nu} : X \to X^{\mathcal{A}_\nu}$, where the algebra $\mathcal{A}_\nu$ is defined as the image $j_\nu(C(Y))$ of the embedding $j_\nu : C(Y) \hookrightarrow C(X), \ \phi \mapsto \phi \circ \nu$.

Remark 3.1. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications then ν_2 *dominates* ν_1, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$. Moreover, if in addition, ν_1 and ν_2 are S-equivariant maps and all s-translations on X, Y_1 and Y_2 are continuous then q is also S-equivariant.

Definition 3.2. Let X be an S-flow.

(1) A *semitopological S-compactification* of X is a continuous S-map $\alpha : X \to Y$ with a dense range into a compact semitopological S-flow Y.

(2) Let $M \subset S$. We say that a semitopological S-compactification $\alpha : X \to Y$ is *M-topological* if the action $S \times Y \to Y$ is continuous at every $(m, y) \in M \times Y$. If $M = S$ then we say *topological S-compactification.*

(3) A flow (S, X) is said to be *compactifiable (semi-compactifiable)* if there exists a *proper* topological (resp.: semitopological) S-compactification $X \hookrightarrow Y$. A topological semigroup S is *compactifiable (semi-compactifiable)* if the flow (S, S), left regular action, is compactifiable (resp.: semi-compactifiable).

Definition 3.3. Let S be a topological semigroup.

(1) [5] A *right topological semigroup compactification* of S is a pair (T, γ) such that T is a compact right topological semigroup, and γ is a continuous homomorphism from S into T, where $\gamma(S)$ is dense in T and the translation $\lambda_s : T \to T, \ x \mapsto \gamma(s)x$ is continuous for every $s \in S$. It follows

that the *associated action* (the *associated flow* in [22])

$$\pi_\gamma : S \times T \to T, \quad (s,x) \mapsto \gamma(s)x = \lambda_s(x)$$

is separately continuous. Moreover, a map $\gamma : S \to T$ is a semigroup compactification iff γ is a semitopological S-compactification of the S-flow S such that at the same time γ is a homomorphism of semigroups.

(2) A *dynamical right topological semigroup compactification* of S in the sense of Ruppert [37] (see also *monoidal compactification* of Lawson [22]) is a right topological semigroup compactification (T,γ) such that γ is a topological S-compactification. That is, the action $\pi_\gamma : S \times T \to T$ is jointly continuous.

Evidently every semi-compactifiable flow, as a space, must be Tychonoff.

Definition 3.4. (1) The *enveloping (or Ellis) semigroup* $E(X) = E(S,X)$ of the semitopological compact flow (S,X) is defined as the closure in X^X (with its compact, pointwise convergence topology) of the set $\breve{S} = \{\breve{s} : X \to X\}_{s \in S}$ considered as a subset of X^X. With the operation of composition of maps this is a right topological semigroup.

(2) The associated homomorphism $j : S \to E(X),\ s \mapsto \breve{s}$ is a right topological semigroup compactification of S. More generally, for every semitopological S-flow X and a semitopological S-compactification $\alpha : X \to Y$ we have the induced right topological semigroup compactification $j_\alpha : S \to E(Y)$ such that the pair

$$(j_\alpha, \alpha) : (S,X) \rightrightarrows (E(Y),Y)$$

is equivariant. The associated action $\pi_j : S \times E(Y) \to E(Y)$ is separately continuous. Furthermore, if Y is a topological S-flow then π_j is jointly continuous.

Proposition 3.5. *Let S be a topological semigroup.*

(1) *S is compactifiable if and only if S has a proper dynamical compactification.*

(2) *S is semicompactifiable if and only if it admits a proper right topological semigroup compactification.*

Proof. (2): Let $\gamma : S \to T$ be a proper right topological semigroup compactification of S. The associated action $\pi_\gamma : S \times T \to T$ is separately continuous. Hence γ is a semitopological (proper) compactification of S.

Conversely, let $\alpha : S \to Y$ be a semitopological S-compactification of S (acting on itself by left translations). We can pass, as in Definition 3.4, to the right topological semigroup compactification $j_\alpha : S \to E(Y)$. We can suppose without restriction of generality that S is a topological monoid (adjoining to S an isolated identity e_S if necessary as in Remark 3.11.1) and $j_\alpha(e_S) = id_Y$. Then we have the continuous map $\hat{e} : E(Y) \to Y,\ p \mapsto p(\alpha(e))$ such that $\hat{e} \circ j_\alpha = \alpha$. It follows that if α is a proper compactification then j_α is also proper.

(1): This is similar. Observe that π_j is jointly continuous if α is a topological S-compactification. □

Remark 3.6.

1. For many natural monoids a separately continuous monoidal action $\pi : S \times Y \to Y$ on arbitrary compact space Y is continuous at every $(e, y) \in \{e\} \times Y$. By a result of Lawson [20, Corollary 5] this happens for instance if S is a *Namioka space* (see also [21, 16] and [5, Theorem 1.4.2]). Every Čech-complete (e.g., locally compact or complete metrizable) space is a Namioka space. It follows that if the monoid S is a Namioka space then every semitopological S-compactification $\alpha : X \to Y$ is $\{e\}$-topological (or, equivalently, $H(e)$-topological, where $H(e)$ denotes the group of all invertible elements in S).
2. Recall also that by a result of Dorroh [11, Theorem 2.2] every separately continuous monoid action of the one-parameter additive monoid $([0, \infty), +)$ on a locally compact space X is jointly continuous.

The following fact is well known.

Lemma 3.7. *Let G be a Čech-complete (e.g., locally compact or complete metrizable) topological group. Then $\gamma : G \to T$ is a right topological semigroup compactification of G if and only if γ is a dynamical compactification of G.*

Proof. In Definition 3.3, (2) implies (1). The converse is true for every topological group S the underlying space of which is Čech-complete (by Remark 3.6.1). ☐

Lemma 3.8. *Every continuous representation* (h, α) *of an* S*-space* X *on a normed space* V *induces the topological* S*-compactification*

$$\alpha : X \to Y := cl(\alpha(X)) \subset B^*$$

where $cl(\alpha(X))$ *is the weak star closure of* $\alpha(X)$ *in* B^**.*

Proof. Indeed, by Lemma 2.4 the action $S \times B^* \to B^*$ is continuous. In particular, the restricted action $S \times Y \to Y$ is continuous, too. ☐

The following definition is well known (under different names and sometimes replacing "right" by "left") for topological group actions [45, 46] and for semigroups [18, 5, 12, 37, 6].

Definition 3.9. Let $\pi : S \times X \to X$ be a given action. A bounded function $f \in C(X)$ is said to be *right uniformly continuous* if the orbit map $\tilde{f} : S \to C(X)$ is continuous. Or, equivalently, for every $s_0 \in S$ and $\varepsilon > 0$ there exists a neighborhood U of s_0 such that $|f(sx) - f(s_0x)| < \varepsilon$ for every $(s, x) \in U \times X$.

For every S-flow X denote by $\mathrm{RUC}_S(X)$, or, by $\mathrm{RUC}(X)$ (where S is understood) the set of all functions on X that are right uniformly continuous. The set $\mathrm{RUC}_S(X)$ is an S-invariant Banach unital subalgebra of $C(X)$. If X is a compact S-space then the standard compactness arguments show that $C(X) = \mathrm{RUC}_S(X)$. If $X = S$ with the left regular action of S on itself by left translations, then we simply write $\mathrm{RUC}(S)$. If $S = G$ is a topological group, then $\mathrm{RUC}(G)$ is the set of all usual right uniformly continuous functions on G.

Let $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ be the canonical $\mathcal{A}$-compactification of X. If the Banach unital subalgebra $\mathcal{A} \subset C(X)$ is S*-invariant* (that is, the function $(fs)(x) := f(sx)$ lies in $\mathcal{A}$ for every $s \in S$) then the spectrum $X^{\mathcal{A}} \subset \mathcal{A}^*$ admits the natural adjoint action $S \times X^{\mathcal{A}} \to X^{\mathcal{A}}$ such that all translations $\breve{s} : X^{\mathcal{A}} \to X^{\mathcal{A}}$ are continuous and $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ is S-equivariant. We get a representation

$$(h, \alpha_{\mathcal{A}}) : (S, X) \rightrightarrows (\Theta(\mathcal{A})^{op}, B^*)$$

on the Banach space $\mathcal{A}$, where $h(s)(f) := fs$ and $\alpha_{\mathcal{A}}(x)(f) := f(x)$. We call it the *canonical $\mathcal{A}$-representation.* Note that this representation is not necessarily *continuous* because h need not continuous.

