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Order and minimality of some topological groups ☆

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A Hausdorff topological group is called minimal if it does not admit a strictly coarser Hausdorff group topology. This paper mostly deals with the topological group $\mathrm{H}_+(X)$ of order-preserving homeomorphisms of a compact linearly ordered connected space X. We provide a sufficient condition on X under which the topological group $\mathrm{H}_+(X)$ is minimal. This condition is satisfied, for example, by: the unit interval, the ordered square, the extended long line and the circle (endowed with its cyclic order). In fact, these groups are even a-minimal, meaning, in this setting, that the compact-open topology on G is the smallest Hausdorff group topology on G. One of the key ideas is to verify that for such X the Zariski and the Markov topologies on the group $\mathrm{H}_+(X)$ coincide with the compact-open topology. The technique in this article is mainly based on a work of Gartside and Glyn [21].

1. Introduction

A Hausdorff topological group G is *minimal* [17,36] if it does not admit a strictly coarser Hausdorff group topology or, equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding.

All topological spaces are assumed to be Hausdorff and completely regular (unless stated otherwise). Let X be a compact topological space. Denote by $\mathrm{H}(X)$ the group of all homeomorphisms of X, endowed with the compact-open topology τ_{co}. In this setting $\mathrm{H}(X)$ is a topological group and the natural action $\mathrm{H}(X) \times X \to X$ is continuous.

Clearly, every compact topological group is minimal. The groups $\mathbb{R}$ and $\mathbb{Z}$, on the other hand, are not minimal. Moreover, Stephenson showed in [36] that an LCA group is minimal if and only if it is compact. Nontrivial examples of minimal groups include $\mathbb{Q}/\mathbb{Z}$ with the quotient topology, [36], and $S(X)$,

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the symmetric group of an infinite set (with the pointwise topology). The minimality of the latter was proved by Gaughan [22] and (independently) by Dierolf and Schwanengel [9]. For more information on minimal groups we refer to the surveys [8,10,11] and the book [12].

The following is a question of Stoyanov (cited in [1], for example):

Question 1.1 *(Stoyanov).* Is it true that for every compact homogeneous space X the topological group $\mathrm{H}(X)$ is minimal?

One important positive example of such a space is the Cantor cube 2^ω. Indeed, in [20] Gamarnik proved that $\mathrm{H}(2^\omega)$ is minimal. Recently van Mill [29] provided a counterexample to Question 1.1 proving that for the n-dimensional Menger universal continuum X, where $n > 0$, the group $\mathrm{H}(X)$ is not minimal.

It is well known that the Hilbert cube $[0,1]^\omega$ is a homogeneous compact space as well. The following question of Uspenskij [39] remains unanswered: is the group $\mathrm{H}([0,1]^\omega)$ minimal?

Definition 1.2.

1. [11] A topological group G is *a-minimal* if its topology is the smallest possible Hausdorff group topology on G.
2. [11] A compact space X is *M-compact* (*aM-compact*) if the topological group $\mathrm{H}(X)$ is minimal (respectively, a-minimal).
3. A compact ordered space X is *M_+-compact* (*aM_+-compact*) if the topological group $\mathrm{H}_+(X)$ of all order-preserving homeomorphisms of X is minimal (respectively, a-minimal).

Several questions naturally arise at this point:

Question 1.3.

1. [11] Which (notable) compact spaces are M-compact? aM-compact?
2. Which compact ordered spaces are M_+-compact? aM_+-compact?

The two point compactification of $\mathbb{Z}$ is a compact LOTS X such that $\mathrm{H}_+(X)$ and $\mathrm{H}(X)$ are not minimal (Example 4.1). Thus not every compact LOTS is M_+-compact or M-compact.

Clearly, every a-minimal group is minimal. It is well known that $(\mathbb{Z}, \tau_p)$ with its p-adic topology is a minimal topological group. Since such topologies are incomparable for different p's, it follows that $(\mathbb{Z}, \tau_p)$ is not a-minimal.

Recall a few results:

1. (Gaughan [22]) The symmetric group $S(X)$ is a-minimal. Since $\mathrm{H}(X^*)$ is precisely $S(X)$ we obtain that the 1-point compactification X^* of a discrete set X is aM-compact.
2. (Banakh–Guran–Protasov [2]) Every subgroup of $S(X)$ that contains $S_\omega(X)$ (permutations of finite support) is a-minimal (answers a question of Dikranjan [28]).
3. (Gamarnik [20]) $[0,1]^n$ is M-compact (for $n \in \mathbb{N}$) if and only if $n = 1$.
4. (Gartside and Glyn [21]) $[0,1]$ and $\mathbb{S}^1$ are aM-compact.
5. (Gamarnik [20]) The Cantor cube 2^ω is M-compact.
6. (Uspenskij [38]) Every h-homogeneous compact space is M-compact.
7. (van Mill [29]) n-dimensional Menger universal continuum X, where $n > 0$, is not M-compact (answers Stoyanov's Question 1.1).

Recall that a zero-dimensional compact space X is *h-homogeneous* if all non-empty clopen subsets of X are homeomorphic to X. In particular, 2^ω is h-homogeneous. Hence, (6) is a generalization of (5).

The concept of an a-minimal group is in fact an intrinsic algebraic property of an abstract group G (underlying a given topological group). a-Minimality is interesting for several reasons. For instance, it is strongly related to some fundamental topics like Markov's and Zariski's topologies.

For additional information about a-minimality (and minimality) see the recent survey [11]. For Markov's and Zariski's topologies see [15,14,2,16]. We recall the definitions.

Definition 1.4. Let G be a group.

