SOME INTRODUCTORY EXERCISES IN TOPOLOGICAL GROUPS
07.02.17

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1.

Exercise 1.1.

(1) Show that every normed (in fact, any seminormed) space \((E, +, \| \cdot \|)\) is a topological group and the family \(\{B_n(0)\}_{n \in \mathbb{N}}\), with \(B_n(0) := \{x \in E : \|x\| < \frac{1}{n}\}\) is a local base at 0.

(2) Show that \((\mathbb{Z}, +, d_5)\), where \(d_5\) is the 5-adic metric \(^1\) is a Hausdorff topological ring and give an example of its local base at 0.

(3) Every group \(G\) with the trivial topology \(\tau_{tr} := \{\emptyset, G\}\) is a topological group. The same is true with respect to the discrete topology \(\tau_{discr} := \mathcal{P}(G)\).

(4) Give an example of a topological group with a nontrivial topology which is not Hausdorff.

Proof. (1) First a general remark: if \(d\) is a metric on a group \(G\) then the continuity of operations can be verified in terms of converging sequences (Heine Principle).

If \(\lim u_n = u, \lim v_n = v\) then \(\lim (u_n + v_n) = u + v\) as it easily follows from the triangle inequality \(\| (u_n + v_n) - (u + v) \| \leq \|u_n - u\| + \|v_n - v\|\). Also, \(\lim (-u_n) = -u\) because \(\| -x \| = \|x\|\).

(2) the standard ultra-metric \(d_5\) is invariant under translations. That is, \(d_5(a + x, a + y) = d(x, y)\). So, \(d_5(x_n, x) = d_5(x_n - x, 0)\). Now in order to check the continuity of operations one may use the same idea as in (1).

The family of open subgroups (in fact, ideals)
\[
\{5\mathbb{Z}, 5^2\mathbb{Z}, 5^3\mathbb{Z}, \ldots\}
\]
is a local base at 0.

(3) Recall that any function into a set with the trivial topology is continuous. Any function from a set with the discrete topology is continuous.

(4) Take any seminormed space which is not a normed space. For example, \(\mathbb{R}^2\) with the seminorm \(\|(x, y)\| := |x|\).

\[\square\]

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\(^1\)Recall that for every distinct \(x, y \in \mathbb{Z}\) we have \(d_5(x, y) = \frac{1}{k(x, y)}\), where \(k = k(x, y) := \max\{i : 5^i|(x - y)\}\)
**Exercise 1.2.** Show that there exist Hausdorff topological groups \((G_1, \tau_1)\) and \((G_2, \tau_2)\) and a continuous onto injective homomorphism \(f : (G_1, \tau_1) \to (G_2, \tau_2)\) such that \(f\) is not a homeomorphism (that is, \(f^{-1}\) is not continuous). This shows that not every continuous algebraic isomorphism is an isomorphism in the category \(TGr\) of topological groups.

**Proof.** Take for example \(id : (\mathbb{Z}, \tau_{\text{discr}}) \to (\mathbb{Z}, \tau(d_5))\). □

Let \(G\) be a topological group.

**Exercise 1.3.** Prove that

1. Every left translation \(l_a : G \to G\) is a homeomorphism (provide a detailed formal proof).
2. The functions
   \[ G^n \to G, (x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \cdots x_n \]
   \[ G \to G, x \mapsto x^k \]
   \[ G \times G \to G, (x, y) \mapsto [x, y] := xy x^{-1} y^{-1} \]
   are continuous for every given \(n \in \mathbb{N}, k \in \mathbb{Z}\).
3. For every nbd \(U \in N(e)\) of the identity \(e \in G\) and every given \(n \in \mathbb{N}\) there exists \(V \in N(e)\) such that and \(V^n := \underbrace{V \cdots V}_n \subset U\).

**Proof.** (1) The restriction \(r : \{a\} \times G \to G\) of the continuous map \(m : G \times G \to G\) on the subspace \(\{a\} \times G\) is continuous. Consider the natural homeomorphism \(u : G \to \{a\} \times G\). Now observe that the composition \(r \circ u\) is exactly \(l_a\). As to the continuity of the inverse note that \(l_a^{-1} = l_{a^{-1}}\).

(2) (Sketch) First of all check by induction that \(G \to G, x \mapsto x^k\) is continuous for every given \(k \in \mathbb{Z}\) (consider separately two cases of natural \(k\) and \(k < 0\)). Then use again induction for \(n \in \mathbb{N}\) to complete the proof in general.

