A NOTE ON SENSITIVITY OF SEMIGROUP ACTIONS

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Abstract. It is well known that for a transitive dynamical system (X, f) sensitivity to initial conditions follows from the assumption that the periodic points are dense. This was done by several authors: Banks, Brooks, Cairns, Davis and Stacey [3], Silverman [10] and Glasner and Weiss [8]. In the latter article Glasner and Weiss established a stronger result (for compact metric systems) which implies that a transitive non-minimal compact metric system (X, f) with dense set of almost periodic points is sensitive. This is true also for group actions as was proved in the book of Glasner [6].

Our aim is to generalize these results in the frame of a unified approach for a wide class of topological semigroup actions including one-parameter semigroup actions on Polish spaces.

1. Introduction

Sensitive dependence on initial conditions is one of the basic ideas in several definitions of chaos (see, for example, [2, 5, 8, 4]).

Definition 1.1. (Devaney's definition of chaos, [5])

Let X be a metric space. A continuous map $f : X \to X$ is said to be chaotic (in the sense of Devaney) on X if the following conditions are fulfilled:

- f is transitive;
- the periodic points of f are dense in X;
- f has sensitive dependence on initial conditions.

It is now well known that the third condition follows from the first two conditions. Recall some relevant results regarding sensitivity of dynamical systems.

Theorem 1.2. (1) (Banks, Brooks, Cairns, Davis and Stacey [3]; Silverman [10] [1]) *Let X be an infinite metric space and $f : X \to X$ be continuous. If f is topologically transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

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[1] replacing the topological transitivity by the point transitivity.

(2) (Glasner and Weiss [8, Theorem 1.3]); see also Akin, Auslander and Berg [1]) *Let X be a compact metric space and (X, f) a non-minimal* M-system. *Then (X, f) is sensitive.*

(3) (Glasner [6, Theorem 1.41]) *Let X be a compact metric space. An almost equicontinuous M-system (G, X), where G is a group, is minimal and equicontinuous. Thus an M-system which is not minimal equicontinuous is sensitive.*

Topological transitivity of $f : X \to X$ as usual, means that for every pair U and V of nonempty subsets of X there exists $n > 0$ such that $f^n(U) \cap V$ is nonempty. A general semigroup action version can be defined analogously (see Definition 3.1.1).

If X is a compact metric space then (2) easily covers (1). In order to explain this recall that an *M-system* means that the set of almost periodic points is dense in X (the *Bronstein condition*) and, in addition, the system is topologically transitive. A very particular case of the Bronstein condition is that X has dense periodic points (the so-called *P-systems*). If now X is infinite then it cannot be minimal.

Our aim is to provide a unified and generalized approach for studying the sensitivity condition and closely related concepts like almost equicontinuity and almost periodicity. We show that (2) and (3) remain true for a large class of *C-semigroups* (which contains cascades, topological groups and one-parameter semigroups) and M-systems (see Definitions 2.1 and 4.1). Our approach allows us also to drop the compactness assumption of X and consider Polish phase spaces. A topological space is *Polish* if it admits a separable complete metric.

We cover also (1) in the case of Polish phase spaces.

Here we formulate one of the main results (Theorem 5.7) of the present article.

Main result: *Let (S, X) be a dynamical system where X is a Polish space and S is a C-semigroup. If X is an M-system which is not minimal or not equicontinuous, then X is sensitive.*

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2. PRELIMINARIES

A *dynamical system* in the present article is a triple (S, X, π), where S is a topological semigroup, X is at least a Hausdorff space and

$$\pi : S \times X \to X, \quad (s, x) \mapsto sx$$

is a continuous action on X. Thus, $s_1(s_2x) = (s_1s_2)x$ holds for every triple (s_1, s_2, x) in $S \times S \times X$. Sometimes we write the dynamical system as a pair (S, X) or even as X, when S is understood. The *orbit* of x is the set $Sx := \{sx : s \in S\}$. By $\overline{A}$ we will denote the closure of a subset $A \subset X$. If (S, X) is a system and Y a closed S-invariant subset, then we say that (S, Y), the restricted action, is a *subsystem* of (S, X). For $U \subset X$ and $s \in S$ denote

$$s^{-1}U := \{x \in X : sx \in U\}.$$

If $S = \{f^n\}_{n \in \mathbb{N}}$ (with $\mathbb{N} := \{1, 2, \cdots\}$) and $f : X \to X$ is a continuous function, then the classical dynamical system (S, X) is called a *cascade*. We use the standard notation: (X, f).