1. The *Zariski topology* $\mathfrak{Z}_G$ is generated by the sub-base consisting of the sets $\{x \in G : x^{\varepsilon_1} g_1 x^{\varepsilon_2} g_2 \cdots x^{\varepsilon_n} g_n \neq e\}$, where e is the unit element of G, $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$, and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$.
2. The *Markov topology* $\mathfrak{M}_G$ is the infimum (taken in the lattice of all topologies on G) of all *Hausdorff* group topologies on G.

Note that $(G, \mathfrak{Z}_G)$ and $(G, \mathfrak{M}_G)$ are quasi-topological groups. That is the inverse and the translations are continuous. They are not necessarily topological groups. In fact, if G is abelian then $\mathfrak{Z}_G$ and $\mathfrak{M}_G$ are not group topologies, unless G is finite, [14, Corollary 3.6]. Here we give some simple properties. Regarding assertion (3) in the following lemma see for example [11, Definition 2.1].

Lemma 1.5. *Let G be an abstract group. Suppose that τ is a Hausdorff group topology on G. Then*

(1) $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \tau$.
(2) $\mathfrak{Z}_G = \mathfrak{M}_G = \tau$ *if and only if* $\tau \subseteq \mathfrak{Z}_G$. *In this case* (G, τ) *is a-minimal.*
(3) $\mathfrak{M}_G$ *is a (not necessarily, Hausdorff) group topology if and only if* $(G, \mathfrak{M}_G)$ *is an a-minimal topological group.*

Proof.

(1) Follows directly from the definitions.
(2) Follows from (1).
(3) Note that $\mathfrak{M}_G$ is always a T_1-topology. Hence, if $\mathfrak{M}_G$ is a topological group topology then it is Hausdorff. Taking into account the definition of $\mathfrak{M}_G$ we can conclude that this topology is the smallest Hausdorff group topology on G. Hence, $(G, \mathfrak{M}_G)$ is a-minimal. □

Question 1.6 *(Markov).* For what groups G the Markov and Zariski topologies coincide?

A review of some old and new partial answers can be found in [16]. Below, in Theorem 3.4, we give additional examples of groups for which $\mathfrak{Z}_G = \mathfrak{M}_G$.

In the present paper we mainly deal with the groups $\mathrm{H}_+(X)$. Given an ordered compact space X, we are interested in the group $\mathrm{H}_+(X)$ of order-preserving homeomorphisms. For a compact space X the group $\mathrm{H}(X)$ is complete (with respect to the two-sided uniformity) and therefore $\mathrm{H}_+(X)$ is also complete (as a closed subgroup of a complete group).

In certain cases the minimality of $\mathrm{H}(X)$ can be deduced from the minimality of $\mathrm{H}_+(X)$, as the following lemma shows.

Lemma 1.7. *Let X be a compact LOTS such that* $\mathrm{H}_+(X)$ *is minimal. If* $\mathrm{H}_+(X)$ *is a co-compact subgroup of* $\mathrm{H}(X)$*, then* $\mathrm{H}(X)$ *is minimal.*

This lemma is a corollary of Lemma 2.2. *Co-compactness* of $\mathrm{H}_+(X)$ in $\mathrm{H}(X)$ means that the coset space $\mathrm{H}(X)/\mathrm{H}_+(X)$ is compact.

If X is a linearly ordered continuum, then by Lemma 2.3 the subgroup $\mathrm{H}_+(X)$ has at most index 2 in $\mathrm{H}(X)$. So, in this case, from the minimality of $\mathrm{H}_+(X)$ we can deduce by Lemma 1.7 the minimality of $\mathrm{H}(X)$. For example, it is true for $X = [0,1]$. Note that $\mathrm{H}[0,1] = \mathrm{H}_+[0,1] \rtimes \mathbb{Z}_2$, the topological semidirect product of $\mathrm{H}_+[0,1]$ and $\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the two element group. However, in general, it is unclear how to infer the minimality of a topological group G from the minimality of $G \rtimes \mathbb{Z}_2$. For instance, in [11, Example 4.7] it is shown that there exists a non-minimal group G such that $G \rtimes \mathbb{Z}_2$ is minimal.

Recall the following result of Gartside and Glyn:

Theorem 1.8. *([21]) For any metric one dimensional manifold (with or without boundary) M, the compact-open topology on the full homeomorphism group $\mathrm{H}(M)$ is the unique minimum Hausdorff group topology on $\mathrm{H}(M)$.*

The one dimensional compact manifolds, up to homeomorphism, are the closed interval $[0,1]$ and the circle $\mathbb{S}^1$. In view of Definition 1.2 this result can be reformulated in the following way.

Theorem 1.9. *([21]) $\mathrm{H}[0,1]$ and $\mathrm{H}(\mathbb{S}^1)$ are a-minimal groups.*

Extending some ideas of Gartside–Glyn [21] to linearly ordered spaces we give some new results about minimality of the groups $\mathrm{H}_+(X)$ of order preserving homeomorphisms.

Theorem. *(See Theorem 3.4.) Let $(X, \tau_\leq)$ be a compact connected LOTS that satisfies the following condition:*

(A) *for every pair of elements $a < b$ in X the group $\mathrm{H}_+[a,b]$ is nontrivial.*

Then:

(1) *For the topological group $G = \mathrm{H}_+(X)$ and $G = \mathrm{H}(X)$ the Zariski and Markov topologies coincide with the compact-open topology. That is, $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$.*
(2) *The topological groups $\mathrm{H}_+(X)$ and $\mathrm{H}(X)$ are a-minimal.*
(3) *X is aM_+-compact and aM-compact.*

According to results of Hart and van Mill [25] (see Section 4.2) there exists a connected compact LOTS X which is H_+-rigid, that is, $\mathrm{H}_+(X)$ is trivial (in fact, $\mathrm{H}(X)$ is trivial). Hence, condition (A) of the theorem above is not always satisfied for general ordered continua. Moreover, one may derive from results of [25] that there exists a connected compact LOTS X for which $\mathrm{H}_+(X) = \mathrm{H}(X) = \mathbb{Z}$, a discrete copy of the integers $\mathbb{Z}$, and $\mathrm{H}[c,d]$ is trivial for some pair $c < d$ in X (Proposition 4.2).