The function
\[ G \times G \to G, (x, y) \mapsto [x, y] := xy x^{-1} y^{-1} \]
can be represented as a composition \(m \circ f_2 \circ f_1\) of continuous functions
\[ f_1 : G \times G \to G^4, (x, y) \mapsto (x, y, x^{-1}, y^{-1}) \]
\[ f_2 : G^4 \to G^2, (a, b, c, d) \mapsto (ab, cd) \]

Alternative (direct) proofs are available also in terms of neighborhoods.

(2) By (1) For every nbd \(U \in N(e)\) of the identity \(e \in G\) and every given natural \(n \in \mathbb{N}\) there exists \(W \in N(e)\) such that and \(W^n \subset U\). Now take the symmetric nbd \(V := W \cap W^{-1}\). □

**Exercise 1.4.** Prove that

1. \(G\) is *strongly homogeneous* as a topological space in the following sense: For every pair \((x, y) \in G \times G\) there exists \(f \in H(G)\) such that \(f(x) = y\) and \(f(y) = x\).
2. Which of the following topological spaces are of the *group type* \(^2\):

\(^2\)Let’s say that a topological space \(X\) is of *group type* if it is homeomorphic to a topological group. Equivalently, if \(X\) admits a group operation which makes it a topological group.
(a) \((0, 1)\);
(b) \(\mathbb{R} \times \mathbb{N}\);
(c) \([0, 1]\);
(d) \(X := \{x \in \mathbb{R}^2 : \|x\| = 5\}\);
(e) \(X := \{x \in \mathbb{R}^3 : \|x\| < 5\}\);
(f) * The integers \(\mathbb{Z}\) with the cofinite topology.

Proof. (1) Consider the homeomorphism \(f : G \to G, f(g) = xg^{-1}y\)

(2)
(a) Yes.
(0, 1) is homeomorphic to \(\mathbb{R}\).
(b) Yes.
\(\mathbb{R} \times \mathbb{N}\) is homeomorphic to \(\mathbb{R} \times \mathbb{Z}\).
(c) No.
\([0, 1]\) is not homogeneous (so it cannot be homeomorphic to a topological group).
(d) Yes.
Homeomorphic to the circle group \(T := \{z \in \mathbb{C} : \|z\| = 1\}\).
(e) Yes.
The open ball \(X := \{x \in \mathbb{R}^3 : \|x\| < 5\}\) is homeomorphic to \(\mathbb{R}^3\). Indeed, the desired homeomorphism is
\[
f : \mathbb{R}^3 \to X, f(x) = \frac{5x}{1 + \|x\|}
\]
Similar arguments work in every normed space for every open ball...
(f) No.
\(\mathbb{Z}\) with the cofinite topology is \(T_1\) but not \(T_2\) (but every \(T_1\) topological group must be \(T_2\)). \(\square\)

Exercise 1.5. Prove that
(1) \(\text{cl}(A^{-1}) = \text{cl}(A)^{-1}\) and \(\text{cl}(A)\text{cl}(B) \subset \text{cl}(AB)\) for every subsets \(A, B\) of \(G\).
(2) If \(H \leq G\) is a subgroup then \(\text{cl}(H) \leq G\) is also a subgroup.
(3) If \(H \trianglelefteq G\) is a normal subgroup then \(\text{cl}(H) \trianglelefteq G\) is also a normal subgroup.
(4) If \(G\), in addition, is abelian and \(H \leq G\) then \(\text{cl}(H) \leq G\) is also an abelian subgroup. Give a counterexample if \(G\) is not Hausdorff.

Proof. (1) The inversion is a homeomorphism. Every homeomorphism preserves the closure operator. This explains why \(\text{cl}(A^{-1}) = \text{cl}(A)^{-1}\).

Let \(x \in \text{cl}(A), y \in \text{cl}(B)\). We have to show that \(xy \in \text{cl}(AB)\). Let \(U \in N(xy)\). By the continuity of the multiplication there exist \(V \in N(x), W \in N(y)\) such that \(VW \subset U\). By our assumptions, \(V \cap A \neq \emptyset, W \cap B \neq \emptyset\). Then \(VW \cap AB \neq \emptyset\). Since \(VW \subset U\) we obtain \(U \cap AB \neq \emptyset\).

Remark: In fact, \(\text{cl}(A)\text{cl}(B) \subset \text{cl}(AB)\) for every semitopological semigroup. Indeed, it is easy to see that \(\text{cl}(A)B \subset \text{cl}(AB)\) for every right topological semigroup. Similarly, \(A\text{cl}(B) \subset \text{cl}(AB)\) for every left topological semigroup. Now use that always \(\text{cl}(\text{cl}(AB)) = \text{cl}(AB)\).