Proposition 3.10. *Let X be an S-flow. Assume that $\mathcal{A}$ is an S-invariant unital Banach subalgebra of $C(X)$.*

1. *$\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ is a topological (i.e. jointly continuous) compactification of the S-flow X if and only if $\mathcal{A} \subset RUC_S(X)$.*
2. *The compactification $\alpha_{RUC} : X \to X^{RUC}$ (for the algebra $\mathcal{A} := RUC_S(X)$) is the maximal topological compactification of the S-flow X.*

Proof. (1): If $\mathcal{A}$ is a subalgebra of $RUC_S(X)$ then by Definition 3.9 the orbit map $\tilde{f} : S \to \mathcal{A}$ is norm continuous for every $f \in \mathcal{A}$. Therefore the canonical representation

$$(h, \alpha_{\mathcal{A}}) : (S, X) \rightrightarrows (\Theta(\mathcal{A}), B^*)$$

is continuous (because h is continuous). By Lemma 3.8 we get that the induced compactification $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ is a topological compactification of the S-flow X.

Conversely, if $\alpha_{\mathcal{A}} : X \to Y := X^{\mathcal{A}}$ is a topological compactification then $C(Y) = \mathrm{RUC}_S(Y)$. This easily implies that $\mathcal{A} \subset \mathrm{RUC}_S(X)$.

(2): Follows from (1) and Remark 3.1. □

The maximal jointly continuous compactification $\alpha_{\mathrm{RUC}} : S \to S^{\mathrm{RUC}}$ defined for the flow (S, S) is the semigroup version of the so-called "greatest ambit". Clearly, S is compactifiable iff α_{RUC} is a proper compactification. Every Hausdorff topological group $G := S$ is compactifiable because the algebra $\mathrm{RUC}(G)$ separates points and closed subsets. It follows that the corresponding canonical representation (call it the *Teleman's representation*)

$$(h, \alpha_{\mathrm{RUC}}) : (G, G) \rightrightarrows (\Theta(V)^{op}, B^*)$$

on $V := \mathrm{RUC}(G)$ is proper and h induces in fact a topological group embedding of G into $Is(V)$. The corresponding proper compactification $\alpha_{\mathrm{RUC}} : G \hookrightarrow G^{\mathrm{RUC}}$ is the *greatest ambit of* G (see, for example, [39, 8, 47, 34, 43]). The induced representation (in the sense of Definition 2.6.2) $(h, \alpha) : (G, G) \rightrightarrows (C(B^*, B^*), B^*)$ on the compact space B^* is also proper and h induces an embedding of topological groups $G \hookrightarrow \mathrm{Homeo}\,(B^*)$.

Note that the maximal S-compactification $\alpha_{\mathrm{RUC}} : X \to X^{\mathrm{RUC}}$ may not be an embedding even for Polish topological group $S := G$ and a Polish phase space X (see [24]); hence X is not G-compactifiable. If S is discrete then $\beta_S X = X^{\mathrm{RUC}}$ coincides with the usual maximal compactification $\beta X = X^{C(X)}$.

Remark 3.11.
1. Every topological semigroup S canonically can be embedded into a topological monoid $S_e := S \sqcup \{e\}$ as a clopen subsemigroup by adjoining to S an isolated identity e. Furthermore, any action $\pi : S \times X \to X$ naturally extended to the monoidal action $\pi_e : S_e \times X \to X$. It is easy to check that $\mathrm{RUC}_{S_e}(X) = \mathrm{RUC}_S(X)$. Therefore, S-space X is compactifiable iff S_e-space X is compactifiable. Similarly, $f \in \mathrm{RUC}(S_e)$ iff $f|_S \in \mathrm{RUC}(S)$. It follows that S_e is compactifiable iff S is compactifiable.
2. Let $Z := X \sqcup Y$ be a disjoint sum of S-spaces. Then $f \in \mathrm{RUC}(Z)$ iff $f|_X \in RUC(X)$ and $f|_Y \in RUC(Y)$. It follows that Z is S-compactifiable iff X and Y are S-compactifiable.

Now we turn to the case of semitopological S-compactifications.

Let $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$ be a representation of a flow (S, X) on a normed space V. Every pair of vectors $(v, \psi) \in V \times V^*$ defines the function

$$m_{v,\psi} : S \to \mathbb{R}, \quad s \mapsto \psi(vs)$$

which is said to be a *matrix coefficient* of the given V-representation.

Lemma 3.12. *Let V be a normed space, X is an S-space and the pair*

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*)$$

is a representation (h is not necessarily continuous). The following conditions are equivalent:

1. *The induced action $S \times Y \to Y$, where $Y := cl(\alpha(X)) \subset B^*$, is separately continuous (equivalently, $\alpha : X \to Y$ is a semitopological S-compactification).*
2. *The matrix coefficient $m_{v,\psi} : S \to \mathbb{R}$ is continuous for every $v \in V$ and $\psi \in Y$.*

Proof. Observe that the orbit map $\tilde{\psi} : S \to Y$ (with $\psi \in Y$) is weak star continuous if and only if the matrix coefficient $m_{v,\psi}$ is continuous for every $v \in V$. $\square$

This lemma naturally leads to the following definition which is well known at least for the particular case of the left action of S on itself. It can be treated as a natural flow generalization of the concept of *LMC-functions* introduced for semigroups by Mitchell (see, for example, [32, 17, 4, 5]). However, in general context of actions, this definition seems to be new even for group actions.

Definition 3.13. (LMC-functions – generalized version) Let X be an S-space. We say that a function $f \in C(X)$ is *left multiplicatively continuous* (notation: $f \in \mathrm{LMC}_S(X)$, or simpler $f \in \mathrm{LMC}(X)$) if for every $\psi \in Y := \beta X$ the matrix coefficient $m_{f,\psi} : S \to \mathbb{R}$ of the canonical $C(X)$-representation of (S, X) is continuous.

We omit a straightforward verification of the following lemma.

Lemma 3.14. *Let X be an S-space. The set $LMC_S(X)$ is an S-invariant Banach subalgebra of $C(X)$ and contains $RUC_S(X)$.*

Proposition 3.15. *Let X be an S-space. Assume that $\mathcal{A}$ is an S-invariant unital Banach subalgebra of $C(X)$ and $f \in \mathcal{A}$.*

1. *$f \in LMC_S(X)$ iff for every $\psi \in X^{\mathcal{A}} \subset B^*$ the matrix coefficient $m_{f,\psi} : S \to \mathbb{R}$ of the canonical $\mathcal{A}$-representation is continuous.*
2. *$\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$ is a semitopological compactification of the S-flow X if and only if $\mathcal{A} \subset LMC_S(X)$. That is, S-invariant unital closed subalgebras of $LMC_S(X)$ correspond to semitopological S-compactifications of X.*
3. *The compactification $\alpha_{LMC} : X \to X^{LMC}$ (defined by the algebra $\mathcal{A} := LMC_S(X)$) is the maximal semitopological compactification of the S-flow X.*
4. *(S, X) is semicompactifiable iff $LMC_S(X)$ separates points and closed subsets in X.*
5. (compare [4, Ch. III, Theorem 4.5]) *A topological semigroup S is semicompactifiable iff $LMC(S)$ separates points and closed subsets in S iff it admits a proper right topological semigroup compactification.*

Proof. (1): The canonical $C(X)$-representation of (S, X) induces the usual maximal compactification $\beta : X \to \beta X$. Denote by

$\alpha_{\mathcal{A}} : X \to Y := cl(\alpha_{\mathcal{A}}(X))$ the induced compactification of the $\mathcal{A}$-representation $(h, \alpha_f) : (S, X) \rightrightarrows (\Theta(\mathcal{A}), B^*)$. Then there exists a continuous S-equivariant onto map $q : \beta X \to Y$ such that $q \circ \beta = \alpha_{\mathcal{A}}$. It follows that the matrix coefficient $m_{f,p}$ coincides with $m_{f,q(p)}$ for every $p \in \beta X$.