In Section 4 we give some concrete examples of spaces that satisfy condition (A) of Theorem 3.4. The following linearly ordered spaces X are aM_+-compact, that is the groups $\mathrm{H}_+(X)$ (and, in fact, also $\mathrm{H}(X)$) are a-minimal:

1. $[0,1]$;
2. the lexicographically ordered square $\mathcal{I}^2$;
3. the extended long line $\mathcal{L}^*$;
4. the ordinal space $[0,\kappa]$;
5. the unit circle $\mathbb{S}^1$ (in this case we work with a *cyclic order*, Definition 2.6).

Note that the groups $\mathrm{H}_+(X)$ play a major role in many research lines. See, for example, [23,32,24].

2. Preliminaries

In what follows, every compact topological space will be considered as a uniform space with respect to its natural (unique) uniformity.

For a topological group (G, γ) and its subgroup H denote by γ/H the natural quotient topology on the coset space G/H.

Lemma 2.1 *(Merson's Lemma). Let (G, γ) be a not necessarily Hausdorff topological group and H be a not necessarily closed subgroup of G. If $\gamma_1 \subseteq \gamma$ is a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1/H = \gamma/H$, then $\gamma_1 = \gamma$.*

Lemma 2.2. *Let H be a co-compact complete subgroup of a topological group G. If H is minimal then G is minimal too.*

Proof. Denote by τ the given topology on G, and let $\gamma \subseteq \tau$ be a coarser Hausdorff group topology. Since H is minimal, we know that $\gamma|_H = \tau|_H$. Furthermore, H is γ-closed in G because H is complete. Since $(G/H, \gamma/H)$ is Hausdorff and $(G/H, \tau/H)$ is compact we have $\gamma/H = \tau/H$. Thus, by Merson's Lemma 2.1, we conclude that $\gamma = \tau$. $\square$

2.1. Ordered topological spaces

A *linear order* on a set X is, as usual, a binary relation $\leq$ which is reflexive, antisymmetric, transitive and satisfies in addition the totality axiom: for all $a, b \in X$ either $a \leq b$ or $b \leq a$.

For a set X equipped with a linear order $\leq$, the *order topology* (or *interval topology*) $\tau_\leq$ on X is generated by the subbase that consists of the intervals $(\leftarrow, a) = \{x \in X : x < a\}$, $(b, \rightarrow) = \{x \in X : b < x\}$. A *linearly ordered topological space* (or LOTS) is a triple $(X, \tau_\leq, \leq)$ where $\leq$ is a linear order on X and $\tau_\leq$ is the order topology on X. For every pair $a < b$ in X the definition of the intervals $(a, b), [a, b]$ is understood. Every linearly ordered compact space X has the smallest and the greatest element; so, $X = [s, t]$ for some $s, t \in X$.

Sometimes we say: *linearly ordered continuum*, instead of *compact and connected LOTS*.

Lemma 2.3. *Let $(X, \tau_\leq)$ be a linearly ordered continuum. Then every $f \in \mathrm{H}(X)$ is either order-preserving or order-reversing. In particular, the index of $\mathrm{H}_+(X)$ in $\mathrm{H}(X)$ is at most 2.*

Proof. Assume for contradiction that there exists $f \in \mathrm{H}(X)$ such that f is neither order-preserving nor order-reversing. Thus there exist three points $x_1, x_2, x_3 \in X$ such that $x_1 < x_2 < x_3$ and either $f(x_1) < f(x_2) \wedge f(x_2) > f(x_3)$ or $f(x_1) > f(x_2) \wedge f(x_2) < f(x_3)$. Both cases lead to a contradiction. We give the details for the first case (the second case is similar).

Suppose $x_1 < x_2 < x_3$ and $f(x_1) < f(x_2) \wedge f(x_2) > f(x_3)$. Since X is linearly ordered there are two possibilities to consider.

1. $f(x_1) < f(x_3) < f(x_2)$: then by the Intermediate Value Theorem (applied to the interval $[x_1, x_2]$) there exists $x_1 < x_0 < x_2$ such that $f(x_0) = f(x_3)$, which is a contradiction since f is $1 - 1$.
2. $f(x_3) < f(x_1) < f(x_2)$: then again by the Intermediate Value Theorem (applied to the interval $[x_2, x_3]$) there exists $x_2 < x_0 < x_3$ such that $f(x_0) = f(x_1)$, which is a contradiction because f is $1 - 1$.

Each case leads to a contradiction, and this fact concludes the proof. $\square$

In the sequel we use several times the following simple "localization lemma".

Lemma 2.4. *Let X be a LOTS and let $a < b$ be a given pair of elements in X. If $h \in \mathrm{H}_+[a,b]$, then for the natural extension $\hat{h}: X \to X$, with $\hat{h}(x) = x$ for every $x \in X \setminus (a,b) = (\leftarrow, a] \cup [b, \to)$, we have $\hat{h} \in \mathrm{H}_+(X)$.*

The idea of the following lemma was kindly provided to us by K.P. Hart.

