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Relative minimality and co-minimality of subgroups in topological groups

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ABSTRACT

We study two properties of subgroups of a topological group (relative minimality and co-minimality), that generalize minimality. Many applications, mostly related to semidirect products and generalized Heisenberg groups are given.

1. Introduction

A Hausdorff topological group G is *minimal* (introduced by Stephenson [22] and Doïchinov [10]) if G does not admit a strictly coarser Hausdorff group topology. Totally minimal groups are defined in [7] as those Hausdorff groups G such that all Hausdorff quotients are minimal (later these groups were studied also by Schwanengel [21] under the name *q-minimal groups*).

Some natural examples of totally minimal groups:

- Compact Hausdorff topological groups.
- Symmetric topological groups S_X (Gaughan).
- $\mathbb{Z}$ with the p-adic topology (Doïchinov, Prodanov).
- The full unitary group $U(H)$ (Stoyanov).
- Every connected semisimple Lie group with finite center, e.g., $SL_n(\mathbb{R})$ (Goto).
- $Homeo(\{0,1\}^{\aleph_0})$ and $Homeo[0,1]$ (Gamarnik).

There are however many minimal groups which are not totally minimal. For example, the semidirect product $\mathbb{R} \rtimes \mathbb{R}_+$ (Dierolf and Schwanengel [3]). More generally, by [13] the same is true also for $\mathbb{R}^n \rtimes \mathbb{R}_+$, where $n \in \mathbb{N}$. Generalized Heisenberg groups (see Section 2 and also [12,14]) provide many examples of this kind. For instance, if G is a locally compact abelian group with the canonical duality mapping $w: G \times G^* \to \mathbb{T}$ then the corresponding "generalized Heisenberg group" $M(w) = (\mathbb{T} \times G) \rtimes G^*$ is minimal. For more information see [8,4,12,13].

Pestov and Morris [17] introduced some time ago *locally minimal* groups (cf. Definition 3.14) as a common generalization of locally compact groups and minimal groups (see also [6], where a joint generalization of total minimality and local compactness is proposed and properties of the locally minimal groups are studied in the spirit of the general problem

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posed in [2]). *Relative minimality* was introduced recently in [14]. In the present paper we investigate this concept in more details and introduce *co-minimal subgroups* as follows:

Definition 1.1.

1. Let X be a subset of a Hausdorff topological group (G,τ). We say that X is relatively minimal in G if every coarser Hausdorff group topology $\sigma \subseteq \tau$ of G induces on X the original topology. That is, $\sigma\restriction_X = \tau\restriction_X$.
2. Let X be a topological subgroup of a Hausdorff topological group (G,τ). We say that X is co-minimal in G if every coarser Hausdorff group topology $\sigma \subseteq \tau$ of G induces on the coset space G/X the original topology. That is, $\sigma/X = \tau/X$.

Obviously, any subgroup of a minimal group is both relatively minimal and co-minimal.

A good motivation to introduce relatively minimal subgroups can be found in [14], where a solution to a problem of Shtern was found by essentially using this concept. In particular, the following result is crucial in [14].

Theorem 1.2. *([14]) The subgroup X is relatively minimal in the Heisenberg group*

$$H(X) = (\mathbb{R} \times X) \leftthreetimes X^*$$

for every normed space X.

For more details about generalized Heisenberg groups see Section 2. Our Theorem 5.1 below generalizes and strengthens Theorem 1.2. In particular, we obtain that the subgroup $\mathbb{R}$ is co-minimal and X and X^* are relatively minimal in the Heisenberg group $H(X)$ for every normed space X. Theorem 5.1 and its corollaries unify also some results of [12]. We answer also two questions posed in [4].

Some interesting applications of generalized Heisenberg groups can be found in the recent papers [9,19,5].

Recently the second named author proved [15] the following theorem answering long standing questions of Arhangelskij and Pestov:

Theorem 1.3. *([15]) Every Hausdorff topological group X is a group retract of a minimal group G.*

It follows that every group X is a *closed* relatively minimal subgroup and a co-minimal subgroup in some group G.

