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REFLEXIVELY BUT NOT UNITARILY REPRESENTABLE TOPOLOGICAL GROUPS

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ABSTRACT. We show that there exists a topological group G (namely, $G := L_4[0,1]$) such that for a certain *reflexive* Banach space X the group G can be represented as a topological subgroup of $Is(X)$ (the group of all linear isometries endowed with the strong operator topology) and such an X never may be Hilbert. This answers a question of V. Pestov and disproves a conjecture of A. Shtern.

1. INTRODUCTION

Let X be a real Banach space. Denote by $Is(X)_s$ ($Is(X)_w$) the group of all linear isometries of X endowed with the strong (resp., weak) operator topologies.

A *representation* of a Hausdorff topological group G in X is a continuous group homomorphism $G \to Is(X)_s$. Let $\mathbf{K}$ be a subclass of the class $\mathbf{Ban}$ of all Banach spaces. We say that G is $\mathbf{K}$-*representable* if for a certain $X \in \mathbf{K}$ there exists a topological group embedding $G \hookrightarrow Is(X)_s$. Denote by $\mathbf{K_R}$ the class of all $\mathbf{K}$-representable groups. For instance, this leads to the definitions of the following classes $\mathbf{Ban_R}$, $\mathbf{Ref_R}$, and $\mathbf{Hilb_R}$, where $\mathbf{Ref}$ and $\mathbf{Hilb}$ denote all reflexive and all Hilbert spaces, respectively. We say that G is *reflexively representable* (*unitarily representable*) if $G \in \mathbf{Ref_R}$ (resp., $G \in \mathbf{Hilb_R}$). Denote by $\mathbf{TopGr}$ the class of all Hausdorff topological groups. We have

$$\mathbf{TopGr} = \mathbf{Ban_R} \supseteq \mathbf{Ref_R} \supseteq \mathbf{Hilb_R}.$$

Indeed, the inclusions are trivial. As to $\mathbf{TopGr} = \mathbf{Ban_R}$, recall that every Hausdorff topological group G can be embedded into the group $Is(E)_s$ of all linear isometries of a suitable Banach space E endowed with the strong operator topology. As in the paper of Teleman [23], take for example $X := C_r^b(G)$, the Banach space of all bounded right uniformly continuous functions on G.

It is also well known that $\mathbf{TopGr} \neq \mathbf{Hilb_R}$. Moreover, there are many examples of so-called *exotic* groups (that is, the groups whose unitary representations are trivial), see Herer-Christensen [12] and Banasczyk [3]. Among many interesting examples from [3], note that l_2/D is exotic for some discrete subgroup D of l_2. Not every separable Banach space (as a topological group) is unitarily representable. It is well known that l_2 is not uniformly universal in the class of separable metrizable uniform spaces (Enflo [10], Aharoni [1]). These arguments lead to the fact that $C[0,1], c_0 \notin \mathbf{Hilb_R}$ (see Proposition 3.7 below).

Recently, it has been proved in [18] that $\mathbf{TopGr} \neq \mathbf{Ref_R}$. Denote by $H_+[0,1]$ the group of all orientation preserving selfhomeomorphisms of the closed interval

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endowed with the compact open topology. It turns out that every (weakly) continuous representation of $H_+[0,1]$ in a reflexive Banach space by linear isometries is trivial. This result answers a question discussed by Ruppert [20] and conjectured by V. Pestov.

A natural question arises about coincidence of $\mathbf{Ref_R}$ and $\mathbf{Hilb_R}$. This question is posed in the recent paper by V. Pestov [19]. Earlier the positive answer was conjectured by A. Shtern [22]. The main result of the present work disproves this conjecture. Theorem 3.1 below implies that $L_4[0,1] \in \mathbf{Ref_R}$ and $L_4[0,1] \notin \mathbf{Hilb_R}$.