Lemma 2.5. *Let X be a linearly ordered continuum. The following conditions are equivalent:*

(A) *for every pair of elements $a < b$ in X the group $\mathrm{H}_+[a,b]$ is nontrivial,*
(B) *for every pair of elements $a < b$ in X the group $\mathrm{H}_+[a,b]$ is nonabelian.*

Proof. Let $a < b$ in X. Assuming (A) there exists a nontrivial $h_1 \in \mathrm{H}_+[a,b]$. So, $h_1(u) \neq u$ for some $u \in (a,b)$. We can suppose that $a < u < h_1(u) < b$ (indeed, if $h_1(u) < u$, replace h_1 by h_1^{-1} and u by $h_1(u)$). Since X is a continuum, the interval $(u, h_1(u))$ is nonempty. Choose an arbitrary $v \in (u, h_1(u))$. By the continuity of h_1 there exists a sufficiently small neighborhood O of u such that

$$s < v < h_1(t)$$

for every $s, t \in O$. Without restriction of generality we can assume that O is the interval $[x_1, x_2]$, where $x_1 < x_2$. Clearly, $h_1(x_1) < h_1(x_2)$, so

$$a < x_1 < x_2 < h_1(x_1) < h_1(x_2) < b.$$

Now apply condition (A) to the interval $[h_1(x_1), h_1(x_2)]$. There exists a nontrivial $h_2 \in \mathrm{H}_+[h_1(x_1), h_1(x_2)]$. Similarly, as for h_1 and $[a,b]$, one may choose, for h_2 and $[h_1(x_1), h_1(x_2)]$, a subinterval $[y_1, y_2]$ of $[h_1(x_1), h_1(x_2)]$ such that

$$h_1(x_1) < y_1 < y_2 < h_2(y_1) < h_2(y_2) < h_1(x_2).$$

We can treat h_2 as an element of $\mathrm{H}_+[a,b]$ by the natural extension (assuming that $h_2(x) = x$ outside of $[h_1(x_1), h_1(x_2)]$).

The interval $[h_1^{-1}(y_1), h_1^{-1}(y_2)]$ is a nonempty subinterval of $[x_1, x_2]$. Now observe that for every $z \in [h_1^{-1}(y_1), h_1^{-1}(y_2)]$ we have $z < h_1(x)$. Therefore, $h_2(z) = z$. So, we get

$$h_1(h_2(z)) = h_1(z) \in [y_1, y_2],$$

while

$$h_2(h_1(z)) \in [h_2(y_1), h_2(y_2)].$$

Since $y_2 < h_2(y_1)$, we can conclude that $h_2 \circ h_1 \neq h_1 \circ h_2$ and $\mathrm{H}_+[a,b]$ is nonabelian. $\square$

Definition 2.6. (See, for example, [6,27].) A ternary relation $R \subseteq X^3$ on a set X is said to be a *cyclic ordering* if:

1. $\begin{cases} a \neq b \neq c \neq a \\ (a,b,c) \notin R \end{cases} \Leftrightarrow (c,b,a) \in R.$
2. $(a,b,c) \in R \Rightarrow (b,c,a) \in R.$
3. $\begin{cases} (a,b,c) \in R \\ (a,c,d) \in R \end{cases} \Rightarrow (a,b,d) \in R.$

Let X be a topological space and R be a cyclic ordering on X. A homeomorphism $f: X \to X$ is *orientation preserving* if f preserves R, meaning that $(z, y, x) \in R$ implies $(f(z), f(y), f(x)) \in R$. The set of all such autohomeomorphisms is a subgroup of $\mathrm{H}(X)$ which we denote by $\mathrm{H}_+(X)$.

3. Order-preserving homeomorphisms and α-minimality

Using some results of Nachbin we extend the ideas of Gartside and Glyn [21] to compact connected linearly ordered spaces (Theorem 3.4).

For the purposes of this section we fix the following notations. Let $(X, \tau_\leq)$ be a compact LOTS with its unique compatible uniform structure μ and denote $s = \min X, t = \max X$. For every $f \in C(X)$ and $\varepsilon > 0$ define

$$U_{f,\varepsilon} := \{(x, y) \in X \times X : \ |f(x) - f(y)| \leq \varepsilon\}.$$

Denote by $C_+(X, [0, 1])$ the set of all continuous order-preserving maps $f: X \to [0, 1]$.

Lemma 3.1. *(Nachbin [31].) Let X be a compact LOTS.*

1. *$C_+(X, [0, 1])$ separates the points of X.*
2. *The family $\{U_{f,\varepsilon} : f \in C_+(X, [0, 1]), \varepsilon > 0\}$ is a subbase of the uniformity μ for every compact LOTS X.*

Proof. (1) It is a fundamental result of Nachbin [31, p. 48 and 113].

(2) Use (1) and the following observation. For every compact space X and a point-separating family F of (uniformly) continuous functions $X \to [0, 1]$, the corresponding weak uniformity μ_F on X is just the natural unique compatible uniformity μ on X. The family of entourages $\{U_{f,\varepsilon} : f \in F, \varepsilon > 0\}$ is a uniform subbase of $\mu = \mu_F$. □

Definition 3.2. Let $\alpha \in \mu$ be an entourage. We say that a finite chain $A := \{c_0, c_1, \cdots, c_n\}$ in X is an *α-connected net* if:

1. $s = c_0 \leq c_1 \leq \cdots \leq c_n = t$;
2. $(x, y) \in \alpha$ for every $x, y \in [c_i, c_{i+1}]$ and $0 \leq i \leq n - 1$.

Notation: $A \in \Gamma(\alpha)$.

Note that $(x, y) \in \alpha^2$ for every $x \in [c_k, c_{k+1}]$ and $y \in [c_{k+1}, c_{k+2}]$.

Lemma 3.3. *Let $(X, \tau_\leq)$ be a compact LOTS with its unique compatible uniform structure μ. The following are equivalent:*

1. *X is connected;*
2. *for every $\alpha \in \mu$ there exists an α-connected net.*

Proof. (1) $\Rightarrow$ (2)

In the setting of Definition 3.2 every finite chain which contains an α-connected net is also an α-connected net. It follows that it is enough to verify the definition for entourages from any given uniform subbase of μ. So, in our case, by Lemma 3.1, it is enough to check that there exists an α-connected net for every $\alpha = U_{f,\varepsilon}$. We have to show that $\Gamma(U_{f,\varepsilon})$ is nonempty for every $f \in C_+(X, [0, 1])$ and every $\varepsilon > 0$.