In Section 3 we introduce a class of closed subgroups that allows for an easier internal description of the co-minimal subgroups (Definition 3.8). This approach gives the possibility to connect co-minimality of open subgroups to local minimality (Proposition 3.15). In Example 3.17 we describe the co-minimal subgroups of an arbitrary infinite cyclic topological group. In Section 4 relative minimality and co-minimality are used to describe minimality of semidirect products.

For a dense subgroup X of G, it was proved by Stephenson that minimality of X is equivalent to minimality of G and essentiality of X in G (see Corollary 6.4). Nevertheless, the very natural weaker condition only G to be minimal seems also worth to be considered (see Lemma 3.4(5)). In the abelian context, it gives the following immediate corollary of Prodanov–Stoyanov's theorem on precompactness of the minimal abelian groups: *for an abelian topological group X the following are equivalent*:

(i) *X is dense and relatively minimal in some group G*;
(ii) *X is relatively minimal in its completion*;
(iii) *X is precompact* (cf. Corollary 6.7).

In the case the pair X, G is chosen with X closed and central in G things change completely. Now relative minimality of X in G is equivalent to minimality of X. This need not be true if X is not central (for example, $\mathbb{R}$ is relatively minimal in the Heisenberg group (by Theorem 1.2), but $\mathbb{R}$ cannot be relatively minimal in any topological abelian group, see Corollary 6.8). Section 6 contains also a criterion for co-minimality of closed subgroups of precompact groups (Theorem 6.9).

2. Generalized Heisenberg groups

Recall that the classical real 3-dimensional Heisenberg group can be defined as a linear group of the following matrices:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. This group is isomorphic to the semidirect product $(\mathbb{R} \times \mathbb{R}) \leftthreetimes \mathbb{R}$ of $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$.

We need a natural generalization (see, for example, [18,16,12,14]) which is based on biadditive mappings. Let E, F, A be abelian groups. A map $w : E \times F \to A$ is said to be *biadditive* if the induced mappings

$$w_x : F \to A, \qquad w_f : E \to A, \qquad w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$. We say that w is *separated* if the induced homomorphisms separate points. That is, for every non-zero $x_0 \in E$, $f_0 \in F$ there exist $f \in F$, $x \in E$ such that $f(x_0) \neq 0_A$, $f_0(x) \neq 0_A$, where 0_A is the zero element of A.

Definition 2.1. Let E, F and A be Hausdorff abelian topological groups and $w : E \times F \to A$ be a continuous biadditive mapping. Define the induced action of F on $A \times E$ by

$$w^{\nabla} : F \times (A \times E) \to (A \times E), \qquad w^{\nabla}\big(f, (a, x)\big) = \big(a + f(x), x\big).$$

Every translation under this action is an automorphism of the group $A \times E$. Denote by

$$H(w) = (A \times E) \leftthreetimes F$$

the semidirect product of F and the group $A \times E$. The resulting group, as a topological space, is the product $A \times E \times F$. This product topology will be denoted by γ. The group operation is defined by the following rule: for a pair

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2)$$

define

$$u_1 \cdot u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big)$$

where $f_1(x_2) = w(x_2, f_1)$.

Then $H(w)$ becomes a two-step nilpotent Hausdorff topological group. We call it the *generalized Heisenberg group* induced by w.

Intuitively we can describe the group $H(w)$ in the matrix form

$$\begin{pmatrix} 1 & F & A \\ 0 & 1 & E \\ 0 & 0 & 1 \end{pmatrix}.$$

Elementary computations for the commutator $[u_1, u_2]$ give

$$[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1} = \big(f_1(x_2) - f_2(x_1), 0_E, 0_F\big).$$

Sometimes we will identify E with $\{0_A\} \times E \times \{0_F\}$, F with $\{0_A\} \times \{0_E\} \times F$ and $E \times F$ with $\{0_A\} \times E \times F$.

In the case of a normed space X and the canonical bilinear function $w : X \times X^* \to \mathbb{R}$ we write $H(X)$ instead of $H(w)$. Clearly, the case of $H(\mathbb{R}^n)$ (induced by the scalar product $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$) gives the classical $(2n+1)$-dimensional Heisenberg group.