(2): Combine Lemma 3.12 and the first assertion.

(3): Easily follows from (2).

(4): Follows from assertion (3).

(5): Use (4) and Proposition 3.5.2. □

Let S be a topological semigroup. Then by results of [4, Chapter III] (or by the results of the present section) we get in fact that the universal LMC-compactification $u_{\mathrm{LMC}} : S \to S^{\mathrm{LMC}}$ (induced by the whole algebra $\mathrm{LMC}(S)$) of the S-flow S is the universal right topological semigroup compactification of S. Therefore our definitions and the traditional *semigroup approach* to LMC-compactifications agree. Recall that if G is a topological group that is a Namioka space then $\mathrm{LMC}(G) = \mathrm{RUC}(G)$ (see [4, Ch. III, Theorem 14.6], Remark 3.6.1 and Lemma 3.7).

4. S-COMPACTIFIABILITY: NECESSARY AND SUFFICIENT CONDITIONS

Let (X, μ) be a uniform space. Denote by j_X or j the completion $(X, \mu) \to (\widehat{X}, \widehat{\mu})$. As usual, (X, μ) is *precompact* (or, *totally bounded*) means that the completion $(\widehat{X}, \widehat{\mu})$ is compact. Every uniform structure μ contains the *precompact replica* of μ. It is the finest precompact uniformity $\mu_{fin} \subset \mu$. Denote by

$$i_{fin} : (X, \mu) \to (X, \mu_{fin}), \ x \mapsto x$$

the corresponding uniform map. This map is a homeomorphism because $top(\mu) = top(\mu_{fin})$. The uniformity μ_{fin} is separated and hence the corresponding completion $(X, \mu_{fin}) \to (\widehat{X}, \widehat{\mu_{fin}}) = (uX, \mu_u)$ (or simply uX) is a proper compactification of the topological space $(X, top(\mu))$. The compactification

$$u_X = u_{(X,\mu)} : X \to uX$$

is the well known *Samuel compactification* (or, *universal uniform compactification*) of (X, μ). The corresponding algebra $\mathcal{A}_\mu \subset C(X)$ consists with all μ-uniformly continuous real valued bounded functions. Here we collect some known auxiliary results.

Lemma 4.1. (1) *For every uniform map* $f : (X, \mu) \to (Y, \xi)$ *the canonically associated maps*

$$f : (X, \mu_{fin}) \to (Y, \xi_{fin}),$$
$$\widehat{f} : (\widehat{X}, \widehat{\mu}) \to (\widehat{Y}, \widehat{\xi})$$
$$f^u : uX \to uY$$

are uniform.

(2) $u_X : X \to uX$ *and* $u_{\widehat{X}} \circ j : X \to u\widehat{X}$ *(for* $u_{\widehat{X}} : \widehat{X} \to u\widehat{X}$*) are equivalent compactifications. More precisely, there exists a unique homeomorphism* $j^u : uX \to u\widehat{X}$ *such that* $j^u \circ u_X = u_{\widehat{X}} \circ j$*. In particular, the natural uniform map*

$$\phi_X := (j^u)^{-1} \circ u_{\widehat{X}} : \widehat{X} \to uX$$

is a topological embedding.

(3)
$$\mathit{Unif}(X, X) \to \mathit{Unif}(\widehat{X}, \widehat{X}), \quad f \mapsto \widehat{f}$$

is a uniform embedding,

$$\mathit{Unif}(X, X) \to \mathit{Unif}(X_{fin}, X_{fin}), \quad f \mapsto f$$

and

$$\mathit{Unif}(X, X) \to \mathit{Unif}(uX, uX), \quad f \mapsto f^u$$

are uniform injective maps.

Proof. (1) and (3) are straightforward. For (2) use the natural map

$$\mathrm{Unif}(X, \mathbb{R}) \to \mathrm{Unif}(\widehat{X}, \mathbb{R}), \ f \mapsto \widehat{f}$$

and observe that the same algebra $\mathrm{Unif}(X, \mathbb{R})$ correspond to the compactifications $u_X : X \to uX$ and $u_{\widehat{X}} \circ j : X \to u\widehat{X}$ and hence these compactifications are equivalent. □

Another direct proof of the fact that $\phi_X : \widehat{X} \to uX$ is a uniform embedding can be found in [13].

Definition 4.2. Let μ be a uniformity on X and $\pi : S \times X \to X$ be a semigroup action. We call this action:

(1) *μ-saturated* if every s-translation $\breve{s} : X \to X$ is μ-uniform (thus the corresponding homomorphism

$$h_\pi : S \to \mathrm{Unif}(X,X),\ \ s \mapsto \breve{s}$$

is well defined).

(2) *μ-bounded at s_0* if for every $\varepsilon \in \mu$ there exists a neighborhood $U(s_0)$ such that $(s_0x, sx) \in \varepsilon$ for each $x \in X$ and $s \in U$. If this condition holds for every $s_0 \in S$ then we simply say: *μ-bounded.*

(3) (see [23]) *μ-equiuniform* if μ is saturated and bounded. It is equivalent to say that the corresponding homomorphism $h_\pi : S \to \mathrm{Unif}(X,X)$ is continuous.

(4) *(ξ,μ)-equiuniform* if ξ is a compatible uniformity on S such that the left actions $\nu : S \times S \to S$ and $\pi : S \times X \to X$ are saturated (with respect to ξ and μ respectively) and the associated homomorphisms $h_\pi : S \to \mathrm{Unif}(X,X)$, $\ \ h_\nu : S \to \mathrm{Unif}(S,S)$ are uniform maps.

Sometimes we say also that the uniformity μ is saturated, bounded and equiuniform, respectively.

For group actions bounded uniformities appear in [45] and in [9] (see also "uniform action" in the sense of [1]). We collect here some simple examples.

Examples 4.3. (1) *Every μ-equiuniform action is continuous.*

(2) *Every compact S-space X is equiuniform (with respect to the unique compatible uniformity on X).*

(3) *For every uniform space (X,μ) and every subsemigroup S of Unif(X,X) endowed with the subspace uniformity ξ inherited from Unif(X,X) the natural action $S \times X \to X$ (see Example 2.2.1) is (ξ,μ)-equiuniform.*

(4) *For every (ξ,μ)-equiuniform action $S \times X \to X$ the left action $S \times S \to S$ is (ξ,ξ)-equiuniform.*

(5) *Let S be a semigroup with a right invariant uniformity ξ on S such that all left translations are uniformly continuous. Then the left action $S \times S \to S$ is (ξ,ξ)-equiuniform.*

We need some notation. Let $S \times X \to X$ be a semigroup action. For every element $s \in S$ and a subset $A \subset X$ define $s^{-1}A := \{x \in X : sx \in A\}$. Let μ be a uniformity on X and $\varepsilon \in \mu$. Then ε is a subset of $X \times X$. For every $s \in S \cup \{id_X\}$ we can define similarly the following set

$$s^{-1}\varepsilon := \{(x,y) \in X \times X : (sx, sy) \in \varepsilon\}$$

where $id_X^{-1}\varepsilon = \varepsilon$.

Lemma 4.4. *Let μ be a uniformity on X such that the semigroup action of a topological semigroup S on $(X, top(\mu))$ is continuous.*

1. *The family $\{s^{-1}\varepsilon : s \in S \cup \{id_X\}, \varepsilon \in \mu\}$ is a subbase of a saturated uniformity $\mu^S \supset \mu$ generating the same topology (that is, $top(\mu) = top(\mu^S)$).*
2. *If the action is μ-bounded then it is also μ^S-bounded (hence, μ^S-equiuniform).*
3. *If the action is μ-bounded (μ-saturated, μ-equiuniform, or (ξ, μ)-equiuniform) then the same action is also μ_{fin}-bounded (μ_{fin}-saturated, μ_{fin}-equiuniform, or (ξ, μ_{fin})-equiuniform respectively).*

Proof. The proofs of (1) and (2) are trivial.

(3): The boundedness of μ_{fin} is clear because $\mu_{fin} \subset \mu$. In order to show that the action is μ_{fin}-saturated we have to check that $\tilde{s} : (X, \mu_{fin}) \to (X, \mu_{fin})$ is uniform for every $s \in S$. Let $\varepsilon \in \mu_{fin}$. Since $s(s^{-1}\varepsilon) \subset \varepsilon$ we have only to show that $s^{-1}\varepsilon \in \mu_{fin}$.