(2) \(\text{cl}(H)\text{cl}(H) \subset \text{cl}(HH) = \text{cl}(H)\) and \(\text{cl}(H)^{-1} = \text{cl}(H^{-1}) = \text{cl}(H)\).

\(^3\)It follows that the closure of a subsemigroup in any semitopological semigroup is a subsemigroup. It is not always true for right topological semigroups.
(3) Let $H \trianglelefteq G$. Then $f_a(H) = H$ for every conjugation $f_a : G \to G, g \mapsto aga^{-1}$. Now use the fact that every conjugation is a homeomorphism for every topological group. So, $f_a(\text{cl}(H)) = \text{cl}(f_a(H)) = \text{cl}(H)$.

(4) The function $f : G \times G \to G, (x, y) \mapsto [x, y] := xyx^{-1}y^{-1}$ is continuous. Since $H$ is abelian, the restriction $f_H : H \times H \to H$ is the constant function. Namely, $f(h_1, h_2) = e$ for every $h_1, h_2 \in H$. Consider the subgroup $\text{cl}(H)$ and the restriction $f_{\text{cl}(H)} : \text{cl}(H) \times \text{cl}(H) \to \text{cl}(H)$ of $f$. Since $H$ is dense in $\text{cl}(H)$ and $\text{cl}(H)$ is Hausdorff, $f_{\text{cl}(H)}$ should also be the constant function.

Hausdorff property is essential. Indeed, take a noncommutative group $G$ with the trivial topology and choose $H := \{e\}$. Then $H$ is abelian but not $\text{cl}(H) = G$. □

Exercise 1.6. Let $A$ and $B$ are subsets of $G$ and $g \in G$. Prove that:

1. If $A$ is open then $gA$, $BA$ and $AB$ are open in $G$.
2. If $A$ and $B$ are compact then $AB$ is also compact.
3. If $A$ and $B$ are connected then $AB$ is also connected.
4. * If $A$ and $B$ are closed then $AB$ need not be closed.
5. * If $A$ is closed and $B$ is compact then $AB$ is closed.

Proof. (1) $gA = l_g(A)$ is open since $l_g$ is a homeomorphism. $AB = \bigcup_{a \in A} aB$ is open as an union of open sets.

(2) $A \times B$ is compact. Thus, its continuous image $m(A \times B) = AB$ is also compact.

(3) $A \times B$ is connected. Thus, its continuous image $m(A \times B) = AB$ is also connected.

(4) In the group $\mathbb{R}$ take the clod subsets $A = \mathbb{Z}, B = \{n + \frac{1}{2n}\}_{n \in \mathbb{N}}$. Then $A + B = \{m + \frac{1}{2n}\}_{m \in \mathbb{Z}, n \in \mathbb{N}}$ and we have $0 = \lim \frac{1}{2n} \in (A + B)$ but $0 \notin A + B$.

(5) Let $c \notin AB$ then $cB^{-1} \cap A = \emptyset$. Thus $\{c\} \times B^{-1} \subseteq m^{-1}(A^c)$, where $A^c := G \setminus A$. Clearly, $\{c\} \times B^{-1}$ is compact and $m^{-1}(A^c)$ is its open nbd in $G \times G$. By the tube lemma (using the standard compactness argument) there exists $U \in N(c)$ with $U \times B^{-1} \subseteq m^{-1}(A^c)$. Hence, $UB^{-1} \cap A = \emptyset$. So, $U \cap AB = \emptyset$. Therefore, $c \notin cl(AB)$.

(6) We show only $cl(A) = \bigcap_{V \in N(e)} VA$.

(\subseteq) For every nbd $V \in N(e)$ we have $V^{-1}x \in N(x)$. So, if $x \in cl(A)$ then $V^{-1}x \cap A \neq \emptyset$. Hence, $x \in VA$.

(\supseteq) Let $x \in \bigcap_{V \in N(e)} VA$. Then $V^{-1}x \cap A \neq \emptyset$ for every $V \in N(e)$. Now observe that for every $U \in N(x)$ there exists $V \in N(e)$ such that $V^{-1}x \subset U$. Hence, $U \cap A \neq \emptyset$. This means that $x \in cl(A)$. □
Exercise 2.1. Let $G$ be a Hausdorff nondiscrete topological group. Show that in the topological group $G^2$ there exist at least 3 distinct but isomorphic proper ($\neq G^2$) nondiscrete closed subgroups.