Definition 2.2. Let (E, σ), (F, τ), (A, ν) be abelian Hausdorff groups such that the separated biadditive mapping

$$w : (E, \sigma) \times (F, \tau) \to (A, \nu) \tag{$*$}$$

is continuous.

(a) A triple $(\sigma_1, \tau_1, \nu_1)$ of coarser Hausdorff group topologies $\sigma_1 \subseteq \sigma$, $\tau_1 \subseteq \tau$, $\nu_1 \subseteq \nu$ on E, F and A, respectively, is called *compatible*, if

$$w : (E, \sigma_1) \times (F, \tau_1) \to (A, \nu_1)$$

is continuous.

(b) The biadditive mapping $(*)$ is called *strongly minimal* if for every compatible triple $(\sigma_1, \tau_1, \nu_1)$ it follows that $\sigma_1 = \sigma$, $\tau_1 = \tau$.

(c) ([12]) We say that the biadditive mapping is *minimal* if $\sigma_1 = \sigma$, $\tau_1 = \tau$ holds for every compatible triple (σ_1, τ_1, ν) (with $\nu_1 := \nu$).

Observe that for every compatible triple $(\sigma_1, \tau_1, \nu_1)$ the groups (E, σ_1) and (F, τ_1) admit continuous injective homomorphisms in the powers $(A, \nu_1)^F$ and $(A, \nu_1)^E$, respectively. Therefore, the topologies σ_1 and τ_1 are automatically Hausdorff assuming only that ν_1 on A is Hausdorff.

Note that the multiplication map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is minimal but not strongly minimal when $\mathbb{Z}$ carries the discrete topology (see Proposition 2.4(3) for a more general statement).

The next lemma collects some trivial, but useful observations.

Lemma 2.3.

(1) *Every strongly minimal map is minimal.*
(2) *Every compatible triple* $(\sigma_1, \tau_1, \nu_1)$ *of topologies gives rise to the corresponding (product) topology* γ_1 *on the Heisenberg group* $H(w) = (A \times E) \leftthreetimes F$ *which is a coarser Hausdorff group topology (that is,* $\gamma_1 \subseteq \gamma$*).*
(3) *If the mapping* w *is minimal and the group* A *is minimal then* w *is strongly minimal.*

Now we provide some examples of (strongly) minimal biadditive maps.

Proposition 2.4.

(1) *The canonical biadditive mapping* $\Delta: G \times G^* \to \mathbb{T}$, $\Delta(g, \chi) = \chi(g)$ *is strongly minimal for every locally compact abelian group* G.
(2) *The canonical bilinear mapping* $V \times V^* \to \mathbb{R}$, $\langle v, f\rangle = f(v)$ *is strongly minimal for all normed spaces* V (*where* V *and its dual* V^* *carry usual norm topologies*).
(3) *The multiplication map* $A \times A \to A$ *is minimal for every Hausdorff topological unital ring* A.

Proof. (1) By [12] the semidirect product $(G \times G^*) \leftthreetimes \mathbb{T}$ is a minimal group. Hence w is strongly minimal by Corollary 5.2 below (or, by assertions (2) and (3) of Lemma 2.3).

(2) We prove that the canonical bilinear mapping $V \times V^* \to \mathbb{R}$ is strongly minimal for every normed space V. Let σ, τ and ν are given topologies on V, V^* and $\mathbb{R}$. Assume that $(\sigma_1, \tau_1, \nu_1)$ is a compatible triple. That is, $\sigma_1 \subseteq \sigma$, $\tau_1 \subseteq \tau$ and $\nu_1 \subseteq \nu$ are coarser Hausdorff group topologies such that the mapping

$$w : (V, \sigma_1) \times (V^*, \tau_1) \to (\mathbb{R}, \nu_1), \qquad (v, f) \mapsto \langle v, f\rangle = f(v)$$

is continuous. We have to show that necessarily $\sigma_1 = \sigma$ and $\tau_1 = \tau$. Consider two cases:

(a) Suppose that σ_1 is strictly coarser than σ. It suffices to show that for every neighborhood P of zero in (V, σ_1) and every neighborhood of zero Q in (V^*, τ_1) we have $\langle P, Q\rangle = \mathbb{R}$ (this will imply that the mapping is not continuous at $(0, 0)$ since ν_1 is Hausdorff). First note that P necessarily is norm unbounded by [12, Lemma 3.5]. By Hahn–Banach theorem for every $x \in V$ there exists $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Therefore

$$\langle P, B^*\rangle = \{\langle v, f\rangle:\ v \in P,\ f \in B^*\}$$

is unbounded in $\mathbb{R}$ (where B^* is the unit ball of V^*). Since $f \in B^*$ implies that $cf \in B^*$ for every $0 < c \leqslant 1$ it follows that in fact $\langle P, B^*\rangle = \mathbb{R}$ and also $\langle P, cB^*\rangle = \mathbb{R}$ for every positive constant $c > 0$. On the other hand, Q contains cB^* for some $c > 0$ (because τ_1 is a coarser topology on V^*). So we obtain that indeed $\langle P, Q\rangle = \mathbb{R}$.

(b) The second case when τ_1 is strictly coarser than τ is very similar (and even easier we do not need even Hahn–Banach theorem).

(3) is trivial. □

3. Relative minimality and co-minimality

Clearly a topological group G is minimal in the usual sense (see Introduction) iff G is relatively minimal in G iff every subgroup H is relatively minimal in G iff every subgroup H is co-minimal in G iff $\{e_G\}$ is co-minimal in G. As a first step we shall push in appropriate way this obvious observation to arbitrary subgroups of H (see Corollary 3.2). To this end we need some preparation.

3.1. Permanence properties of relatively minimal subgroups

Let H be a subgroup of a topological group (G, γ). Then $\gamma\restriction_H$ will mean the subspace topology on H. The quotient topology on the left coset space $G/H := \{gH\}_{g\in G}$ will be denoted by γ/H.

The following well-known result is very useful (see for instance [8]).

Lemma 3.1 *(Merson's Lemma). Let* (G, γ) *be a* (*not necessarily Hausdorff*) *topological group and* H *be a* (*not necessarily closed*) *subgroup of* G. *Assume that* $\gamma_1 \subseteq \gamma$ *be a coarser group topology on* G *such that* $\gamma_1\restriction_H = \gamma\restriction_H$ *and* $\gamma_1/H = \gamma/H$. *Then* $\gamma_1 = \gamma$.

We shall use frequently the following immediate corollaries of Merson's lemma. The first one is straightforward (it is the promised generalization of the starting observation above):

Corollary 3.2. *A topological group* G *is minimal if and only if it contains a subgroup* H *which is both relatively minimal and co-minimal in* G.

Corollary 3.3. *Let* (G, γ) *be a* (*not necessarily Hausdorff*) *topological group and* H *be a dense subgroup of* G. *If* $\gamma_1 \subseteq \gamma$ *is a coarser group topology on* G *with* $\gamma_1\restriction_H = \gamma\restriction_H$, *then* $\gamma_1 = \gamma$.

Proof. Note that both γ_1/H and γ/H are indiscrete (so coincide), as H is dense in both topologies. Now Merson's lemma applies. □

We collect here some other useful properties of relative minimality.

Lemma 3.4.

(1) *If G is minimal, then every subset X is relatively minimal in G.*
(2) *Any compact subset X is relatively minimal in any Hausdorff group.*
(3) *Suppose $H \leqslant K \leqslant G$. Then H is relatively minimal in G, whenever*
  - *K is relatively minimal in G; or*
  - *H is relatively minimal in K.*

(4) *A subgroup H of a topological group G is relatively minimal in G iff its closure is relatively minimal in G.*
(5) *A dense subgroup H of a group G is relatively minimal iff G is minimal.*

Proof. (1), (2) and (3) are trivial.

(4) Let H be relatively minimal in (G,γ). Denote by $\overline{H}$ the corresponding γ-closure of H in G. Assume that $\gamma_1 \subseteq \gamma$ is a coarser Hausdorff group topology on G. By our assumption we have $\gamma\restriction_H = \gamma_1\restriction_H$. From Corollary 3.3 we get $\gamma\restriction_{\overline{H}} = \gamma_1\restriction_{\overline{H}}$.