Pick a symmetric entourage $\delta \in \mu_{fin}$ such that $\delta \circ \delta \subset \varepsilon$. Since $\delta \in \mu_{fin}$ there exists a finite subset $\{y_1, y_2, \cdots, y_n\}$ in X which is δ-dense in X (that is, $\cup_{i=1}^n \delta(y_i) = X$, where $\delta(y) := \{x \in X : (x,y) \in \delta\}$). Passing to a subfamily if necessary we can suppose in addition that $\delta(y_i) \cap sX \neq \emptyset$ for every $i \in \{1, 2, \cdots, n\}$. Choose $z_i \in X$ such that $sz_i \in \delta(y_i)$ for each i. Then $\{z_1, z_2, \cdots, z_n\}$ is a finite $s^{-1}\varepsilon$-dense subset in X. Indeed, for every $x_0 \in X$ there exists i_0 such that $(y_{i_0}, sx_0) \in \delta$. Since $(sz_{i_0}, y_{i_0}) \in \delta$ we get $(sz_{i_0}, sx_0) \in \delta \circ \delta \subset \varepsilon$. Thus, $(z_{i_0}, x_0) \in s^{-1}\varepsilon$.

Checking that the action is (ξ, μ_{fin})-equiuniform (provided that it is (ξ, μ)- equiuniform) observe that the map

$$\mathrm{Unif}(X, X) \to \mathrm{Unif}(X_{fin}, X_{fin}), \ f \mapsto f$$

is uniform. This implies that the homomorphism

$$(S, \xi) \to \mathrm{Unif}(X_{fin}, X_{fin})$$

is also uniform. □

Lemma 4.5. (1) *Let μ be a saturated uniformity on X with respect to the action $S \times X \to X$. Let Y be an S-invariant dense subset of X such that the induced action $S \times Y \to Y$ is $\mu|_Y$-bounded. Then the given action $S \times X \to X$ is μ-equiuniform and continuous.*

(2) *Let $\pi : S \times X \to X$ be a continuous μ-equiuniform action. Then the induced action on the completion $\widehat{\pi} : S \times \widehat{X} \to \widehat{X}$ is well-defined, $\widehat{\mu}$-equiuniform and continuous.*

Proof. (1) Let $s_0 \in S$ and $\varepsilon \in \mu$. There exists an element $\varepsilon_1 \in \mu$ such that $\varepsilon_1 \subset \varepsilon$ and ε_1 is a closed subset of $X \times X$. Choose a neighborhood $U(s_0)$ such that $(s_0 y, sy) \in \varepsilon_1$ for every $s \in U$ and $y \in Y$. Then $(s_0 x, sx) \in \varepsilon$ for every $s \in U$ and $x \in X$. Thus the given (saturated) action is μ-bounded. The action is continuous by Example 4.3.1.

(2) Easily follows from (1). □

Lemma 4.6. *Let X and P be Hausdorff spaces. Assume that:*

(i) *S is a dense subset of P.*
(ii) *S is a semigroup w.r.t. the operation $w_S : S \times S \to S$.*
(iii) *$\vartheta : S \times P \to P$ is a semigroup action with continuous translations.*
(iv) *$m : P \times P \to P$ is a right continuous mapping which extends w_S and ϑ.*
(v) *$\pi_S : S \times X \to X$ is a semigroup action with continuous translations.*
(vi) *$\pi_P : P \times X \to X$ is a right continuous mapping which extends π_S.*

Then we have:

(1) *(P, m) is a right topological semigroup.*
(2) *$\pi_P : P \times X \to X$ is a semigroup action.*
(3) *If X is regular and π_S is continuous at (s_0, x_0) with some $(s_0, x_0) \in S \times X$ then π_P remains continuous at (s_0, x_0).*

Proof. First of all we check the associativity

$$(p_1 p_2)x = p_1(p_2 x)$$

for every given triple $(p_1, p_2, x) \in P \times P \times X$, where $(p_1 p_2)x := \pi_P(m(p_1, p_2), x)$ and $p_1(p_2 x) := \pi_P(p_1, \pi_P(p_2, x))$.

Choose nets a_i and b_j in P such that $a_i, b_j \in S$ and $\lim_i a_i = p_1, \lim_j b_j = p_2$. Then by our assumptions we have

$$(p_1p_2)x = \lim_i (a_i p_2)x = (\lim_i \lim_j (a_i b_j))x =$$

$$= \lim_i \lim_j (a_i(b_j x)) = \lim_i a_i(\lim_j (b_j x)) = \lim_i (a_i(p_2 x)) = p_1(p_2 x).$$

Apply this formula to the particular case of the map $m : P \times P \to P$ (where $X := P$). Then we get that $(p_1p_2)p_3) = p_1(p_2p_3)$ for all triples $(p_1, p_2, p_3) \in P^3$. This proves (1). Moreover, now the general formula means that π_P is a semigroup action.

For (3) use Lemma 2.1. □

Proposition 4.7. *Let ξ be a compatible uniformity on a topological semigroup S such that the left action $\nu : S \times S \to S$ is (ξ, ξ)-equiuniform. Identify S with its image under the completion map $j : S \to \widehat{S}$. Then there exists a map $m : \widehat{S} \times \widehat{S} \to \widehat{S}$ such that $(\widehat{S}, m)$ is a topological semigroup, S is a subsemigroup of $\widehat{S}$ and the left action m is $(\widehat{\xi}, \widehat{\xi})$-equiuniform.*

Proof. The natural homomorphism

$$h_\nu : (S, \xi) \to \mathrm{Unif}(S, S), \ s \mapsto \lambda_s$$

is uniform. Consider the uniform embedding

$$\mathrm{Unif}(S, S) \to \mathrm{Unif}(\widehat{S}, \widehat{S}), \quad f \mapsto \widehat{f}.$$

Denote by h the corresponding uniform composition $h : S \to \mathrm{Unif}(\widehat{S}, \widehat{S})$. Since the uniform space $\mathrm{Unif}(\widehat{S}, \widehat{S})$ is complete there exists a unique uniform extension $\widehat{h} : \widehat{S} \to \mathrm{Unif}(\widehat{S}, \widehat{S})$ of h. Then the evaluation map $m : \widehat{S} \times \widehat{S} \to \widehat{S}$, $m(t,p) = \widehat{h}(t)(p)$ is jointly continuous and extends the original multiplication ν on S. On the other hand by Lemma 4.5.2 we get that there exists a uniquely determined continuous semigroup action $\vartheta : S \times \widehat{S} \to \widehat{S}$ which also extends ν. It follows that m extends ϑ. By Lemma 4.6 (for the setting $P := \widehat{S}, X := \widehat{S}$) we obtain that $(\widehat{S}, m)$ is a semigroup and S is its subsemigroup. Furthermore, $\widehat{S}$ is a topological semigroup because m is continuous. Since h_ν is a uniform homomorphism and $\widehat{h}|_S = h_\nu$ it follows that the uniform map $\widehat{h}$ also is a *homomorphism* of semigroups. This means that the left action m is $(\widehat{\xi}, \widehat{\xi})$-equiuniform. □

Proposition 4.8. *Let $\pi : S \times X \to X$ be a (ξ, μ)-equiuniform action. Then there exist (uniquely determined) continuous semigroup actions:*

(i) $\widehat{\pi} : \widehat{S} \times \widehat{X} \to \widehat{X}$ *which is $(\widehat{\xi}, \widehat{\mu})$-equiuniform and naturally extends π;*
(ii) $\pi : S \times X_{fin} \to X_{fin}$ *which is (ξ, μ_{fin})-equiuniform;*
(iii) $\widehat{\pi}_u : \widehat{S} \times uX \to uX$ *which is $(\widehat{\xi}, \mu_u)$-equiuniform and naturally extends $\widehat{\pi}$.*

Proof. (i) By Proposition 4.7 we know that the left action is $(\widehat{\xi}, \widehat{\xi})$-equiuniform on the topological semigroup $\widehat{S}$. Since $\mathrm{Unif}(X, X) \to \mathrm{Unif}(\widehat{X}, \widehat{X}), f \mapsto \widehat{f}$ is a uniform embedding and $\mathrm{Unif}(\widehat{X}, \widehat{X})$ is complete there exists a (unique) uniform map $\widehat{h} : \widehat{S} \to \mathrm{Unif}(\widehat{X}, \widehat{X})$ which extends the homomorphism $h = h_\pi : S \to \mathrm{Unif}(X, X)$. In fact $\widehat{h}$ is a homomorphism because h and $\widehat{h}$ agree on a a dense subsemigroup S of $\widehat{S}$ and $\mathrm{Unif}(\widehat{X}, \widehat{X})$, $\widehat{S}$ are Hausdorff topological semigroups. This proves that the corresponding (evaluation) action $\widehat{\pi}$ is $(\widehat{\xi}, \widehat{\mu})$-equiuniform. The action $\widehat{\pi}$ extends the original action π because $\widehat{h}$ extends h.