Proof. For $G \times G$ consider the projections $\pi_1 : G \times G \to G, \pi_2 : G \times G \to G$ and the following proper subgroups:
$$\{e\} \times G = \pi_1^{-1}(e)$$
$$G \times \{e\} = \pi_2^{-1}(e)$$
$$\Delta := \{(x, x) : x \in G\}$$
It is a standard exercise that for every Hausdorff space $X$ the diagonal $\Delta \subset X \times X$ is closed (and vice versa).
All these three groups are isomorphic (in TGr category) to $G$, hence non-discrete. They are distinct, otherwise $G$ is trivial, hence discrete, a contradiction. □

Exercise 2.2. Show that there exist: an abelian topological group $(G, +)$ and closed subgroups $H_1$ and $H_2$ of $G$, such that the subgroup $H_1 + H_2$ is not closed. Show also that it is impossible for $G := \mathbb{T}$.

Proof. (1) $H_1 := \mathbb{Z}$ and $H_2 := \alpha \mathbb{Z}$ with any given irrational $\alpha$. Then $H_1 + H_2$ is a noncyclic subgroup of $\mathbb{R}$. By the dichotomy theorem we obtain that $H_1 + H_2$ is dense in $\mathbb{R}$. $H_1 + H_2$ clearly is countable. Therefore, $H_1 + H_2 \neq \mathbb{R}$.

(2) In $\mathbb{T}$ we have the following closed subgroups $\Omega_n, n \in \mathbb{N}$ (besides $\mathbb{T}$). So for every proper closed subgroups $H_1, H_2$ each of them is finite. Then $H_1 + H_2$ is finite, too. Therefore it is closed and cannot be dense in $\mathbb{T}$. □

Exercise 2.3. Let $G$ be a topological group and $H$ be its normal subgroup. Prove:

1. $G/H$ is discrete iff $H$ is open in $G$.
2. If the normal subgroup $H \leq G$ is closed and has the finite index $[G : H] < \infty$ then $H$ is clopen.
3. If $G$ is LC then $G/H$ is also LC.

Proof. (1) $H$ is open in $G$ iff $e_{G/H}$ is an isolated point in $G/H$.

(2) If $G/H$ is finite and Hausdorff then it is discrete. So, $H$ is clopen as a preimage of the isolated point $e_{G/H}$.

(3) Continuous open map preserves LC. □

Exercise 2.4. Prove:

1. The topological groups $\mathbb{C}^*$ and $\mathbb{R}_+ \times \mathbb{T}$ (where, $\mathbb{R}_+ := (0, \infty)$) are isomorphic in TGr.

2. The topological group $\mathbb{T}^2$ (two-dimensional torus) is a quotient (topological factor group) of the group $\mathbb{C}^*$.

Proof. (1) $\mathbb{C}^*$ is topologically isomorphic to $\mathbb{R}_+ \times \mathbb{T}$. Indeed, the functions
$$f : \mathbb{C}^* \to \mathbb{R}_+ \times \mathbb{T}, \ z \mapsto (|z|, \frac{\zeta}{|z|})$$
\[ f^{-1} : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{C}^*, \ (r, c\alpha) \to r c\alpha \]

are well defined continuous homomorphisms.

(2) It is equivalent to show that there exists an open continuous onto homomorphism \( \mathbb{C}^* \to \mathbb{T} \times \mathbb{T} \). Second observation: \( \mathbb{R}_+ \) and \( \mathbb{R} \) are topologically isomorphic. Indeed, consider for example the exponential function.

The third observation. As we already know the natural homomorphism \( q : \mathbb{R} \to \mathbb{T} \) is an open map. Then the induced onto homomorphism \( q \times id : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{T} \times \mathbb{T} \) is also open.

Summing up all three observations we obtain an open continuous onto homomorphism \( \mathbb{C}^* \to \mathbb{T} \times \mathbb{T} \), as desired.

\[ \square \]

Exercise 2.5. Prove or disprove: \( GL_n(\mathbb{R})/D \) is a locally compact Hausdorff topological group, where \( D \) denotes the subgroup of all invertible scalar matrices in \( GL_n(\mathbb{R}) \) for every \( n \in \mathbb{N} \).