The reverse direction follows from item (3).

(5) If H is a dense relatively minimal subgroup in G then by (3), G as the closure of H is relatively minimal in G. So G is minimal. Conversely, if G is minimal, then every subgroup of G is relatively minimal. □

3.2. Co-minimality and strongly closed subgroups

Here we study permanence properties of co-minimality and its connection to an appropriately defined class of closed subgroups (see Definition 3.8).

Let us start with some basic properties of co-minimality.

Lemma 3.5.

(1) *If $H \leqslant K \leqslant G$ and H is co-minimal in G then K is co-minimal in G.*
(2) *A subgroup H of a topological group G is co-minimal in G iff its closure is co-minimal in G.*
(3) *A dense subgroup K of G is always co-minimal in G.*
(4) *If τ' is a finer group topology on (G,τ), then the co-minimal subgroups of (G,τ') are also co-minimal in (G,τ) as well.*

Proof. (1) Take a Hausdorff group topology $\sigma \leqslant \tau$ on G, where τ is the original topology of G. Then co-minimality of H yields that $\sigma/H = \tau/H$ on G/H. Since G/K is in its turn a quotient of G/H, the equality $\sigma/H = \tau/H$ yields $\sigma/K = \tau/K$.

(2) According to (1) it suffices to show that H is co-minimal in G whenever its closure $N := \overline{H}$ is co-minimal in G. Take any Hausdorff group topology $\sigma \leqslant \tau$ on G, where τ is the original topology of G. By our hypothesis we get $\sigma/N = \tau/N$ on G/N. To show that $\sigma/H = \tau/H$ let us use the fact that both spaces are homogeneous and $\sigma/H \leqslant \tau/H$. Thus it suffices to see that if O is a neighborhood of $\varepsilon = e \cdot H$ in $(G/H, \tau/H)$, then O is a neighborhood of ε in $(G/H, \sigma/H)$ as well. Let U be a neighborhood of e in G such that $O = U \cdot H$ and pick a neighborhood V of e in G such that $V^2 \subseteq U$. Then $V \cdot N$ is a neighborhood of $e \cdot N$ in $(G/N, \tau/N)$. As $\sigma/N = \tau/N$, the set $V \cdot N$ is a neighborhood of $e \cdot N$ in $(G/N, \sigma/N)$ as well. Then there exists a neighborhood W of e in (G,σ) such that $W \subseteq W \cdot N \subseteq V \cdot N$. Thus $W \subseteq V \cdot N \subseteq V^2 \cdot H \subseteq U \cdot H$, since H is dense in N. This proves that $W \cdot H \subseteq U \cdot H$. Therefore, $\sigma/H = \tau/H$.

(3) Use (2) and the fact that G is co-minimal in G.

(4) is obvious. □

Lemmas 3.4 and 3.5 allow us to consider mainly closed subgroups in questions related to co-minimality and relative minimality.

Example 3.6.

(a) Let X and Y be co-minimal subgroups in G. Then $X \cap Y$ need not be a co-minimal subgroup in G. Indeed take X and Y to be dense subgroups of a non-minimal group G with trivial $X \cap Y$. Then $X \cap Y$ cannot be co-minimal since G is not minimal, while X and Y are co-minimal by item (3) of Lemma 3.5. (For example, let G be a subgroup of the circle group $\mathbb{T}$ generated by two rationally independent irrationals $\alpha, \beta \in T$. Then the decomposition $G = X \oplus Y$ with $X = \langle\alpha\rangle$, $Y = \langle\beta\rangle$ works.)
(b) (Groups with many co-minimal subgroups) This can be realized in a trivial way when the group itself is minimal, so all its subgroups are co-minimal. Other source of such groups are groups G with all proper subgroups dense. For example, if τ is the topology on $\mathbb{Z}$ obtained by an arbitrary embedding in $\mathbb{T}$, then every non-zero subgroup of $G = (\mathbb{Z},\tau)$ is dense (hence co-minimal) in G, while (0) is not co-minimal. More generally, one can take groups a monothetic group G with connected completion. If C is the dense cyclic subgroup of G, then every non-zero subgroup of C is dense, so co-minimal (both in C and G).