Definition 2.1. Let S be a topological semigroup.
(1) We say that S is a *F-semigroup* if for every $s_0 \in S$ the subset $S \setminus Ss_0 = \{s \in S : \ s \notin Ss_0\}$ is finite.
(2) We say that S is a *C-semigroup* if $S \setminus Ss_0$ is relatively compact (that is, its closure is compact) in S.

Example 2.2.
(1) The standard one-parameter additive semigroup $S := ([0, \infty), +)$ is a C-semigroup.
(2) Every cyclic "positive" semigroup $M := \{s^n : n \in \mathbb{N}\}$ is an F-semigroup. In particular, for every cascade (X, f) the corresponding semigroup $S = \{f^n\}_{n \in \mathbb{N}}$ is an F-semigroup (and hence also a C-semigroup).
(3) Every topological group is of course an F-semigroup.
(4) Every compact semigroup is a C-semigroup.

Definition 2.3. Let (S, X) be a dynamical system where (X, d) is a metric space.
(1) A subset A of S acts *equicontinuously* at $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies $d(ax_0, ax) < \epsilon$ for every $a \in A$.
(2) A point $x_0 \in X$ is called an *equicontinuity point* (notation: $x_0 \in Eq(X)$) if $A := S$ acts equicontinuously at x_0. If $Eq(X) = X$ then (S, X) is *equicontinuous*.
(3) (S, X) is called *almost equicontinuous* (see [1, 6]) if the subset $Eq(X)$ of equicontinuity points is a dense subset of X.

The following lemma is standard.

Lemma 2.4. *Let (S, X) be a dynamical system where (X, d) is a metric space. Let $A \subset S$ be a relatively compact subset. Then A acts equicontinuously on (X, d).*

3. Transitivity conditions of semigroup actions

Definition 3.1. The dynamical system (S, X) is called:
(1) *topologically transitive* (in short: TT) if for every pair (U, V) of non-empty open sets U, V in X there exists $s \in S$ such that

$U \cap sV \neq \varnothing$. Since $s(s^{-1}U \cap V) = U \cap sV$, it is equivalent to say that $s^{-1}U \cap V \neq \varnothing$.

(2) *point transitive* (PT) if there exists a point x_0 with dense orbit $Sx_0 := \{sx_0 : s \in S\}$. Such a point is called *transitive point.* Notation: $x_0 \in Trans(X)$.

(3) *densely point transitive* (DPT) if there exists a dense set $Y \subset X$ of transitive points.

Of course always (DPT) implies (PT). In general, (TT) and (PT) are independent properties. For a detailed discussion of transitivity conditions (for cascades) see a review paper by Kolyada and Snoha [9].

As usual, X is *perfect* means that X is a space without isolated points. Assertions (1) and (2) in the following proposition are very close to Silverman's observation [10, Proposition 1.1] (for cascades).

Proposition 3.2. (1) *If X is a perfect topological space and S is an F-semigroup, then (PT) implies (TT).*

(2) *If X is a Polish space then every (TT) system (S, X) is (DPT) (and hence also (PT)).*

(3) *Every (DPT) system (S, X) is (TT).*

Proof. (1) Let x be a transitive point with orbit Sx. Now, let U and V be nonempty open subsets of X. There exists $s_1 \in S$ such that $s_1x \in V$. The subset $S \setminus Ss_1$ is finite because S is an almost F-group. Since X is perfect, removing the finite subset $(S \setminus Ss_1)x$ from the dense subset Sx we get again a dense subset. Therefore, Ss_1x is a dense subset of X. Then there exists $s_2 \in S$ such that $s_2s_1x \in U$. Thus $s_2^{-1}U \cap V \neq \varnothing$. By Definition 3.1.1 this means that (S, X) is a (TT) dynamical system.

(2) If (S, X) is topologically transitive, then $S^{-1}U$ is a dense subset of X for every open set U. We know that X is Polish. Then there exists a countable open base $\mathcal{B}$ of the given topology. By the Baire theorem, $\bigcap\{S^{-1}U : U \in \mathcal{B}\}$ is dense in X and every point of this set is a transitive point of the dynamical system X.