(*ii*) Is clear by Lemma 4.4.3.

(*iii*) Combine (*i*) and (*ii*) taking into account that uX is the completion of μ_{fin}.

The continuity of these actions are trivial by Example 4.3.1.

□

Proposition 4.9. (1) *If the semigroup action $\pi : S \times X \to X$ is μ-equiuniform then the induced action $\pi_u : S \times uX \to uX$ on the Samuel compactification $uX := u(X, \mu)$ is a proper S-compactification of X.*

(2) *(S, X) is compactifiable iff the action on X is μ-bounded with respect to some compatible uniformity μ.*

Proof. (1) The action is μ-equiuniform means that the homomorphism $h_\pi : S \to \mathrm{Unif}(X, X)$ is continuous. It suffices to prove our assertion for the action of $h_\pi(S) \times X \to X$. Hence we can suppose that in fact S is the semigroup $h_\pi(S)$. Now the action is (ξ, μ)-equiuniform where ξ is the uniformity induced on $h_\pi(S)$ from $\mathrm{Unif}(X, X)$. Using Proposition 4.8(iii) we get a continuous action

$\widehat{\pi}_u : \widehat{S} \times uX \to uX$ which is $(\widehat{\xi}, \mu_u)$-equiuniform and naturally extends $\widehat{\pi}$. Then its restriction $\pi_u : S \times uX \to uX$ is continuous, too. Hence $u : X \to uX$ is a (proper) S-compactification of X.

(2) Assume that X is μ-bounded. Then by Lemma 4.4 the action is μ^S-equiuniform (which is a compatible uniformity). Now by the first assertion X is S-compactifiable. For the converse use Example 4.3.2. ☐

Corollary 4.10. *There exists a 1-1 correspondence between proper topological S-compactifications of X and precompact compatible equiuniformities on X.*

Note that Corollary 4.10 is well known for group actions [8, 23].

Theorem 4.11. *Let $\pi : S \times X \to X$ be a (ξ, π)-equiuniform semigroup action. Then*

(a) *$u : S \to uS$ is a proper right topological semigroup compactification of S.*

(b) *There exists a right continuous semigroup action*

$$\pi_u^u : uS \times uX \to uX$$

which extends the action

$$\widehat{\pi}_u : \widehat{S} \times uX \to uX$$

(hence also $\widehat{\pi} : \widehat{S} \times \widehat{X} \to \widehat{X}$) and is continuous at every $(p, z) \in \widehat{S} \times uX$.

Proof. By Proposition 4.8(*iii*) there exists a continuous action $\widehat{\pi}_u : \widehat{S} \times uX \to uX$ which extends $\widehat{\pi}$ and is $(\widehat{\xi}, \mu_u)$-equiuniform. Then, in particular, the orbit map $\tilde{z} : \widehat{S} \to uX$, $t \mapsto tz$ is uniform for every $z \in uX$. By the universality of Samuel compactifications there exists a uniquely defined continuous extension $u\widehat{S} \to uX$ of $\tilde{z}$. The compactifications $S \to uS$ and $S \to u\widehat{S}$ are naturally equivalent (Lemma 4.1.2). Hence we have a continuous function $\tilde{z}_u : uS \to uX$ which extends the map $\tilde{z} : \widehat{S} \to uX$, where $\widehat{S}$ is treated as a topological subspace of uS.

Now we define $\pi_u^u : uS \times uX \to uX$ by $\pi_u^u(p, z) := \tilde{z}_u(p)$ for every $p \in uS$ and $z \in uX$. Clearly, π_u^u is right continuous and $\pi_u^u(t, z) = \widehat{\pi}_u(t, z)$ for every $t \in \widehat{S}$. On the other hand again by Proposition 4.8(*iii*) (for $X := S$) we have the continuous action

$\widehat{S} \times uS \to uS$ which extends the multiplication $\widehat{m} : \widehat{S} \times \widehat{S} \to \widehat{S}$ (via the natural dense embedding $\widehat{S} = \phi_S(\widehat{S}) \hookrightarrow uS$). We can apply Lemma 4.6 (for the dense subset $\widehat{S} = \phi_S(\widehat{S})$ of uS and natural maps $\widehat{\pi}_u$ and π_u^u). It follows that uS is a right topological semigroup with the subsemigroup $\widehat{S}$ and $\pi_u^u : uS \times uX \to uX$ is a right continuous semigroup action extending $\widehat{\pi}_u$. By Lemma 4.6.3 we get that π_u^u is jointly continuous at every $(p, z) \in \widehat{S} \times uX$. □

Corollary 4.12. *Let S be a semigroup with a right invariant uniformity ξ on S such that all left translations are uniformly continuous.*

1. (Kocak and Strauss [19]) *$S \to uS$ is a right topological semigroup compactification of S.*
2. (Ferri and Strauss [13]) *The multiplication $uS \times uS \to uS$ is jointly continuous at every $(p, z) \in \widehat{S} \times uS$.*

Proof. The left action $S \times S \to S$ is (ξ, ξ)-equiuniform by Example 4.3.5. Now apply Theorem 4.11. □

Now we give a compactifiability criterion for semigroup actions.

Theorem 4.13. *For every S-space X the following conditions are equivalent:*

1. *X is S-compactifiable.*
2. *$RUC_S(X)$ separates points from closed subsets.*
3. *There exists a Banach space V and a proper continuous representation*
$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{op}, B^*).$$
4. *There exists a compact space Y and a proper representation*
$$(h, \alpha) : (S, X) \rightrightarrows (C(Y, Y), Y).$$
5. *There exists a uniform space (Y, μ) and a proper representation*
$$(h, \alpha) : (S, X) \rightrightarrows (\mathit{Unif}(Y, Y), Y).$$

Proof. (1) ⇒ (2): Let $\nu : X \hookrightarrow Y$ be a proper S-compactification. Then $C(Y) = \mathrm{RUC}_S(Y)$. Now use the obvious hereditarity property of right uniformly continuous functions. That is the fact that $f \circ \nu \in \mathrm{RUC}_S(X)$ for every $f \in \mathrm{RUC}_S(Y)$.

(2) ⇒ (3): Consider the canonical $V := \mathrm{RUC}_S(X)$-representation of (S, X) on V and apply Proposition 3.10.

(3) ⇒ (4): Apply Lemma 2.4 to $V := \mathrm{RUC}_S(X)$ and define $Y := B^*$.

(4) ⇒ (5): For a compact space K (and its uniquie compatible uniformity) the uniform spaces $\mathrm{Unif}(K, K)$ and $C(K, K)$ are the same.

(5) ⇒ (1): By Example 4.3.3 there exists a compatible uniformity μ on X such that the action is μ-equiuniform. Then the corresponding Samuel compactification of (X, μ) is an S-compactification by virtue of Proposition 4.9.1. □

The following theorem shows that a topological semigroup S is compactifiable iff S "lives in natural monoids".