Proof. As we already know the natural projection \( q : G \to G/H \) onto the topological factor group is open. Clearly, every open map preserves the local compactness. So, we have only to show that \( G/H \) is Hausdorff. As we know it is equivalent to say that \( H \) is closed in \( G \). Finally observe that \( H := D \) is closed in \( G := GL_n(\mathbb{R}) \). This can be checked directly (sequentially closed in a metric space) or by mentioning that \( D := \{ cI : c \in \mathbb{R}^* \} \) is topologically isomorphic to the LC group \( \mathbb{R}^* \) and hence must be closed in every Hausdorff topological group.

Exercise 2.6. Give a concrete example of a continuous onto homomorphism \( f : G_1 \to G_2 \) of Hausdorff separable metrizable topological groups which is not a quotient map.

Proof. Take for example \( id : (\mathbb{Z}, \tau_{\text{discr}}) \to (\mathbb{Z}, \tau(d_5)) \).

\[ \square \]

Exercise 2.7.

(1) Let \( f : G_1 \to G_2 \) be a continuous onto homomorphism of topological groups. Assume that \( G_1 \) is compact and \( G_2 \) is Hausdorff. Show that \( f \) is an open map.

(2) Give an example of a continuous onto map \( f : K \to Y \) between topological spaces with \( K \) compact space and \( Y \) is Hausdorff such that \( f \) is not open.

Proof. (1) \( f : G_1 \to G_2 \) is a continuous closed function because \( G_1 \) is compact and \( G_2 \) is Hausdorff. Since this map is also onto and continuous then it is a quotient map. On the other hand every quotient map. This implies that the natural map \( G_1/\ker f \to G_2 \) is a homeomorphism. Now use that the canonical projection \( p : G_1 \to G_1/\ker f \) on the quotient group is open.

(2) The standard quotient map \([0, 1] \to \mathbb{T} \) is closed but not open.

Exercise 2.8. (Coset spaces) Let \( H \) be a (not necessarily normal) subgroup of a topological group \( G \). Consider the set of all left cosets \( G/H := \{ gH : g \in G \} \) and the natural function \( p : G \to G/H, g \mapsto gH \). Define on \( G/H \) the quotient topology. Show:

(1) \( p : G \to G/H \) is a continuous open function.
(2) The natural action \( G \times G/H \to G/H \) is continuous.
(3) The topological space \( G/H \) is homogeneous.
(4) If \( H \) is a normal subgroup of \( G \) then \( G/H \) is a topological group.
Proof. (1) Continuity comes from the definition of quotient topology. We show that $p$ is open. Observe that

$$p^{-1}(p(U)) = UH = \cup \{uH : u \in U\} = \cup \{Uh : h \in H\}$$

is open for every open $U \subset G$ as a union of open sets.

(2) Let $(g, t) \in G \times G$ be a given pair of points. We show the continuity of the action

$$\pi : G \times G/H \to G/H, \ (x, yH) \mapsto xyH$$

at the point $(g, tH) \in G \times G/H$. Let $O$ be a neighborhood of the point $gtH$ in $G/H$. Then $gt \in p^{-1}(O)$. By the continuity, $p^{-1}(O)$ is open in $G$. Choose $U \in N(e)$ such that $Ugt \subset p^{-1}(O)$. That is, $UgtH \subset O$. By the continuity of group operations in $G$ one may choose neighborhoods $V, W \in N(e)$ in $G$ such that $V(gWg^{-1}) \subset U$. Then

$$(Vg)(WtH) \subset UgtH \subset O.$$ 

Since $Vg \in N(g)$ and $WtH$ is a neighborhood of $tH$ in $G/H$ ($p$ is open) we obtain the continuity of $\pi$ at the point $(g, tH)$.

(3) Every $g$-translation $l_g : G/H \to G/H, \ xH \mapsto gxH$ is continuous by (2). Now, observe that $l_{g^{-1}} = (l_g)^{-1}$ and the action is transitive.

(4) As in (2) use the fact that the sets $UxH$, where $U \in N(e)$ and $x \in G$ form a topological basis of $G/H$. Since $H$ is a normal subgroup of $G$ we have $j(UxH) = (UxH)^{-1} = x^{-1}U^{-1}H$. This easily implies that the inversion $j : G/H \to G/H$ is continuous. The continuity of the group multiplication in $G/H$ can be derived from (2). 

\[\square\]

\[\text{4this argument shows that the sets } UxH, \text{ where } U \in N(e) \text{ and } x \in G \text{ form a topological basis of } G/H\]
3.

**Exercise 3.1.** Let $K$ be a compact subgroup in a topological group $G$.

(1) Show that the natural (continuous open) quotient map $p : G \to G/K$ on the coset space $G/K$ is closed.

(2) Give an example which shows that the compactness is essential.

