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Some examples in General Topology

www.math.biu.ac.il/~megereli/TOP.html

אקסיומות הפרדה Separation Axioms

$$TOP \supset T_0 \supset T_1 \supset T_2 \supset T_3 \supset T_{3\frac{1}{2}} \supset T_4 \supset Metr$$

1. $(\mathbb{R}, \tau_{tr}) \in TOP \setminus T_0$
2. **Sierpinski's space** $(\{0,1\}, \tau_{\leq}) \in T_0 \setminus T_1$
3. $(\mathbb{R}, \tau_{cofinite}) \in T_1 \setminus T_2$
4. **Smirnov's space** $\in T_2 \setminus T_3$
5. **Mysior's space** $\in T_3 \setminus T_{3\frac{1}{2}}$
6. $\in T_{3\frac{1}{2}} \setminus T_4$
   a. ***Sorgenfrey* plane** $P := (\mathbb{R}, \tau_s) \times (\mathbb{R}, \tau_s)$ http://en.wikipedia.org/wiki/Sorgenfrey_plane
   b. **Moore *plane*** (with tangent disc topology) http://en.wikipedia.org/wiki/Moore_plane
   c. $\mathbb{Z}^{\mathbb{R}}$
7. $\in T_4 \setminus Metr$
   a. ***Sorgenfrey* line** $(\mathbb{R}, \tau_s)$ http://en.wikipedia.org/wiki/Sorgenfrey_line
   b. $[0,1]^{\mathbb{R}}$

Definitions and some explanations:

1. (X, τ_{tr}), $\tau_{tr} := \{X, \varnothing\}$ trivial space.
2. $(\{0,1\}, \tau_{\leq})$, $\tau_{\leq} := \{\varnothing, \{0\}, \{0,1\}\}$ Sierpinski's space.
3. Exercise.

4. **Smirnov's space**

$(\mathbb{R},\sigma)$

$O\in\sigma \equiv x\in O \Rightarrow \exists U\in\tau(d)\,\exists C\subset\mathbb{R}, |C|\le\aleph_0: \quad x\in U\setminus C\subseteq O.$

Then $\tau(d)\subset\sigma$, $\sigma\in T_2$, $A:=\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is closed, A and the point 0 cannot be separated by disjoint neighborhoods.

5. **Mysior's space** (X,τ) (Proceedings Amer. Math. Soc., 81:4 1981)

$X:=\{a\}\cup\{(x,y)\in\mathbb{R}^2: \quad y\ge 0\}$

$O\in\tau \equiv if\ a\in O\ then\ \exists n\in\mathbb{N}\ \{(x,y): x>n\}\subseteq O$

$if\ (x,0)\in O\ then\ O\ contains\ almost\ all\ members\ of\ I_x\cup J_x$

$I_x:=\{(x,y): 0\le y<2\} \quad J_x:=\{(x+y,y): 0\le y<2\}$

The closed subset $A:=\{(x,0): x\le 1\}$ and the point a

cannot be separated by a continuous function $f: X\to\mathbb{R}$.

6. a.

$P\in T_{3\frac{1}{2}}$.

Clearly, $P\in T_1$. Let $B\subset P$ a closed subset of P and $a\in P$. There exists a clopen subset (use $\dim(\mathbb{R},\tau_s)=0$) $D\subset P$ such that $a\in D\wedge D\cap B=\varnothing$. Now the characteristic function $\chi_D: P\to\{0,1\}$ separates point a and the closed subset B.

$P\notin T_4$.

The closed subspace $Y := \{(x,-x) \in \mathbb{R}\times\mathbb{R}\}$ of the ***Sorgenfre*y plan**e =
$P := (\mathbb{R},\tau_s)\times(\mathbb{R},\tau_s)$ is **discrete !.**

So $C(Y,[0,1]) = [0,1]^Y$. Then $|C(Y,[0,1])| = |[0,1]^Y| = c^c = 2^c$.

$\mathbb{Q}\times\mathbb{Q}$ is dense in $P := (\mathbb{R},\tau_s)\times(\mathbb{R},\tau_s)$. So
$|C(P,[0,1])| \le |C([0,1]^{\aleph_0})| \le c^{\aleph_0} = c.$

Assuming the contrary that $P \in T_4$, by Tietze's extension theorem we get that

$|C(Y,[0,1])| \le |C(P,[0,1])|$

Then we should have $|C(Y,[0,1])| = 2^c \le c.$ This contradiction completes the proof.

<u>Some other properties</u>

$FU \supset B_1 \supset Metr \qquad B_1 \cap Sep \supset B_2 \supset Comp \cap Metr$

Definition (Frechet-Urysohn space): $X \in FU \equiv scl(A) = cl(A) \quad \forall A \subseteq X$,

where $scl(A) = \{x = \lim a_n \in X : \quad a_n \in A\}$ is the *sequential closure*.

8. $(\mathbb{R},\tau_s) \in B_1 \setminus Metr \qquad (\mathbb{R},\tau_s) \in B_1 \setminus B_2$

9. $(\mathbb{R},\tau_0) \notin FU$

   $U \in \tau_0 \equiv 0 \notin U \vee |\mathbb{R}\setminus U| \le \aleph_0$

10. $(l_\infty, \|\,\|_{\sup}) \notin Sep$

    Hint: There exists an isometric embedding $(\{0,1\}^{\mathbb{N}}, d_\Delta) \to l_\infty \qquad \ldots$

11. $(X,\tau) \in FU \setminus B_1$

(X, τ) is the quotient space of $\mathbb{R}$ under identifying all the points of $\mathbb{N}$

(R. Engelking, General Topology, II edition, Example 1.6.18)

Some recommended books:

J. Kelley, General Topology.

L. Steen and J. Seebach, *Counterexamples in Topology*.