(3) Let U and V be nonempty open subsets in X. Since the set Y of point transitive points is dense in X, it intersects V. Therefore, we can choose a transitive point $y \in V$. Now by the transitivity of y there exists $s \in S$ such that sy belongs to U. Hence, sy is a common point of U and sV. □

Lemma 3.3. *Let (X, d) be a metric S-system which is (TT). Then $Eq(X) \subset Trans(X)$.*

Proof. Let $x_0 \in Eq(X)$ and $y \in X$. We have to show that the orbit Sx_0 intersects the ε-neighborhood $B_\varepsilon(y) := \{x \in X : d(x, y) < \varepsilon\}$ of y for every given $\varepsilon > 0$. Since $x_0 \in Eq(X)$ there exists a neighborhood U of x_0 such that $d(sx_0, sx) < \frac{\varepsilon}{2}$ for every $(s, x) \in S \times U$. Since X is (TT) we can choose $s_0 \in S$ such that $s_0U \cap B_{\frac{\varepsilon}{2}}(y) \neq \emptyset$. This means that $d(s_0x, y) < \frac{\varepsilon}{2}$ for some $x \in U$. Then $d(s_0x_0, y) < \varepsilon$. □

4. Minimality conditions

The following definitions are standard for compact X.

Definition 4.1. Let X be a not necessarily compact S-dynamical system.

(1) X is called *minimal*, if $\overline{Sx} = X$ for every $x \in X$. In other words, all points of X are transitive points.
(2) A point x is called *minimal* if the subsystem $\overline{Sx}$ is minimal.
(3) A point x is called *almost periodic* if the subsystem $\overline{Sx}$ is minimal and compact.
(4) If the set of almost periodic points is dense in X, we say that (S, X) satisfies the *Bronstein condition.* If, in addition, the system (S, X) is (TT), we say that it is an *M-system.*
(5) Let (X, f) be a cascade. As usual, a point $x \in X$ is *periodic*, if there exists a natural $n \in \mathbb{N}$ such that $f^n x = x$. If X is a (TT) cascade and the set of periodic points is dense in X, then we say that it is a *P-system*, [8].

If X is compact then a point in X is minimal iff it is almost periodic. Every periodic point is of course almost periodic. Therefore it is also obvious that every P-system is an M-system.

For a system (S, X) and a subset $B \subset X$, we use the following notation

$$N(x, B) = \{s \in S : sx \in B\}.$$

The following definition is also standard.

Definition 4.2. A subset $P \subset S$ is (left) *syndetic*, if there exists a finite set $F \subset S$ such that $F^{-1}P = S$.

The following lemma is a slightly generalized version of a well known criterion for almost periodic points (cf. Definition 4.1.3).

Lemma 4.3. *Let X be a regular (not necessarily compact) S-dynamical system and $x_0 \in X$. Consider the following conditions:*

(1) *x_0 is an almost periodic point.*
(2) *For every open neighborhood V of x_0 in X there exists a finite set $F \subset S$ such that $F^{-1}V \supseteq Y := \overline{Sx_0}$.*
(3) *For every neighborhood V of x_0 in X the set $N(x_0, V)$ is syndetic.*
(4) *x_0 is a minimal point (i.e., the subsystem $\overline{Sx_0}$ is minimal).*

Then (1) $\Rightarrow$(2) $\Rightarrow$ (3) $\Rightarrow$ (4).
If X is compact then all four conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) : Suppose that (S, Y) is minimal and compact. Then for every open neighborhood V of x_0 in X and for every $y \in Y$ there exists $s \in S$ such that $sy \in V$. Equivalently, $y \in s^{-1}V$. Therefore, $\bigcup_{s\in S} s^{-1}V \supseteq Y$. By compactness of Y we can choose a finite set $F \subseteq S$ such that $F^{-1}V \supseteq Y$.

(2) $\Rightarrow$ (3) : It suffices to show that $F^{-1}N(x_0, V) = S$, where F is a subset of S defined in (2). Assume otherwise, so that there exists $s \in S$ such that $s \notin F^{-1}N(x_0, V)$. Then $sx_0 \notin F^{-1}V$. On the other hand clearly, $sx_0 \in Y$, contrary to our condition that $F^{-1}V \supseteq Y$.