Since X is connected and compact the continuous image $f(X) \subseteq [0,1]$ is a closed subinterval, say $f(X) = [u, v]$.

Fix $n \in \mathbb{N}$ large enough such that $\frac{v-u}{n} \leq \varepsilon$. For every natural i with $0 < i < n$ choose $c_i \in X$ with $f(c_i) = \frac{(v-u)i}{n} + u$ and $c_0 = s$, $c_n = t$. Then

$$A := \{c_0, c_1, \ldots, c_n\} \in \Gamma(U_{f,\varepsilon}).$$

Indeed, since f is order-preserving, for every $x, y \in X$ with $x, y \in [c_i, c_{i+1}]$ we have

$$f(x), f(y) \in [f(c_i), f(c_{i+1})].$$

So $|f(x) - f(y)| \leq \frac{v-u}{n} \leq \varepsilon$. Therefore, $(x, y) \in \alpha = U_{f,\varepsilon}$.

(2) ⇒ (1)

Assume to the contrary that X is not connected. Since X is a compact LOTS it follows that the order is not dense. That is, there exist $a < b$ in X such that the interval (a, b) is empty. Then the function $f : X \to [0,1]$, where $f(x) = 0$ for $x \leq a$ and $f(x) = 1$ for $b \leq x$ is continuous. Choose any $0 < \varepsilon < 1$ and define $\alpha := U_{f,\varepsilon} \in \mu$. Then $\Gamma(\alpha)$ is empty. □

Assertion (2) of the following theorem for $X := [0,1]$ generalizes a result of [21] mentioned above in Theorem 1.9. We modify the arguments of [21] and use Lemmas 1.5, 2.5 and 3.3.

Theorem 3.4. *Let $(X, \tau_{\leq})$ be a compact connected LOTS that satisfies the following condition:*

(A) *for every pair of elements $a < b$ in X the group $\mathrm{H}_+[a,b]$ is nontrivial.*

Then:

(1) *For the topological groups $G = \mathrm{H}_+(X)$ and $G = \mathrm{H}(X)$ the Zariski and Markov topologies coincide with the compact-open topology. That is, $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$.*
(2) *The topological groups $\mathrm{H}_+(X)$ and $\mathrm{H}(X)$ are a-minimal.*
(3) *X is aM_+-compact and aM-compact.*

Proof. Assertion (2) follows from (1) by applying Lemma 1.5. By Definition 1.2 assertion (3) is a reformulation of (2). So it is enough to prove (1).

Below G denotes one of the groups $\mathrm{H}_+(X)$ or $\mathrm{H}(X)$. Denote by τ_{co} the (compact-open) topology on G. By Lemma 1.5 it is equivalent to show that $\tau_{co} \subseteq \mathfrak{Z}_G$.

For every interval $(a, b) \subseteq X$ (with $a < b$) the group $\mathrm{H}_+[a,b]$ is nontrivial (condition (A)) and thus, by Lemma 2.5, this group is nonabelian. Taking into account Lemmas 2.5 and 2.4 choose $p, q \in \mathrm{H}_+(X)$ such that $pq \neq qp$ and $p(x) = q(x) = x$ for every $x \notin (a, b)$. Define

$$T(a,b) := \{g \in G : gpg^{-1} \text{ does not commute with } q\}. \tag{3.1}$$

Claim 3.5. $e \in T(a,b) \in \mathfrak{Z}_G$.

Proof. Indeed, rewrite the definition of $T(a,b)$ to obtain

$$T(a,b) = \{g \in G : (gpg^{-1})q(gpg^{-1})^{-1}q^{-1} \neq e\}$$

and use Definition 1.4 to conclude that $T(a,b) \in \mathfrak{Z}_G$. The fact that $e \in T(a,b)$ is trivial by the choice of p, q. □

Claim 3.6. *For every $g \in T(a,b)$ there exists $x \in (a,b)$ such that $g(x) \in (a,b)$. That is, $g(a,b) \cap (a,b) \neq \emptyset$ $\forall g \in T(a,b)$.*

Proof. Assuming the contrary, there exists $g \in T(a,b)$ such that $g(a,b) \cap (a,b) = \emptyset$. Equivalently, $(a,b) \cap g^{-1}(a,b) = \emptyset$. Hence, $g^{-1}(x) \notin (a,b)$ for every $x \in (a,b)$. By the choice of p we have $pg^{-1}(x) = g^{-1}(x)$ and so $gpg^{-1}(x) = x$ for every $x \in (a,b)$. On the other hand, $q(x) = x$ for every $x \in X \setminus (a,b)$ (by the choice of q). It follows that gpg^{-1} and q commute, which contradicts the definition of $T(a,b)$ in (3.1). □

Let α be the collection of all finite intersections of $T(a,b)$'s. By Claim 3.5 (using that $\mathfrak{Z}_G$ is a topology) we obtain $\alpha \subseteq \mathfrak{Z}_G$. Both τ_{co} and $\mathfrak{Z}_G$ are completely determined by the neighborhood base at $e \in G$. So, in order to see that $\tau_{co} \subseteq \mathfrak{Z}_G$ it suffices to show the following.

Claim 3.7. *Every open neighborhood U of e in G, with the compact-open topology τ_{co}, contains an element T from α.*

Proof. Let μ be the unique compatible uniformity on X. A basic neighborhood of e has the form:

$$O_\varepsilon := \{g \in G : (g(x), x) \in \varepsilon \quad \forall\, x \in X\},$$

where $\varepsilon \in \mu$. Choose a symmetric entourage $\varepsilon_1 \in \mu$ such that $\varepsilon_1^2 \subseteq \varepsilon$. For ε_1 by Lemma 3.3 choose an ε_1-connected net

$$c_0 < c_1 < \cdots < c_n$$

of X. We can suppose that X is nontrivial and $n > 0$.