Theorem 4.14. *Let S be a topological semigroup. The following are equivalent:*

1. *S is compactifiable;*
2. *$RUC(S)$ determines the topology of S.*
3. *The monoid S_e (from Remark 3.11.1) is compactifiable;*
4. *S has a proper dynamical compactification.*
5. *S^{op} (the opposite semigroup of S) is a topological subsemigroup of $\Theta(V)$ for some normed (equivalently, Banach) space V;*
6. *S^{op} is a topological subsemigroup of $\Theta(M, d)$ for some metric space (M, d);*
7. *S is a topological subsemigroup of $C(Y, Y)$ for some compact space Y;*
8. *S is a topological subsemigroup of Unif(Y, Y) for some uniform space (Y, μ).*
9. *There exists a compatible right invariant uniformity μ on S.*
10. *There exists a compatible uniformity μ on S such that the right action of S on (S, μ) is equicontinuous.*
11. *The topology of S can be generated by a family $\{d_i\}_{i \in I}$ of right contractive pseudometrics on S.*

If S is a monoid then we can ensure in the assertions (5), (6), (7) *and* (8) *that S is a topological submonoid of the corresponding topological monoid.*

Proof. (1) $\Leftrightarrow$ (2): Follows from Proposition 3.10.

(1) $\Leftrightarrow$ (3): See Remark 3.11.1.

(2) $\Leftrightarrow$ (4): RUC(S) determines the topology of S iff the universal dynamical compactification $u_{\mathrm{RUC}} : S \to S^{\mathrm{RUC}}$ is *proper*.

(1) $\Rightarrow$ (5): First of all observe that by Remark 2.3.2, "normed" and "Banach" cases of (5) are equivalent.

By our assumption S is S-compactifiable. Theorem 4.13 implies that there exists a proper continuous representation

$$(h, \alpha) : (S, S) \rightrightarrows (\Theta(V)^{op}, B^*)$$

where $V :=$ RUC(S). By (1) $\Leftrightarrow$ (3) we can assume that S is a monoid. Since $\alpha : S \to B^*$ is an S-embedding and the pair (h, α) is equivariant it follows that the homomorphism $h : S \to \Theta(V)^{op}$ is a topological embedding, too.

(5) $\Rightarrow$ (6): $\Theta(V)$ is embedded into $\Theta(M, d)$ where $M := V$ and $d(x, y) := ||x - y||$.

(5) $\Rightarrow$ (7): Immediate by Lemma 2.4.

(7) $\Rightarrow$ (8): Trivial.

(8) $\Rightarrow$ (9): Follows by Example 2.9.2.

(9) $\Rightarrow$ (10): Trivial by Definition 2.7.

(10) $\Rightarrow$ (11): If a family $\{d_i\}$ of bounded pseudometrics generates an equicontinuous uniformity μ then the family $\{d_i^*\}$ of right contractive pseudometrics

$$d_i^*(x, y) := \max\{\sup_{s \in S} d_i(xs, ys), d_i(x, y)\}$$

generates a uniformity μ^* which is topologically equivalent to μ.

(11) $\Rightarrow$ (1): Let μ be the uniformity generated by the given family of pseudometrics on S. Since the pseudometrics are right contractive it follows that the action of S on S is μ-bounded. Now Proposition 4.9.2 implies that S is a compactifiable S-flow.

(6) $\Rightarrow$ (11): Denote by $(S, *)$ the opposite semigroup $\Theta(M, d)^{op}$ of $\Theta(M, d)$. The family of pseudometrics $\{\rho_m\}_{m \in M}$ generates the topology of S where

$$\rho_m(s_1, s_2) := d(s_1 m, s_2 m).$$

Now observe that each ρ_m is right contractive on the topological semigroup S. Indeed, for every triple $t, s_1, s_2 \in S$ we have $\rho_m(s_1 * t, s_2 * t) = \rho_m(ts_1, ts_2) = d(ts_1 m, ts_2 m) \leq d(s_1 m, s_2 m) = \rho_m(s_1, s_2)$.

Finally, note that if S is a monoid then by the proof of (1) $\Rightarrow$ (5) the homomorphism $h : S \to \Theta(V)^{op}$ is a topological embedding of monoids. $\square$

Corollary 4.15. *Each of the following semigroups is compactifiable:*

1. $\Theta(X, d)^{op}$ *for every metric space* (X, d)*. In particular,* $\Theta(V)^{op}$ *(endowed with the strong operator topology) for every normed space* V*.*
2. *Unif*(Y, Y) *for every uniform space* (Y, μ)*.*
3. $C(Y, Y)$ *for every compact space* Y*.*
4. *The multiplicative monoid* (B_V, τ_u) *endowed with the uniform topology for every normed algebra* V *(e.g., for the algebra* $V := L(E)$ *for arbitrary normed space* E*).*
5. *Let* G *be a topological group and* $\mathcal{R}$ *its right uniformity. Then the completion* $S := (\widehat{G}, \widehat{\mathcal{R}})$ *is a topological semigroup and this semigroup is compactifiable.*

Proof. All assertions easily follow from Theorem 4.14. For (4) observe that the original metric of the original norm on B_V is right (and also left) contractive $||xs - ys|| \leq ||x - y|| \cdot ||s|| \leq ||x - y||$ for every $x, y, s \in B_V$.

(5): $G \times G \to G$ is $\mathcal{R}(G)$-equiuniform. Apply now Propositions 4.7 and 4.9. $\square$

It is well known that $(\widehat{G}, \widehat{\mathcal{R}})$ is a topological semigroup (see for example [36, Proposition 10.12(a)]) containing G as a subsemigroup. For several important semigroups of the form $S := (\widehat{G}, \widehat{\mathcal{R}})$ see Pestov [35, Ch. 8].

Remark 4.16.

1. Kocak and Strauss proved in [19, Theorem 14] that if a topological semigroup S admits a right invariant left saturated uniformity then S is compactifiable. One can remove "saturated" as Theorem 4.14 shows. Furthermore by assertion (9) the existence of right invariant uniformity is also a necessary condition.
2. As we already have seen $\Theta(E)^{op}$ is compactifiable for every normed space E. It is not true for $\Theta(E)$, in general, as we will see later in Examples 6.3. So we cannot substitute $\Theta(E)^{op}$ by $\Theta(E)$ in Theorem 4.14. However, we can repair

this situation for involutive subsemigroups S of $\Theta(E)$ (see Corollary 4.18).

(3) We cannot replace B_V by the multiplicative monoid V in Corollary 4.15.4 as the example of the multiplicative monoid $V := \mathbb{R}$ shows (Examples 6.3.2).

(4) Our results suggest a semigroup version of the right uniformities $\mathcal{R}(S)$. For a compactifiable topological semigroup one can define $\mathcal{R}(S)$ as the finest right invariant compatible uniformity on S. Then Corollary 4.15.5 admits a natural semigroup generalization for the completion of $(S, \mathcal{R}(S))$.

Theorem 4.17. *Let G be a paratopological group. Then G is compactifiable iff G is a topological group.*

Proof. If G is compactifiable then by Theorem 4.14 we have an embedding $h : G \to C(K, K)$ of topological monoids. Then $h(G) \subset \mathrm{Homeo}\,(K)$, where $\mathrm{Homeo}\,(K)$ is a topological group. The converse is clear by the Teleman's representation. $\square$

Recall that a semigroup S is said to be an *inverse semigroup* if for every $s \in S$ there exists a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. *Topological inverse semigroup* will mean that the multiplication is continuous and in addition the map $S \to S$, $s \mapsto s^*$ is continuous.

By an *involution* on a semigroup S we mean a map $i : S \to S$ such that $i(i(s)) = s$ and $i(s_1 s_2) = i(s_2) i(s_1)$. If S admits a continuous involution then we say that S is *topologically involutive.* Actually, topologically involutive semigroup S is just a semigroup which is topologically isomorphic with the opposite semigroup S^{op}. For example, S is involutive if S is a topological inverse semigroup;. This happens in particular if either S is a commutative topological semigroup or a topological group.

Proposition 4.18. *Let S be a topological subsemigroup of $\Theta(E)$ for a normed space E. Suppose that S is topologically involutive. Then S is compactifiable.*

Proof. S is topologically involutive means that S is topologically isomorphic to the opposite semigroup S^{op} which is compactifiable by Corollary 4.15.1 and the assumption that S is embedded into $\Theta(E)$. $\square$

5. A universal compactifiable semigroup

Denote by U the topological monoid $C(I^\omega, I^\omega)$, where $I := [0,1]$ is the closed interval. Theorem 4.14 implies that U is compactifiable. It contains the subgroup $\mathrm{Homeo}\,(I^\omega)$ of all selfhomeomorphisms of the Hilbert cube I^ω. Recall that $\mathrm{Homeo}\,(I^\omega)$ is a universal second countable topological group (see Uspenskij [41]). Moreover, by [26] the group action $\mathrm{Homeo}\,(I^\omega) \times I^\omega \to I^\omega$ is universal for all second countable compactifiable G-flows X with a second countable acting group G. We can now give a natural generalization for semigroups and semigroup actions.