(3) $\Rightarrow$ (4) : $Y = \overline{Sx_0}$ is non-empty, closed and invariant. It remains to show that if $y \in Y$ then $x_0 \in \overline{Sy}$. Assume otherwise, so that $x_0 \notin \overline{Sy}$. By the regularity of X we can choose an open neighborhood V of x_0 in X such that $\overline{V} \cap \overline{Sy} = \varnothing$. By our assumption the set $N(x_0, V)$ is syndetic. Therefore there is a finite set $F := \{s_1, ..., s_n\}$ so that for each $s \in S$ some $s_i s x_0 \in V$. That is each sx_0 belongs to $F^{-1}V = \bigcup_{i=1}^n s_i^{-1}V$ for every $s \in S$. Hence, $Sx_0 \subseteq \bigcup_{i=1}^n s_i^{-1}V$. Then

$$y \in \overline{Sx_0} \subset \overline{\bigcup_{i=1}^n s_i^{-1}V} = \bigcup_{i=1}^n \overline{s_i^{-1}V} \subset \bigcup_{i=1}^n s_i^{-1}\overline{V}.$$

But then $\overline{V} \cap Sy \neq \varnothing$ contrary to our assumption.

If X is compact then by Definition 4.1 it follows that (4) $\Rightarrow$ (1). $\square$

5. Sensitivity and other conditions

Proposition 5.1. *Let S be an C-semigroup. Assume that (X, d) is a point transitive S-system such that $Eq(X) \neq \emptyset$. Then every transitive point is an equicontinuity point. That is, $Trans(X) \subset Eq(X)$ holds.*

Proof. Let y be a transitive point and $x \in Eq(X)$ be an equicontinuity point. We have to show that $y \in Eq(X)$. For a given $\varepsilon > 0$ there exists a neighborhood $O(x)$ of x such that

$$d(sx'', sx') < \epsilon \qquad \forall\ s \in S \quad \forall\ x', x'' \in O(x).$$

Since y is a transitive point then there exists $s_0 \in S$ such that $s_0 y \in O(x)$. Then $O(y) := s_0^{-1}O(x)$ is a neighborhood of y. We have

$$d(ss_0y', ss_0y'') < \epsilon \qquad \forall\ s \in S \quad \forall\ y', y'' \in O(y).$$

Since S is a C-semigroup the subset $M := \overline{S\backslash Ss_0}$ is compact. Hence by Lemma 2.4 it acts equicontinuously on X. We can choose a neighborhood $U(y)$ of y such that

$$d(ty', ty'') < \epsilon \qquad \forall\ t \in M \quad \forall\ y', y'' \in U(y).$$

Then $V := O(y) \cap U(y)$ is a neighborhood of y. Since $S = M \cup Ss_0$ we obtain that $d(sy', sy'') < \epsilon$ for every $s \in S$ and $y', y'' \in V$. This proves that $y \in Eq(X)$. $\square$

Proposition 5.2. *Let S be an C-semigroup. Assume that (X, d) is a metric S-system which is minimal and $Eq(X) \neq \emptyset$. Then X is equicontinuous.*

Proof. If (S, X) is a minimal system then $Trans(X) = X$. Then if $Eq(X) \neq \emptyset$ every point is an equicontinuity point by Proposition 5.1. Thus, $Eq(X) = X$. □

Proposition 5.3. *Let S be an C-semigroup. Assume that (X, d) is a Polish (TT) S-system. Then X is almost equicontinuous if and only if $Eq(X) \neq \emptyset$.*

Proof. X is (DPT) by Proposition 3.2.2. That is, $Trans(X)$ is dense in X. Assuming that $Eq(X) \neq \emptyset$ we obtain by Proposition 5.1 that $Trans(X) \subset Eq(X)$. It follows that $Eq(X)$ is also dense in X. Thus, X is almost equicontinuous. This proves "if" part. The remaining direction is trivial. □

The following natural definition plays a fundamental role in many investigations about chaotic systems. The present form is a generalized version of existing definitions for cascades (see also [8, 7]).

Definition 5.4. (sensitive dependence on initial conditions) A metric S-system (X, d) is *sensitive* if it satisfies the following condition: there exists a (*sensitivity constant*) $c > 0$ such that for all $x \in X$ and all $\delta > 0$ there are some $y \in B_\delta(x)$ and $s \in S$ with $d(sx, sy) > c$.