2. Weakly almost periodic functions

Recall that a continuous bounded function $f \in C^b(G)$ on a topological group G is called *weakly almost periodic* (in short: *wap*) in the sense of Eberlein [9, 8] if the orbit of f in $C^b(G)$ is relatively weakly compact. The subset $WAP(G)$ of all wap functions in $C^b(G)$ forms a closed (left and right) translation-invariant subalgebra. Every positive definite function is wap [5].

A semigroup S is *semitopological* if the multiplication $S \times S \to S$ is separately continuous. The compactification $j : G \to G^w$ induced by the algebra $WAP(G)$ is the *universal semitopological compactification* of G.

For every reflexive Banach space E, the semigroup

$$\Theta(E)_w := \{s \in L(E,E) : \|s\| \leq 1\}$$

of all contractive linear operators forms a compact semitopological semigroup in the weak operator topology [8]. Hence, the same is true for its closed subsemigroups. Conversely, an arbitrary compact Hausdorff semitopological semigroup can be obtained in this way (see Fact 2.2). By a result of Lawson [7, Corollary 6.3] every subgroup of a compact Hausdorff semitopological semigroup is a topological group.

Fact 2.1. *Let G be a Hausdorff topological group. Then the following conditions are equivalent:*

(i) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_s$ (equivalently, G is reflexively representable);

(ii) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_w$;

(iii) The algebra $WAP(G)$ separates points and closed subsets;

(iv) The canonical map $j : G \to G^w$ is a topological embedding;

(v) G is a topological subgroup of a Hausdorff compact semitopological semigroup.

The equivalence of (iii), (iv) and (v) is well known [20, 5].

The part (ii) $\Longrightarrow$ (iii) follows from the fact that for every reflexive Banach space E and a norm-bounded semigroup S of linear operators on E the *generalized matrix coefficients*

$$\{m_{v,f} : S \to \mathbb{R}\}_{f \in E^*, v \in E} \qquad m_{v,f}(s) = f(sv)$$

all are wap.

The part (iii) $\Longrightarrow$ (ii), is a direct consequence of the following result of A. Shtern.

Fact 2.2. *(*[22, 15, 17]*) The following conditions are equivalent:*
(a) $WAP(S)$ separates points and closed subsets;
(b) S can be embedded into $\Theta(E)_w$ for a certain reflexive Banach space E.

As to the equivalence (i) $\iff$ (ii), note that by [16, 17], strong and weak operator topologies coincide on $Is(X)$ for a wide class of Banach spaces with PCP (the *point of continuity property*) including the class of all reflexive Banach spaces.

3. Main Results

Let (Ω, B, μ) be a measure space. The corresponding standard space $L_p(\Omega, B, \mu)$ $(1 \leq p < \infty)$ will be denoted simply by $L_p(\mu)$.

Theorem 3.1. (i) *$L_{2k}(\mu) \in \mathbf{Ref_R}$ for every natural $k \in \mathbb{N}$;*
(ii) *$L_p[0,1] \notin \mathbf{Hilb_R}$ for every $2 < p < \infty$.*

We say that a function $F : A \times B \to \mathbb{R}$ has *Double Limit Property* (in short: DLP) if for every pair of sequences $\{a_n\}, \{b_m\}$ in A and B respectively,

$$\lim_n \lim_m F(a_n, b_m) = \lim_m \lim_n F(a_n, b_m)$$

whenever both of the limits exist.

We need *Grothendieck's characterization of wap* in terms of Double Limit Property.

Fact 3.2. *(*[5, 20]*) A function $f \in C^b(G)$ is wap iff the induced map $F : G \times G \to \mathbb{R}$ defined by $F(g, h) := f(gh)$ has DLP, that is, for every pair of sequences $\{g_n\}, \{h_m\}$ in G,*

$$\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m)$$

whenever both of the limits exist.

Lemma 3.3. *Let a topological group G admit a left-invariant metric d with DLP. Then G is reflexively representable.*

Proof. Define the norm $\|g\| := d(e, g)$. Then $\|g_n h_m\| = d(g_n^{-1}, h_m)$.