Theorem 5.1. *Let S be a compactifiable second countable semigroup. Then every compactifiable second countable S-flow X is a part of the flow (U, I^ω). That is, there exists a representation $(h, \alpha) : (S, X) \rightrightarrows (U, I^\omega)$ such that $h : S \hookrightarrow U$ is an embedding of topological semigroups and $\alpha : X \hookrightarrow I^\omega$ is a topological embedding.*

Proof. By Remark 3.11.1 we can assume that S is a monoid with the identity e and $S \times X \to X$ is a monoidal action.

Furthermore, we can suppose in addition that the action is *topologically exact.* This means (see [26]) that: (a) $sx = x$ for all $x \in X$ implies that $s = e$; (b) there exists no strictly weaker topology on S which makes the action on X continuous. Indeed, we can pass, if necessary, to the following new (but still S-compactifiable by Remark 3.11.2) second countable phase space $X' := X \sqcup S$, a disjoint sum of the S-flows X and S, where the monoid S acts on itself by left multiplications. Thus, by our assumption X is a compactifiable S-flow with the topologically exact action. The algebra $\mathrm{RUC}(X)$ separates points and closed subsets of X. Since X is second countable we can choose a separable closed subalgebra $\mathcal{A}$ of $RUC(X)$ having the same property. Moreover since S is also second countable we can assume that $\mathcal{A}$ is even S-invariant. Indeed if $T \subset \mathcal{A}$ and $S_1 \subset S$ are countable dense subsets then TS_1 is a countable dense subset in the S-invariant closed subalgebra $\mathcal{A}' \supset \mathrm{RUC}(X)$ topologically generated by $\mathcal{A}S$.

Now consider the corresponding representation

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(\mathcal{A})^{op}, B^*)$$

of the flow (S, X) on the separable Banach space $\mathcal{A}$. Now, as in [41], we use the fact that the unit ball B^* being a convex compact subset of a separable Frechet space $(\mathcal{A}, weak^*)$ is homeomorphic by Keller's theorem (see for example [7]) to the Hilbert cube I^ω. By our assumption $\mathcal{A}$ separates points from closed subsets in X. Therefore the map $\alpha : X \hookrightarrow B^*$ is a topological embedding. Moreover, since the action of S on X is topologically exact and the pair (h, α) is equivariant it follows that the homomorphism $h : S \to \Theta(\mathcal{A})^{op}$ is in fact an embedding of topological monoids. Observe that

$$(\gamma, id) : (\Theta(\mathcal{A})^{op}, B^*) \rightrightarrows (C(B^*, B^*), B^*)$$

is an equivariant pair with the embedding γ of topological monoids (see Lemma 2.4). Now substituting B^* by the Hilbert cube I^ω we complete the proof. □

As a corollary we get

Theorem 5.2. (Semigroup version of Uspenskij's theorem) *The monoid* $U := C(I^\omega, I^\omega)$ *is universal in the class of all separable metrizable compactifiable semigroups.*

6. Some examples

Recall that if G is a Hausdorff (Tychonoff) topological group then a Tychonoff G-flow X is compactifiable in each of the following cases:

- G is locally compact [46];
- X is locally compact [44];
- X admits a G-invariant metric [48];
- X is a normed space and each g-translation $X \to X$ is linear [25];
- G is second category, (X, d) is a metric G-space and each $\breve{g} : X \to X$ is d-uniformly continuous [25].
- X is Baire, G is separable or Lindelöf and acts transitively on X [42].

Examples below show that for the case of monoidal actions analogous results do not remain true, in general.

Answering de Vries' "compactification problem" negatively in [24] we construct a noncompactifiable Polish G-space X with a Polish acting group G. Moreover by [30] for every Polish group G which is not locally compact there exists a suitable noncompactifiable Polish G-space. We can use this fact below (see Example 6.3.10) providing many non-semi-compactifiable Polish topological semigroups.

Lemma 6.1. *Let $S \times X \to X$ be a monoidal action of a monoid S (with the identity e). Assume that there exists a proper semitopological compactification $\nu : X \hookrightarrow Y$ of X which is $\{e\}$-topological (that is, the action $S \times Y \to Y$ is continuous at every (e, y)). If $F \subset X$ is a closed subset and $a \notin F$ then there exist neighborhoods $U(e)$, $V(F)$ and $O(a)$ such that $UV \cap UO = \emptyset$.*

Proof. Since $\nu : X \hookrightarrow Y$ is an embedding the closure $cl(\nu(F))$ of $\nu(F)$ in Y does not contain the point $\nu(a)$. By the continuity of the action at every point (e, y) (making use the Hausdorff axiom) it follows that for every $b \in cl(\nu(F))$ there exist a neighborhood U_b of e and neighborhoods O_b of $\nu(a)$ and V_b of b in Y such that $U_b V_b \cap U_b O_b = \emptyset$. Now the standard compactness argument easily completes the proof.

□

Let $\pi : S \times X \to X$ be a jointly continuous semigroup action. Up to an S-isomorphisms we can assume that S and X are disjoint sets. Denote by $S \sqcup_\pi X$ a new semigroup defined as follows. As a set it is a *disjoint union* $S \cup X$. The multiplication is defined by setting:

$a \circ b := sx$ if $a = s \in S,\ b = x \in X$

$a \circ b := s_1 s_2$ if $a = s_1 \in S,\ b = s_2 \in S$

and

$a \circ b := a$ if $a \in X$.

Then $(S \sqcup_\pi X, \circ)$ is a topological semigroup which we call a π-*generated semigroup.*

Lemma 6.2. *Let X be an S-space.*

1. *The topological semigroup $P := S \sqcup_\pi X$ is compactifiable (semi-compactifiable) if and only if (S, X) is a compactifiable (resp.: semi-compactifiable) flow and at the same time S is a compactifiable (resp.: semi-compactifiable) semigroup.*

(2) *The opposite topological semigroup* $P^{op} := (S \sqcup_\pi X)^{op}$ *is compactifiable if and only if* S^{op} *is a compactifiable semigroup and the topology of* X *admits a system of* S*-contractive pseudometrics.*

Proof. (1): Observe that we have naturally defined equivariant inclusion of flows

$$(h, \alpha) : (S, X) \rightrightarrows (P, P) = (S \sqcup_\pi X, S \sqcup_\pi X).$$

Therefore if (P, P) is compactifiable then the same is true for (S, X) and (S, S).

Conversely, every pair $\psi_1 : S \hookrightarrow Y_1$ and $\psi_2 : X \hookrightarrow Y_2$ of proper S-compactifications (one may assume that Y_1 and Y_2 are disjoint) defines a proper P-compactification $\psi : P = S \sqcup_\pi X \hookrightarrow Y_1 \sqcup Y_2$.

(2): If P^{op} is compactifiable then S^{op} being a subsemigroup of P^{op} is also compactifiable. Moreover, by Theorem 4.14 there exists a system of right contractive pseudometrics on $P^{op} = (S \sqcup_\pi X)^{op}$. Such a system is clearly left contractive on P. It induces the desired system of S-contractive pseudometrics on X.

Conversely, suppose that S^{op} is compactifiable and the topology of X is generated by a family $\mathcal{F}_1 := \{d_i\}_{i \in I}$ of S-contractive pseudometrics. By the first assumption and Theorem 4.14 there exists a system $\mathcal{F}_2 := \{\rho_j\}_{j \in J}$ of left contractive pseudometrics on S. One can suppose in addition that $d_i \leq 1$ and $\rho_j \leq 1$ for every $(i, j) \in I \times J$.