It seems a rather natural question to ask whether co-minimality of a closed normal subgroup H of a group G is related to the minimality of the quotient group G/H. In the sequel we discuss the precise relation between these two properties.

Remark 3.7.

(a) The quotient G/H w.r.t. a closed normal co-minimal subgroup H need not be minimal. Indeed, take any minimal non-totally minimal group G (one may choose simply $\mathbb{R} \rtimes \mathbb{R}_+$). Any closed normal subgroup H witnessing non-total minimality of G is co-minimal, but G/H fails to be minimal.
(b) Theorem 3.10 and Example 3.17 give a large supply of closed (normal) subgroups H of finite index that are not co-minimal. This shows, in particular, that the minimality of G/H alone (in the specific case G/H is even finite) does not guarantee co-minimality of H in G.

The notion of co-minimality contains an intrinsic difficulty related to the recourse to *quotients*, i.e., constructions leading out of the group. In order to understand better this property, we consider now a related property that allows one to remain in the group itself (it has also a natural connection to minimality). This new property will help us to describe properly the relation between co-minimality of a subgroup H of a group G and the minimality of the quotient group G/H (see item (b) of the above remark).

Definition 3.8. A subgroup H of a Hausdorff topological group (G, τ) is called *strongly closed* (resp., *strongly open*), if H is σ-closed (resp., σ-open) for every Hausdorff group topology $\sigma \subseteq \tau$ on G.

Example 3.9.

(a) Every compact subgroup is strongly closed.
(b) According to Markov's terminology, a subgroup H of a group G is *unconditionally closed*, if H is closed in any Hausdorff group topology on G (e.g., the center $Z(G)$, or the subgroups of the form $G[m] = \{x \in G\colon\ mx = 0\}$ of an abelian group G, where $m \in \mathbb{N}$). Obviously every unconditionally closed subgroup is strongly closed in any Hausdorff group topology.

Theorem 3.10. *Let H be a closed subgroup of a Hausdorff topological group (G, τ).*

(a) *If H is co-minimal, then H is strongly closed.*
(b) *If H is co-minimal and open, then H is strongly open.*
(c) *If H is a strongly closed normal subgroup of G and G/H is minimal, then H is co-minimal.*

Proof. (a) To check H is strongly closed consider a Hausdorff group topology $\sigma \subseteq \tau$ on G. Then $\sigma/H = \tau/H$. Since H is τ-closed, τ/H is Hausdorff. This yields that also σ/H is Hausdorff. This implies that H is σ-closed.

(b) Assume now that N is co-minimal and τ-open in G. In other words $(G/H, \tau/H)$ is discrete. Then $(G/H, \sigma/H)$ is discrete as well, so H is σ-open.

(c) If $\sigma \subseteq \tau$ is a Hausdorff group topology, then H is σ-closed by hypothesis. Hence σ/H is a Hausdorff group topology on G/H coarser than τ/H. Now the minimality of τ/H yields $\sigma/H = \tau/H$. □

Remark 3.11.

(a) Item (a) in Theorem 3.10 cannot be inverted, i.e., strongly closed normal subgroups need not be co-minimal.
(b) Item (c) cannot be inverted either. There exists a minimal group G, so that the center $H = Z(G)$ is minimal and co-minimal, yet G/H is not minimal. (Take the generalized Heisenberg group $H(w)$ obtained from the topological ring $(\mathbb{Z}, \tau_p)$ and the multiplication $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.)

Corollary 3.12. *If H is a closed normal subgroup of finite index of a topological group G, then H is co-minimal iff H is strongly closed.*

Remark 3.13. We do not know whether every strongly closed subgroup of a topological abelian group has finite index (so is automatically strongly open). A counter-example in the non-abelian case is given below (Example 4.9).