**Proof.** (1) For every closed subset $A \subset G$ its image $p(A)$ is also closed in $G/K$. Indeed, $p^{-1}p(A) = AK$. On the other hand, $AK$ is closed in $G$ by Exercise 1.6.5. Since $p$ is a quotient map this means (by equivalent definition of quotient maps) that $p(A)$ is closed.

(2) First of all note that $\mathbb{T}$ is naturally isomorphic in $TGr$ to the quotient group $\mathbb{R}/\mathbb{Z}$. Indeed, the standard continuous projection $p : \mathbb{R} \to \mathbb{T}$ is open. However this map is not closed. Indeed, $A := \{ n + \frac{1}{n} : n \in \mathbb{N} \}$ is closed in $\mathbb{R}$ but $p(A)$ is not closed in $\mathbb{T}$ (observe that $1 \in cl(p(A)$ and $1 \notin p(A)$).

**Exercise 3.2.** A Hausdorff topological group $G$ is said to be minimal if there is no strictly coarser Hausdorff group topology on $G$. Which of the following topological groups are minimal: $\mathbb{Z}, \mathbb{Z}^2, \mathbb{R}, \mathbb{T}$?

**Proof.** $\mathbb{T}$ is minimal being a compact group. Others groups in this list are not minimal.

For (discrete) $\mathbb{Z}$ take the $p$-adic (for some concrete $p$) topology which is a Hausdorff topological group topology. The same argument leads easily to the conclusion that $\mathbb{Z}^2$ is not minimal. For $\mathbb{R}$ use the following

$$f : \mathbb{R} \to \mathbb{T}^2, \ x \mapsto ([x], [\alpha x]) = (cis(2\pi x), cis(\alpha 2\pi x))$$

continuous not onto homomorphism with dense image. Take into account that if $f : \mathbb{R} \to f(\mathbb{R})$ is a homeomorphism then $f(\mathbb{R})$ being a locally compact subgroup of the compact group $\mathbb{T}^2$ is closed, hence also compact. This contradicts to the fact that $\mathbb{R}$ in its usual topology is not compact.

**Definition 3.3.** Let $X$ be a topological space. A compactification of $X$ is a continuous map $f : X \to Y$ where $Y$ is a compact Hausdorff space and $f(X)$ is dense in $Y$. We say: proper compactification when, in addition, $f$ is required to be a topological embedding.

One of the standard examples of a proper compactification is the so-called 1-point compactification $\nu : X \hookrightarrow X_\infty := X \cup \{ \infty \}$ defined for every locally compact non-compact Hausdorff space $(X, \tau)$. Recall the topology

$$\tau_\infty := \tau \cup \{ X_\infty \setminus K : K \text{ is compact in } X \}.$$  

\footnote{Another important example of a compactification is the so-called maximal (or, Stone-Chech) compactification $\beta : X \to \beta X$ which is proper iff $X \in T_{3.5}$. See for example the file of Doron Ben Hadar downloadable from the course homepage}
Exercise 3.4.

(1) Let $S := \mathbb{R} \cup \{\infty\}$ be the 1-point compactification of $\mathbb{R}$. Define the "usual" operation $+ \colon S \times S \to S$ by $x + y$ is already defined for $x, y \in \mathbb{R}$. Otherwise, $x + y = \infty$ (that is, $x + \infty = \infty + x = \infty + \infty = \infty$). Show that $(S, +)$ is a semitopological (meaning that left and right translations are continuous) but not topological (meaning that the multiplication is not continuous) semigroup.

(2) More generally, let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of $G$.

* Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not topological monoid.

Proof. (2) First we show that $S$ is semitopological. Let $a \in S$. We have to show that $l_a : S \to S$ and $r_a : S \to S$ are continuous. We consider only the case of $l_a$. The second case is similar. So, we have to check that $l_a : S \to S$ is continuous at every $y \in S$. For $a = \infty$ the transition $l_a$ is a constant map. WRG assume that $a \neq \infty$, hence $a \in G$. We have two cases for $y \in S$:

(a) If $y \neq \infty$ then for every open nbhd $U \subseteq N(y), U \subseteq G$ just take the open nbhd $V : = a^{-1}U \subseteq N(a^{-1}y)$. Then $l_a(V) = U$. 

(b) Let $y = \infty$ and $U \subseteq N(\infty)$ is an open nbhd. Then by the definition of the 1-point compactification topology, $U = S \setminus K$, where $K$ is compact in $G$. Then $a^{-1}K$ is also compact in $G$. So, $V := S \setminus a^{-1}K \subseteq N(\infty)$ and $l_a(V) = U$.