By Equation (3.1), we have the corresponding $T(c_i, c_{i+1}) \subseteq G$ for every index $0 \leq i \leq n-1$. Define

$$T := \bigcap_{i=0}^{n-1} T(c_i, c_{i+1}).$$

Now it is enough to show:

$$T \subseteq O_\varepsilon. \tag{3.2}$$

Assuming the contrary let $h \in T$ but $h \notin O_\varepsilon$. Then there exists $x \in X$ such that $(h(x), x) \notin \varepsilon$. Pick minimal index k between 0 and $n-1$ such that $x \in [c_k, c_{k+1}]$. Then by a remark after Definition 3.2 we have $(x, y) \in \varepsilon_1^2 \subseteq \varepsilon$ for every $y \in [c_{k-1}, c_{k+2}]$. If $k = 0$, we replace c_{k-1} by c_0. Similarly, we replace c_{k+2} by c_n if $k = n-1$.

Hence,

$$h(x) \in X \setminus [c_{k-1}, c_{k+2}] = [c_0, c_{k-1}) \cup (c_{k+2}, c_n]. \tag{3.3}$$

Note that one of the intervals in the union can be empty.

From Claim 3.6 for every index $0 \leq i \leq n-1$ choose x_i such that

$$x_i, h(x_i) \in (c_i, c_{i+1}). \tag{3.4}$$

We show that there is no such $h \in G$. By Lemma 2.3 any autohomeomorphism $h \in \mathrm{H}(X)$ is either order-preserving or order-reversing. By Equation (3.4) we have $h(x_i) < h(x_{i+1})$, where $x_i < x_{i+1}$. So, h can be only order-preserving.

Now, we show that h is not order-preserving. Indeed, we have the following two cases:

(1) $h(x) \in (c_{k+2}, c_n]$.
Then, $h(x_{k+1}) < h(x)$, while $x < x_{k+1}$.
(2) $h(x) \in [c_0, c_{k-1})$.
Then, $h(x) < h(x_{k-1})$, while $x_{k-1} < x$.

In both cases we get a contradiction. This completes the proof of Equation (3.2) and hence of our theorem. □

Corollary 3.8. *Let $(X, \tau_{\leq})$ be a compact connected LOTS that satisfies the following condition:*

(C) *for every pair of elements $a < b$ in X there exist $c, d \in X$ with $a \leq c < d \leq b$ such that $[c, d]$ is separable (equivalently, the subspace $[c, d] \subseteq X$ is homeomorphic to the real unit interval $[0, 1]$).*

Then $\mathfrak{Z}_G = \mathfrak{M}_G = \tau_{co}$ and the groups $G = \mathrm{H}_+(X)$, $G = \mathrm{H}(X)$ are a-minimal (that is, X is aM_+-compact and aM-compact).

Proof. Recall (see, for example, [18, Exercise 6.3.2]) that a separable linearly ordered continuum is homeomorphic to $[0, 1]$. It is well known and easy to see that $\mathrm{H}_+[0, 1]$ is nonabelian (Section 4.4). Also, up to the inversion, there exists only one linear order on $[0, 1]$ inducing the natural topology [27, Cor. 4.1]. We see that (C) implies that $\mathrm{H}_+[c, d]$ (being a copy of $\mathrm{H}_+[0, 1]$) is nonabelian. So, we can apply Theorem 3.4. □

4. Some examples

4.1. Not every compact LOTS is M_+-compact

The following example shows that $\mathrm{H}_+(X)$ is not necessarily minimal.

Example 4.1. Denote by $\mathbb{Z}^*$ the two-point compactification of $\mathbb{Z}$. One can easily verify that $\mathrm{H}_+(\mathbb{Z}^*)$ is a discrete copy of $\mathbb{Z}$ and thus not minimal. That is, the compact LOTS $\mathbb{Z}^*$ is not M_+-compact. Note that $\mathbb{Z}^*$ is also not M-compact as it directly follows from [11, Theorem 4.25].

4.2. Rigid ordered compact spaces

Let us say that a topological space X is H-*rigid* if the group $\mathrm{H}(X)$ is trivial. Similarly, let us say that a linearly ordered space X is H_+-*rigid* if the group $\mathrm{H}_+(X)$ is trivial. Certainly, if X is H-rigid then it is also H_+-rigid. There are many known examples of H-rigid compact spaces, and in particular of compact ordered H-rigid spaces. Most of the examples of the latter kind (Jonsson, Rieger, de Groot-Maurice) are zero-dimensional. It seems that the first ("naive") example of a nontrivial *connected* compact ordered H-rigid space was constructed by Hart and van Mill [25]. Note also that, under the *diamond principle*, there exists an H-rigid Suslin continuum (Jensen, see in [37, p. 268]).

Using results of [25], one may show the following.

Proposition 4.2. *There exists an ordered continuum X with $\mathrm{H}_+(X) = \mathrm{H}(X) = \mathbb{Z}$, a discrete copy of the integers.*

So, we get a *connected* compact LOTS X such that $\mathrm{H}_+(X)$ is not minimal (or, X is not M_+-compact). Hence, Theorem 3.4 does not remain true for general ordered continua.

We sketch the proof of Proposition 4.2. Let $L := [a, b]$ be the ordered continuum constructed in [25, Section 5]. This space has very few continuous selfmaps. Any continuous map $f : L \to L$ is a *canonical retraction*. That is, there exists a pair $u \leq v \in L$ such that

$$f(x) = u \ \forall x \leq u, f(x) = x \ \forall u \leq x \leq v, \ f(x) = v \ \forall x \geq v.$$

In particular, L is H-rigid. Moreover, for every topological embedding $f : L \to L$ we have $f = id$. Note also the following *special property* which we use below: if $f(a) = a$ then either $f(x) = x$ for every $x \in U$ on some neighborhood U of a, or f is the constant map $f(x) = a$ for every $x \in L$.