Now define a new system $\mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{D\}$ on $P = S \sqcup_\pi X$ by setting $D(s, x) = D(x, s) = 1$ for every $s \in S,\ x \in X$ and $D(s_1, s_2) = D(x_1, x_2) = 0$ for every $s_1, s_2 \in S,\ x_1,\ x_2 \in X$. It is easy to verify that $\mathcal{F}_3$ is a system of left contractive pseudometrics on P generating its topology. The same system is right contractive on P^{op}. Hence by Theorem 4.14 we can conclude that P^{op} is compactifiable. □

Examples 6.3. *Here we give some examples of noncompactifiable topological semigroups and actions.*

(1) *The linear action of the compact multiplicative monoid* $S := ([0, 1], \cdot)$ *on* $X := [0, \infty)$ *is not compactifiable. Moreover, every* $f \in RUC_S(X)$ *is necessarily constant.*

Assuming the contrary let $f \in RUC_S(X)$ *be nonconstant. Then* $f(a) - f(b) = \varepsilon > 0$ *for a pair* $a, b \in X$*. By definition of* $RUC_S(X)$ *there exists* $\delta > 0$ *such that* $|f(u_1x) - f(u_2x)| < \varepsilon$ *for every triple* $(u_1, u_2, x) \in U \times U \times X$*, where* $U := [0, \delta)$*. Choose* $x_0 \in X$ *such that* $a < \delta x_0$ *and* $b < \delta x_0$*. Take* $u_1 := \frac{a}{x_0}$ *and* $u_2 := \frac{b}{x_0}$*. Then* $(u_1, u_2, x_0) \in U \times U \times X$ *but* $|f(u_1x_0) - f(u_2x_0)| = \varepsilon$*.*

Note that in this example the acting monoid is a submonoid of $\Theta(V)$ *for* $V := \mathbb{R}$*. As a corollary we get that the action* $\Theta(V) \times V \to V$ *is not compactifiable for any nontrivial normed space* V*.*

(2) *The multiplicative monoid* $S := ([0, \infty), \cdot)$ *(and hence also the multiplicative monoid* $(\mathbb{R}, \cdot)$ *of all reals) is not compactifiable. In fact the corresponding universal dynamical compactification* $S \to S^{RUC}$ *is a singleton.*

This follows directly from example (1).

Since $\Theta(V)^{op}$ *is compactifiable and* $\mathbb{R}$ *is involutive (being commutative) by Proposition 4.18 we get that* $(\mathbb{R}, \cdot)$ *is not embedded into* $\Theta(V)$ *for arbitrary normed space* V*. As well as* $(\mathbb{R}, \cdot)$ *is not embedded as a topological subsemigroup into* $U := C(I^\omega, I^\omega)$ *(Corollary 4.15.3).*

(3) *The universal right topological semigroup compactification* $S \to S^{LMC}$ *of* $S := ([0, \infty), \cdot)$ *is injective but not proper (that is,* $LMC(S)$ *separates the points but does not determine the original topology). Hence,* $[0, \infty), \cdot)$ *is not semicompactifiable.*

Let M *be the additive topological monoid* $\mathbb{R} \cup \{\theta\}$ *where topologically* θ *is a point at* $+\infty$ *and algebraically* $\theta + x = x + \theta = \theta$ *for every* $x \in M$*. In fact this semigroup* M *is a copy of the multiplicative semigroup* $([0, \infty), \cdot)$ *via the topological isomorphism* $\mathbb{R} \cup \{\theta\} \to [0, \infty),\ \alpha(\theta) = 0, \alpha(x) = 2^{-x}$ *for all* $x \in \mathbb{R}$*. Now note that by results of Hindman and Milnes* [17, Chapter 5] *the algebra* $LMC(M)$ *separates the points but does not determine the original topology (see also the results of Section 3).*

(4) *A one-parameter additive semigroup action on a Polish phase space which is not semi-compactifiable.*

This construction was inspired by Ruppert [37, Ch. II, Example 7.8]. *Let* $\mathbb{R}_+ = ([0,\infty),+)$ *be the one parameter additive semigroup. Denote by* $[0,\infty]$ *the Alexandrov compactification of* $\mathbb{R}_+$. *In the product space* $[0,\infty] \times [0,\infty]$ *consider the following subspace*

$$X := [0,\infty) \times [0,\infty) \cup \{(\infty,\infty)\}$$

Then X *is Polish being homeomorphic to a* G_δ*-subset of the 2-cell* $[0,1] \times [0,1]$. *Define now the desired continuous action by*

$$\pi : \mathbb{R}_+ \times X \to X, \quad t(x,y) = (x, tx+y), \ t(\infty,\infty) = (\infty,\infty)$$

Define in X *the closed subset* $F := [0,\infty) \times \{0\}$ *and the point* $a := (\infty,\infty)$. *Then for every neighborhood* $O(a)$ *of* a *in* X *and every neighborhood* $U(0)$ *of* 0 *in* $\mathbb{R}_+$ *we have* $UF \cap O \neq \emptyset$. *Now Remark 3.6.2 and Lemma 6.1 imply that* X *is not semi-compactifiable.*

(5) *A compact monoid action on a discrete space which is not semi-compactifiable.*

Consider the 2-point multiplicative monoid $\{0,1\}$ *and endow the Cantor cube* $C := \{0,1\}^{\mathbb{N}_0}$ *with the topological monoid structure of pointwise multiplication. Let* $X = \mathbb{N}_0 := \mathbb{N} \cup 0$. *Define an action by:*

$$\pi : C \times \mathbb{N}_0 \to \mathbb{N}_0, \quad \pi(c,n) = c_n n,$$

where $c = (c_k)_{k \in \mathbb{N}_0} \in C$. *In* $\mathbb{N}_0$ *choose the point* $a := 0$ *and the closed subset* $F := \mathbb{N}$. *Let* $\mathbf{1} := (1,1,\cdots)$ *denote the identity of* C. *Then for every neighborhood* U *of* $\mathbf{1}$ *we have* $a \in UF$. *Hence Lemma 6.1 and Remark 3.6.1 finish the proof.*

(6) *A topological semigroup* Q *such that* Q *is compactifiable and the opposite semigroup* Q^{op} *is not semi-compactifiable.*

We construct the desired semigroup as the π*-generated semigroup* $P := \{0,1\}^{\mathbb{N}_0} \sqcup_\pi \mathbb{N}_0$ *for the flow* $(S,X) = (C,\mathbb{N}_0)$

described in (5). Now P *is not semi-compactifiable by Lemma 6.2.1. Then the opposite semigroup* $Q := P^{op}$ *is the desired one. Indeed, first of all* $Q^{op} = P$ *is not semi-compactifiable.*

Clearly, $S = \{0,1\}^{\mathbb{N}_0}$ *is compactifiable being a compact semigroup. Define the standard* $0, 1$ *metric on the discrete space* $X := \mathbb{N}_0$*. Then this metric is contractive with respect to the action of* S *on* X*. By Lemma 6.2.2 we conclude that* $P^{op} = Q$ *is compactifiable.*

(7) *There exists a Banach space* V *such that the monoid* $\Theta(V)$ *is not semi-compactifiable.*

Let Q *be the topological semigroup defined in (6). Then* $P := Q^{op}$ *is not semi-compactifiable. On the other hand* P*, being the opposite semigroup of a compactifiable semigroup* Q*, is a topological subsemigroup of* $\Theta(V)$ *for some Banach space* V *(see Theorem 4.14). Therefore* $\Theta(V)$ *is not semi-compactifiable, too.*

(8) *Sorgenfrey line* $(\mathbb{R}_s, +)$ *is a noncompactifiable topological monoid.*

This follows directly from Theorem 4.17. Moreover it is not hard to see that $RUC(\mathbb{R}_s) = RUC(\mathbb{R})$*. That is, the universal dynamical compactification* $\mathbb{R}_s^{RUC}$ *is just the greatest ambit* $\mathbb{R}^{RUC}$ *(for the usual topological group* $\mathbb{R}$ *of the reals).*

(9) *For every Polish not locally compact topological group* G *there exists a continuous action* $\pi : G \times X \to X$ *on a Polish space* X *such that the corresponding* π*-generated Polish semigroup* $P := G \sqcup_\pi X$ *is not semi-compactifiable.*

By [30] *there exists a noncompactifiable Polish* G*-space* X*. Then the semigroup* $P := G \sqcup_\pi X$ *is not semi-compactifiable. Indeed assuming the contrary it follows by Lemma 6.2.1 that* (G, X) *is semi-compactifiable. Since* G *is Čech-complete we get (see Remark 3.6.1) that* X *is* G*-compactifiable, a contradiction.*

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