Now we show that $S$ is not topological. That is, the multiplication is not continuous. Indeed, we show that the multiplication $m : S \times S \to S$ is not continuous at the point $(\infty, \infty)$.

Assuming the contrary let $m$ be continuous at $(\infty, \infty)$. Choose the open nbhd $U := S \setminus \{e\}$ of $\infty$. Since $\infty \cdot \infty = \infty$, by the continuity assumption there exists nbhd $V$ of $\infty$ such that $VV \subseteq U$. By definition of $\tau_\infty$ there exists a compact subset $K$ of $G$ such that $S \setminus K \subseteq V$. Since $K \cup K^{-1}$ is also compact in $G$ the set

$$W := S \setminus (K \cup K^{-1})$$

is also nbhd of $\infty$ in $S$ such that $W \subseteq V$. Then $W$ is symmetric (i.e., $W^{-1} = W$). Take any $w \in W$. Then $w^{-1} \in W$ and

$$e = w w^{-1} \in WW \subseteq VV \subseteq U.$$ 

This contradicts to the choice of $U := S \setminus \{e\}$.

□

Remark 3.5. As we know a locally compact Hausdorff group $G$ admits an embedding into a compact Hausdorff group iff $G$ is compact. Exercise 3.4 shows that such $G$ at least admits a proper semigroup compactification $\nu : G \hookrightarrow S$ such that $S$ is a compact semitopological monoid.

Exercise 3.6. Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{Z}$. Extend the usual addition by:

$$n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \; s, t \in \{-\infty, \infty\}$$

Show: $(S, +)$ is a noncommutative compact right topological (meaning that all right translations are continuous) monoid. $S$ is not semitopological.
Proof. First of all it is straightforward to see that \((S, +)\) is a monoid and \((\mathbb{Z}, +)\) its submonoid.

\((S, +)\) is noncommutative because \(\infty + (-\infty) = -\infty\) and \((-\infty) + \infty = \infty\).

\(S := \mathbb{Z} \cup \{-\infty, \infty\}\) carries the topology of the natural linear order. A natural subbase for the topology of \(S\) is the following family

\[ A_n := \{x \in S : x < n\}, \quad B_m := \{x \in S : m < x\}, \quad n, m \in \mathbb{Z} \]

Clearly, \(\mathbb{Z}\) is dense in \(S\) and every \(x \in \mathbb{Z}\) is an isolated point in \(S\). The space \(S\) is homeomorphic to the following closed subset

\[ Y := \{-1\} \cup \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1\} \]

of \([-1, 1]\), hence compact.

The right translations \(r_t : S \to S\) are continuous. Indeed, \(r_\infty\) is the constant function \(r_\infty(x) = \infty\) for every \(x \in S\). \(r_{-\infty}\) is the constant function \(r_{-\infty}(x) = -\infty\) for every \(x \in S\). \(r_k^{-1}(A_n) = A_{n-k}, \quad r_k^{-1}(B_m) = B_{m-k}\) for every \(k \in \mathbb{Z}\).

\(l_\infty : S \to S\) is not continuous at the point \(s = -\infty\). Take a sequence \(\{-k\}_{k \in \mathbb{N}}\). Then

\[ \lim(-k) = -\infty \text{ but } \lim l_\infty(-k) = \infty \neq l_\infty(-\infty) = \infty + (-\infty) = -\infty \]

Similarly, \(l_{-\infty} : S \to S\) is not continuous at the point \(s = \infty\).

\(\Box\)

Exercise 3.7. * Let \(S\) be a right topological semigroup. Show that if \(S\) is compact then it contains at least one idempotent.

Proof. We have to show that there exists \(m \in S\) such that \(m^2 = m\).

By Zorn’s Lemma (and compactness of \(S\)) there exists a minimal compact subsemigroup \(M \subseteq S\) (indeed, for any chain of compact subsemigroups the intersection is a nonempty compact subsemigroup (by the compactness) of \(S\)). Take arbitrary \(m \in M\). Our aim is to show that \(m^2 = m\). Consider the set \(Mm\). Then \(Mm \subseteq M\) is a subsemigroup. Observe also that \(Mm\) is compact. By the minimality of \(M\) we necessarily have \(Mm = M\). This implies that \(um = m\) for some \(u \in S\). Hence, the following set

\[ K := \{x \in M : xm = m\} \]

is nonempty. Moreover, \(K\) is a subsemigroup and compact (again use the continuity of right translations). Since \(K \subseteq M\) we necessarily have \(K = M\). This implies that \(m \in K\). Therefore, \(m^2 = m\).