Now the desired continuum X will be the two point compactification of some locally compact connected LOTS Y, the "long L". More precisely, the corresponding linearly ordered set Y is the lexicographically ordered set $\mathbb{Z} \times [a, b)$. Endow Y with its usual interval topology. Every subinterval in Y of the form

$$L_n := [(n, a), (n + 1, a)] = \{(n, x) : \ x \in [a, b)]\} \cup \{(n + 1, a)\}$$

is naturally order isomorphic with L for every $n \in \mathbb{Z}$. Our aim is to show that $\mathrm{H}_+(X) = \mathrm{H}(X) = \mathbb{Z}$. First of all we have a naturally defined (shift) homeomorphism $\sigma : X \to X$ where $\sigma(n, x) = (n + 1, x)$ for every $n \in \mathbb{Z}, x \in [a, b)$. We claim that any other homeomorphism $f : X \to X$ is σ^k (the k-th iteration) for some $k \in \mathbb{Z}$. Indeed, if $f(L_0) \subseteq L_k$ for some $k \in \mathbb{Z}$ then $f(L_0) = L_k$. Moreover it is easy to see that $f = \sigma^k$. Now assume that $f(L_0) \subseteq L_k$ is not true for every $k \in \mathbb{Z}$. Then there exists $k \in \mathbb{N}$ such that $f(0, a) < (k, a) < f(0, b)$. Consider the retraction

$$h : X \to X, h(z) = (k, a) \ \forall z \leq (k, a), \text{ and } h(z) = z \ \forall z > (k, a).$$

Then the composition $h \circ f$ restricted on L_0 defines a nonconstant continuous map $L_0 \to L_k$ which moves $(0, a)$ to (k, a). This induces a continuous nonconstant selfmap $q : L \to L$ such that $q(x) = a \quad \forall \ x \in U$ for some neighborhood U of a. By the *special property* of L mentioned above, we get a contradiction. These arguments show that algebraically $\mathrm{H}_+(X) = \mathrm{H}(X) = \mathbb{Z}$. Finally observe that $\mathrm{H}_+(X)$ is discrete in the compact-open topology.

4.3. The ordinal space

For every ordinal number κ the space $[0, \kappa]$ is a compact LOTS. This space is scattered and hence not connected for every $\kappa > 0$. Nonetheless, one can show that $\mathrm{H}_+[0, \kappa]$ is trivial (hence a-minimal). We start by noting that $[0, \kappa]$ is certainly a well-ordered set.

Lemma 4.3. *([7, Corollary 4.1.9]) If two well-ordered sets A and B are order-isomorphic, then the isomorphism is unique.*

It follows from Lemma 4.3 that the identity is the only order-preserving automorphism of a well-ordered set.

Corollary 4.4. *Every well-ordered compact LOTS X (e.g., the ordinal space $X = [0, \kappa]$) is H_+-rigid. That is, $\mathrm{H}_+(X) = \{e\}$ (thus X is aM_+-compact).*

This example shows that the condition of Theorem 3.4 is not necessary.

4.4. The unit interval

The group $\mathrm{H}_+[0,1]$ (and, hence, also any $\mathrm{H}_+[a,b]$ for every two reals $a < b$) is not abelian. Take, for example, the following pair f, h of noncommuting elements. Define $f(x) = x^2$, $h(x) = 0.5x$ for $0 \leq x \leq 0.5$ and $h(x) = 1.5x - 0.5$ for $0.5 \leq x \leq 1$. So, the continuum $[0,1]$ clearly satisfies the conditions of Theorem 3.4. Therefore, the groups $\mathrm{H}_+[0,1]$ and $\mathrm{H}[0,1]$ are a-minimal.

4.5. The ordered square

Let $I = [0,1]$ and define the lexicographic order on $I \times I$. Then $\mathcal{I}^2 = (I \times I, \tau_{\leq})$, the unit square with the order topology, is a compact and not metrizable space. We show that it satisfies the conditions of Corollary 3.8. It is connected (see [35, Section 48]). As to the second condition, let $K = [(a_1,b_1),(a_2,b_2)] \subseteq \mathcal{I}^2$ be a closed interval. If $a_1 = a_2$ then K is homeomorphic to $[0,1] \subseteq \mathbb{R}$. Otherwise, if $a_1 < a_2$, K contains an interval homeomorphic to $[0,1] \subseteq \mathbb{R}$ (for example $[(\frac{a_1+a_2}{2},0),(\frac{a_1+a_2}{2},1)]$). Thus condition (C) of Corollary 3.8 is satisfied. Hence, $\mathrm{H}_+(\mathcal{I}^2)$ and $\mathrm{H}(\mathcal{I}^2)$ are a-minimal (and $\mathcal{I}^2$ is both aM_+-compact and aM-compact).

4.6. The extended long line

Let $\mathcal{L}$ be the set $[0,\omega_1) \times [0,1)$ where ω_1 is the least uncountable ordinal. Considering $\mathcal{L}$ with the lexicographic order, the set $\mathcal{L}$ with the topology induced by this order is called *the long line*. Let $\mathcal{L}^* = \mathcal{L} \cup \{\omega_1\}$ and extend the ordering on $\mathcal{L}$ to $\mathcal{L}^*$ by letting $a < \omega_1$ for all $a \in \mathcal{L}$. The space $\mathcal{L}^*$ with the order topology is a compact space called the *extended long line*. In fact, $\mathcal{L}^*$ is the one point compactification of $\mathcal{L}$.