We say that (S, X) is *non-sensitive* otherwise.

Proposition 5.5. *Let S be an C-semigroup. Assume that (X, d) is a (TT) Polish S-system. Then the system is almost equicontinuous if and only if it is non-sensitive.*

Proof. Clearly an almost equicontinuous system is always non-sensitive.

Conversely, the non-sensitivity means that for every $n \in \mathbb{N}$ there exists a nonempty open subset $V_n \subset X$ such that

$$diam(sV_n) < \frac{1}{n} \quad \forall\ (s, n) \in S \times \mathbb{N}.$$

Define

$$U_n := S^{-1}V_n \qquad R := \bigcap_{n \in \mathbb{N}} U_n.$$

Then every U_n is open. Moreover, since X is (TT), for every nonempty open subset $O \subset X$ there exists $s \in S$ such that $O \cap s^{-1}U_n \neq \emptyset$. This means that every U_n is dense in X. Consequently, by Baire theorem (making use that X is Polish), R is also dense. It is enough now to show that $R \subset Eq(X)$. Suppose $x \in R$ and $\epsilon > 0$. Choose n so that $\frac{1}{n} < \epsilon$; then $x \in U_n$ implies the existence of $s_0 \in S$ such that $s_0 x \in V_n$.

Put $V = s_0^{-1}V_n$. Therefore for $y \in V$ and every $s := s's_0 \in Ss_0$ we get

$$d(sx, sy) = d(s's_0x, s's_0y) < \frac{1}{n} < \epsilon.$$

But $S \setminus Ss_0$ is relatively precompact in S because S is a C-semigroup. Then by Lemma 2.4 the set $S \setminus Ss_0$ acts on (X, d) equicontinuously. We have an open neighborhood O of x such that $d(sx, sy) < \epsilon$ holds for all $y \in O$ and for every $s \in \overline{S \setminus Ss_0}$. Define an open neighborhood $M := O \cap V$ of x. Then $d(sx, sy) < \epsilon$ for every $s \in S$ and all $y \in M$. Thus, $x \in Eq(X)$. □

Theorem 5.6. *Let (X, d) be a metric S-system where S is a C-semigroup. If X is an M-system and $Eq(X) \neq \emptyset$ then X is minimal and equicontinuous.*

Proof. Let $x_0 \in X$ be an equicontinuity point. Since every M-system is (TT), by Lemma 3.3 we know that $x_0 \in Trans(X)$. Thus, $\overline{Sx_0} = X$. Therefore, for the minimality of X it is enough to show that x_0 is a minimal point.

Since $x_0 \in Eq(X)$, given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < \delta < \frac{\epsilon}{2}$ and $x \in B_\delta(x_0)$ implies $d(sx_0, sx) < \frac{\epsilon}{2}$ for every $s \in S$. Since X is an M-system the set Y of all almost periodic points is dense. Choose $y \in B_\delta(x_0) \cap Y$. Then the set

$$N(y, B_\delta(x_0)) := \{s \in S : sy \in B_\delta(x_0)\}$$

is a syndetic subset of S by Lemma 4.3. Clearly, $N(y, B_\delta(x_0))$ is a subset of the set

$$N(x_0, B_\epsilon(x_0)) = \{s \in S : d(sx_0, x_0) \leq \varepsilon\}.$$

Then $N_\varepsilon := N(x_0, B_\epsilon(x_0))$ is also syndetic (for every given $\varepsilon > 0$). Using one more time Lemma 4.3 we conclude that x_0 is a minimal point, as desired. Now the equicontinuity of X follows by Proposition 5.2. □

Theorem 5.7. *Let (X, d) be a Polish S-system where S is a C-semigroup. If X is an M-system which is not minimal or not equicontinuous, then X is sensitive.*

Proof. If X is non-sensitive then by Proposition 5.5 the system is almost equicontinuous. Theorem 5.6 implies that X is minimal and equicontinuous. This contradicts our assumption. □

Now if the action is a cascade (X, f) or if S is a topological group (both are the case of C-semigroups, see Example 2.2) then we get, as a direct corollary, assertions (2) and (3) of Theorem 1.2. The assertion (1) is also covered in the case of Polish phase spaces X. Furthermore the main results are valid for a quite large class of actions including the actions of one-parameter semigroups on Polish spaces.

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