\(\Box\)
4.

**Exercise 4.1.** Show that the class NA of all non-archimedean groups is closed under formation of: subgroups, factor-groups, products.

**Proof.** Subgroups.
Let $G \in NA$ and $H \leq G$. There exists a local base $\gamma$ at $e$ in $G$ which consist by subgroups. Then $\gamma|_H := \{ U \cap H \}$ is a local base at $e$ in $H$ which consist by subgroups.

Factor-groups.
Let $G \in NA$, $H \unlhd G$, $G/H$ is the factor-group and $q : G \to G/H$ is the quotient homomorphism. There exists a local base $\gamma$ at $e$ in $G$ which consist by subgroups. Then $q(\gamma) := \{ q(U) \leq G/H \}$ is a local base at $e$ in $G/H$ which consist by subgroups. Here we use the fact that $q$ is open.

Products.
Straightforward. Easy to see also using weak topologies.

**Exercise 4.2.** Show that every LCA group $G$ is a subgroup of a self-dual LCA group $H$ (self-dual means that $H^*$ is isomorphic to $H$).

**Proof.** Take $H := G \times G^*$. Then $H^* = G^* \times G \cong G \times G^* = H$.

$$G \cong G \times \{e\} \leq H := G \times G^*.$$ 

**Exercise 4.3.** Prove that every compact abelian group $G$ is a subgroup of some (maybe infinite-dimensional) torus $\mathbb{T}^S$.

**Proof.** There exists a set of continuous homomorphisms (characters) $S := \{ f_i : G \to \mathbb{T} \}$ which separates the points of $G$. Indeed, take for example, $S := G^*$. The diagonal map $f : G \to \mathbb{T}^S$ is a continuous homomorphism which is injective. Since $G$ is compact and $\mathbb{T}^S$ is Hausdorff we obtain that $f$ is an embedding (of topological groups).

**Exercise 4.4.** (Stephenson’s Theorem) Every LCA minimal group is necessarily compact.

**Proof.** By Pontryagin’s theory the set $S := G^*$ separates the points of $G$. Hence the diagonal function $f : G \to \mathbb{T}^S$ is a continuous injective homomorphism. Since $G$ is a minimal group $f$ is a topological embedding (as it follows from the equivalent definition for minimality of topological groups). Since $G$ is LC then its isomorphic copy (in TGr) $f(G)$ is also LC. As we know this implies that $f(G)$ is a closed subgroup of (Hausdorff) compact group $\mathbb{T}^S$. Hence, $f(G)$ is also compact. So, we can conclude that $G$ is compact, too.

A topological group $G$ is said to be **compactly generated** if there exists a compact subset $K \subset G$ which algebraically generates $G$; that is $< K >= G$.

A topological space $X$ is **$\sigma$-compact** if $X = \bigcup_{n \in \mathbb{N}} K_n$ where each $K_n \subset X$ is compact.

**Exercise 4.5.**
(1) Show that every connected LC topological group $G$ is compactly generated and $\sigma$-compact.

(2) Give an example of a $\sigma$-compact topological group which is not compactly generated.

Proof. (1) $G$ is compactly generated.

Since $G$ is LC one may choose a compact (not necessarily, open) nbd $K \in N(e)$. Now it is enough to show that $K$ algebraically generates $G$. In fact we can show the following

CLAIM: every connected topological group $G$ and every nbd $U \in N(e)$ we have

$$G = \bigcup_{n \in \mathbb{N}} U^n.$$ 

So, $U$ algebraically generates $G$.

Proof of the Claim:

Choose a symmetric nbd $V \in N(e)$ such that $V \subset U$. By results from lecture notes we know that $H := \bigcup_{n \in \mathbb{N}} V^n = < V >$ is an open subgroup of $G$ and in fact $H$ is clopen. Since, $G$ is connected we obtain that necessarily $H = G$. So, $< V > = G$. Then $< V > \subset < U > = G$.

$G$ is $\sigma$-compact.

Observe that $G = \bigcup_{n \in \mathbb{N}} U^n$ and each $K^n := KK \cdots K$ is compact being a continuous image of $\underbrace{K \times \cdots \times K}_n$.

(2) Is based on the following two simple facts: for discrete groups compactly generated = finitely generated and $\sigma$-compactness is exactly the countability. Now, take any countable but not finitely generated group $G$ (say, $\mathbb{Q}$) with the discrete topology. $\square$