By Grothendieck's characterization (Fact 3.2), the bounded function

$$\phi_e : G \to \mathbb{R} \quad g \mapsto \frac{1}{1 + \|g\|}$$

is wap. Then its left or right translations are also wap. Therefore, for every fixed $z \in G$ the function

$$\phi_z : G \to \mathbb{R} \quad g \mapsto \frac{1}{1 + \|gz\|}$$

is a wap function. Since the norm generates the original topology on G, the family $\{\phi_z\}_{z \in G}$ of wap functions separates points and closed subsets of G. Hence, by Fact 2.1 we can conclude that $G \in \mathbf{Ref_R}$. □

Lemma 3.4. *The norm in the Banach space $L_{2k}(\mu)$ $(k \in \mathbb{N})$ has DLP.*

Proof. We have to show that for every pair $\{u_n(t)\}, \{v_m(t)\}$ of sequences in $L_{2k}(\mu)$

$$\lim_n \lim_m \|u_n + v_m\| = \lim_m \lim_n \|u_n + v_m\|$$

whenever both of the limits exist. We can suppose that the sequences are norm-bounded. Computing the norm $\|u_n + v_m\|$, we get

$$\|u_n + v_m\| = (\int_\Omega (u_n(t) + v_m(t))^{2k} d\mu)^{\frac{1}{2k}}$$

$$= (\|u_n\|^{2k} + \sum_{i=1}^{2k-1} C_{2k}^i < u_n^{2k-i}, v_m^i > + \|v_m\|^{2k})^{\frac{1}{2k}}$$

where

$$< u_n^{2k-i}, v_m^i >= \int_\Omega u_n^{2k-i}(t) v_m^i(t) d\mu$$

and

$$u_n^{2k-i} \in L_{\frac{2k}{2k-i}}(\mu), \qquad v_m^i \in L_{\frac{2k}{i}}(\mu) = L^*_{\frac{2k}{2k-i}}(\mu).$$

Passing to subsequences if necessary, we may assume that there exist

$$\lim_n \|u_n\|, \qquad \lim_m \|v_m\|.$$

By reflexivity, every bounded subset of $L_p(\mu)$ $(p > 1)$ is relatively weakly compact and hence, relatively sequentially compact (as it follows by the classical Eberlein-Šmulian theorem). Therefore, we can suppose in addition (again by passing to subsequences) that there exist weak limits

$$weak - \lim_n u_n^{2k-i}, \qquad weak - \lim_m v_m^i$$

for every $i \in \{1, 2, \cdots, 2k-1\}$.

Now, in order to complete the proof of the lemma it remains to observe that for every *reflexive* Banach space X and bounded subsets $A \subset X$, $B \subset X^*$, the canonical duality

$$A \times B \to \mathbf{R}, \qquad < x, f >= f(x)$$

has DLP. This fact easily follows using once again the Eberlein-Šmulian theorem. □

By Lemmas 3.4 and 3.3 we get $\boxed{L_{2k}(\mu) \in \mathbf{Ref_R} \quad \forall k \in N}$.

Now, we prove the second part of Theorem 3.1.

We need the following fundamental fact.

Fact 3.5. *(Aharoni-Maurey-Mityagin* [2]*)*
For $2 < p < \infty$, an infinite-dimensional $L_p(\mu)$ space is not uniformly embedded into a Hilbert space.

Additional information about uniform embeddings into Hilbert spaces can be found in [4].

The following Lemma is inspired by [14, Counterexample 2.13].