Several properties of this space can be found in [26,30,33]. The extended long line satisfies the conditions of Corollary 3.8. Indeed, it is well known that $\mathcal{L}^*$ is a compact connected LOTS. Also, $\mathcal{L}$ (the long line) is locally homeomorphic (by an order-preserving homeomorphism) to the interval $(0,1)$. In case the interval in question is of the form $[a,\omega_1]$, we can verify condition (C) for a subinterval $[a,b]$ of $[a,\omega_1]$, where $b \neq \omega_1$. So, $\mathrm{H}_+(\mathcal{L}^*)$ and $\mathrm{H}(\mathcal{L}^*)$ are a-minimal. Hence, $\mathcal{L}^*$ is both aM_+-compact and aM-compact.

4.7. The circle

Recall the definition of the natural cyclic ordering (Definition 2.6) on the unit circle $\mathbb{S}^1$. Identify $\mathbb{S}^1$, as a set, with $[0,1)$ and define a ternary relation $R \subseteq [0,1)^3$ as follows: $(z,y,x) \in R$ if and only if $(x-y)(y-z)(x-z) > 0$. Denote by $\mathrm{H}_+(\mathbb{S}^1)$ the Polish group of all orientation preserving homeomorphisms of the circle $\mathbb{S}^1$.

The arguments of Theorem 3.4 (or, of [21, Theorem 1]) can be easily modified for the circle $\mathbb{S}^1$, hence:

Theorem 4.5. *The group* $\mathrm{H}_+(\mathbb{S}^1)$ *is a-minimal.*

Note that the coset space $\mathrm{H}_+(\mathbb{S}^1)/\mathrm{St}(z)$ is naturally homeomorphic to the circle, where $\mathrm{St}(z)$ is the stabilizer group of any given $z \in \mathbb{S}^1$. So the minimality of $\mathrm{H}_+(\mathbb{S}^1)$ can be derived from the minimality of $\mathrm{H}_+[0,1]$ using Lemma 2.2 and the fact that $\mathrm{St}(z)$ is topologically isomorphic to $\mathrm{H}_+[0,1]$.

Since $\mathrm{H}_+(\mathbb{S}^1)$ is a closed normal subgroup of $\mathrm{H}(\mathbb{S}^1)$, and $\mathrm{H}(\mathbb{S}^1)/\mathrm{H}_+(\mathbb{S}^1) \cong \mathbb{Z}_2$, we can use Lemma 2.2 one more time to deduce the minimality of $\mathrm{H}(\mathbb{S}^1)$.

A Hausdorff topological group is *totally minimal* if every Hausdorff quotient is minimal [13]. Every minimal algebraically (or, at least, topologically) simple minimal group is totally minimal. $\mathrm{H}_+(\mathbb{S}^1)$ is algebraically simple as can be seen (for example) in [34,23]. Although the group $\mathrm{H}_+[0,1]$ is not algebraically simple, it is topologically simple. Indeed, by [19, Theorem 14], $\mathrm{H}_+[0,1]$ has exactly five normal subgroups: $\{e\}$, $\mathrm{H}_+[0,1]$,

Q_1, Q_0, $Q := Q_0 \cap Q_1$. It is easy to see that Q is dense in $\mathrm{H}_+[0,1]$. This yields that $\mathrm{H}_+[0,1]$ is topologically simple.

Corollary 4.6. $\mathrm{H}_+(\mathbb{S}^1)$ *and* $\mathrm{H}_+[0,1]$ *are totally minimal groups.*

5. Some questions

A more general version of Question 1.3 is the following.

Question 5.1. When appropriate subgroups G of $\mathrm{H}(X)$ (say, the automorphism groups of some structures on X) are minimal (a-minimal)?

We already know that the Cantor cube 2^ω is M-compact [20].

Question 5.2.

1. Is the Cantor cube 2^ω aM-compact?
2. Is the Cantor set $X \subseteq [0,1]$, as a linearly ordered compact LOTS, M_+-compact? aM_+-compact?
3. Is the space 2^λ M-compact (or, aM-compact) for every cardinal λ?

Question 5.3.

1. Is it true that every M-compact space is also aM-compact?
2. Is it true that every linearly ordered connected M_+-compact space is aM_+-compact?
3. Is it true that for ordered continua condition (A) of Theorem 3.4 is really weaker than condition (C) of Corollary 3.8?

In view of Markov's Question 1.6 and Theorem 3.4 we have several good reasons to pose the following question.

Question 5.4. For what compact (linearly ordered) spaces X the Markov and Zariski topologies coincide on the group $G = \mathrm{H}(X)$ (resp., $G = \mathrm{H}_+(X)$)?

Various properties of the homeomorphism group $\mathrm{H}(X)$ of several important 1-dimensional continua X were intensively studied from several points of view. Among others is the case where X is the *pseudo-arc* or the *Lelek Fan*. About the latter case, see, for example, the very recent works of Bartošova–Kwiatkowska [3,4] and Ben Yaacov–Tsankov [5].

Question 5.5. Let X be the pseudo-arc or the Lelek fan. Is it true that $\mathrm{H}(X)$ is minimal? a-minimal?

It is well known that the pseudo-arc is a homogeneous compactum. So the previous question is related to Stoyanov's Question 1.1. Another property of the pseudo-arc is that it is a chainable continuum. Recall that a compact space X is *chainable* if every (finite) open cover ε has a finite open refinement α that is an ε-*small chain*, that is, $\alpha = \{O_1, \dots, O_n\}$, where $O_i \cap O_j \neq \emptyset \Leftrightarrow |i-j| \leq 1$ and every O_i is ε-small. Every linearly ordered continuum is chainable. This follows, for example, by Lemma 3.3. Therefore, it would be interesting to extend Theorem 3.4 to some broader class of chainable continua.

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