Lemma 3.6. *The uniform space $(Is(l_2)_s, \mathcal{L})$, where $\mathcal{L}$ denotes the left uniformity on $Is(l_2)_s$, can be uniformly embedded into l_2.*

Proof. Since $Is(l_2)_s$ is separable and metrizable, there exists a sequence $\{v_n\}$ in l_2 such that:

(a): $\|v_n\| = \frac{1}{2^n}$
(b): $\{\tilde{v}_n : Is(l_2) \to l_2, \quad g \mapsto gv_n\}_n$ generates the left uniformity on $Is(l_2)$.

Denote by $B_{\frac{1}{2^n}}$ the closed $\frac{1}{2^n}$-ball centered at the origin with its usual uniformity. Then the uniform product $\prod_n B_{\frac{1}{2^n}}$ is uniformly embedded in a natural way into the l_2-sum $(\sum_n (l_2)_n)_{l_2}$ of countably many copies of the Hilbert space l_2. Eventually we have

$$Is(l_2)_s \overset{unif}{\hookrightarrow} \prod_n B_{\frac{1}{2^n}} \overset{unif}{\hookrightarrow} (\sum_n (l_2)_n)_{l_2} \longleftrightarrow l_2$$

□

Now we prove the second part (ii) of Theorem 3.1. Assuming the contrary, suppose that $L_p[0,1]$ $(2 < p < \infty)$ is unitarily representable in the (infinite-dimensional) Hilbert space H. Then, using the separability of $L_p[0,1]$ and passing to an appropriate *separable* infinite-dimensional closed linear subspace E of H, we can suppose even that there exists a topological group embedding $L_p[0,1] \hookrightarrow Is(E)_s$. Since $Is(E)_s$ and $Is(l_2)_s$ are topologically isomorphic, it is clear that Fact 3.5 and Lemma 3.6 will lead to a contradiction.

Therefore, we have $\boxed{L_p[0,1] \notin \mathbf{Hilb_R} \quad (2 < p < \infty)}$.

Theorem 3.1 is proved.

Proposition 3.7. $C[0,1], c_0 \notin \mathbf{Hilb_R}$.

Proof. Follows from Lemma 3.6 because l_2 is not uniformly universal space for separable Banach spaces (Enflo [10]) in contrast to c_0 (Aharoni [1]) or $C[0,1]$ (Banach-Mazur). □

4. QUESTIONS

Question 4.1. Is it true that every Banach space X, as a topological group, is reflexively representable ? Or, equivalently, does $WAP(X)$ separate points and closed subsets ?

By Lemma 3.4, the answer is yes for $L_p(\mu)$ spaces, where $p = 2k$ is an arbitrary even integer. Using results of Shoenberg [21], we easily can extend this result to the case of $1 \leq p \leq 2$. Indeed, the function $f(v) = e^{-\|x\|^p}$ is positive definite (and hence, wap) on $L_p(\mu)$ spaces for every $1 \leq p \leq 2$. Moreover, Chaatit [6] proved that every *stable Banach space* in the sense of Krivine-Maurey [13], in particular, every separable $L_p(\mu)$ space $(1 \leq p < \infty)$, is reflexively representable.

By Lemma 3.3, a closely related question is: for which Banach spaces does the original norm (or its some renorming) satisfy DLP ?

It is easy to show that the original norm of the Banach space c_0 does not satisfy DLP. Indeed, define $u_n := e_n$ (the standard basis vectors) and $v_m := \Sigma_{i=1}^m e_i$. Then the corresponding double limits are 1 and 2.

Lemma 3.3 suggests also the following questions:

Question 4.2. Let G be a reflexively representable group. Is it true that the topology of G is generated by a family of left-invariant pseudometrics with DLP ?

Question 4.3. Which (non-locally compact) metrizable topological groups admit a left-invariant metric with DLP ?

For instance, it would be interesting to know for which metric spaces (X, d) does the corresponding *Graev's metric* ([11, 24]) on the free group $F(X)$ satisfy DLP ?

Lemma 3.6 leads to the following natural question.

Question 4.4. Suppose a topological group G is such that the left uniformity of G admits a uniform embedding into l_2. Is then G unitarily representable ?

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