Definition 3.14. A topological group (G, τ) is *locally minimal* if there exists a neighborhood V of e such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of e, then $\sigma = \tau$. To underline that the neighborhood V witnesses local minimality for (G, τ), we say sometimes (G, τ) is *V-locally minimal* ([1]).

The following general properties connect co-minimality of open subgroups to local minimality:

Proposition 3.15. *Let H be an open subgroup of a topological group G. Then the following are equivalent:*

(a) *H is co-minimal and G is H-locally minimal*;
(b) *H is strongly open and G is H-locally minimal*;
(c) *G is minimal.*

Proof. The implication (a) → (b) follows from Theorem 3.10. To prove the implication (b) → (c) assume H is strongly open and G is H-locally minimal. Assume $\sigma \leqslant \tau$ is a Hausdorff group topology, where τ is the original topology of G. Then H is σ-open. Now the H-local minimality of (G, τ) implies $\sigma = \tau$. Finally, the minimality of G implies that H is co-minimal and G is H-locally minimal. This proves (c) → (a). □

Here is a corollary of the proposition.

Corollary 3.16. *An open subgroup H of a locally compact group G is co-minimal iff G is minimal.*

Proof. Let U be a compact neighborhood of e_G in G contained in H. Then G is U-locally minimal [6], so also H-locally minimal. Now the above proposition applies. □

One can put it also in this way: a locally compact group is minimal iff it admits an open co-minimal subgroup.

For every group G one has the non-empty family $\mathcal{C}(G)$ of subgroups that are co-minimal w.r.t. G. This family is up-ward closed and contains the family $\mathcal{D}(G)$ of all dense subgroups of G. Clearly, G is minimal iff $\mathcal{C}(G)$ is the family of *all* subgroups of G (iff $\mathcal{C}(G)$ contains some finite subgroup of G). For the sake of simplicity we consider also the family $\mathcal{C}_c(G)$ of all closed co-minimal subgroups of G.

Now we provide a large supply of non-co-minimal subgroups of $\mathbb{Z}$ in the next example.

Example 3.17. We describe here the closed co-minimal subgroups in a non-discrete Hausdorff group topology τ on $\mathbb{Z}$. To this end we need some invariants. For $p \in P$ let τ_p denote the p-adic topology, let $supp(\tau) = \{p \in P\colon\ p\mathbb{Z} \in \tau\}$ and $supp_\infty(\tau) = \{p \in P\colon\ \tau_p \leqslant \tau\}$. Let $o(\mathbb{Z}, \tau)$ be the intersection of all τ-open subgroups, and let $o_s(\mathbb{Z}, \tau)$ be the intersection of all strongly τ-open subgroups. Then $o(\mathbb{Z}, \tau) \leqslant o_s(\mathbb{Z}, \tau)$.

(a) $o_s(\mathbb{Z}, \tau) \neq 0$. Clearly, this occurs iff there are only finitely many strongly τ-open (i.e., co-minimal) subgroups. In this case there exists a smallest strongly τ-open subgroup $m\mathbb{Z} = o_s(\mathbb{Z}, \tau)$. All subgroups $d\mathbb{Z} \supseteq m\mathbb{Z}$ (with $d|m$) are closed and co-minimal.
(b) $o_s(\mathbb{Z}, \tau) = 0$. Now there are infinitely many strongly τ-open subgroups. They form a base of neighborhoods of 0 in a linear Hausdorff group topology $\tau_s \subseteq \tau$. By item (4) of Lemma 3.5, $\mathcal{C}_c(\mathbb{Z}, \tau) \subseteq \mathcal{C}_c(\mathbb{Z}, \tau_s)$. Actually, the strongly τ-open subgroup are precisely the τ_s-open subgroups of $\mathbb{Z}$. We prove that $supp_\infty(\tau)$ consits of a single prime p and $\mathcal{C}_c(\mathbb{Z}, \tau) = \{p^n\mathbb{Z}\colon\ n \in \omega\}$. Let us consider the following cases, of which the first two cannot occur:
   (i) $supp(\tau_s)$ is infinite. Now there are infinitely many primes (p_n) such that $p_n\mathbb{Z} \in \tau_s$. To get a contradiction fix p_0 and take the linear topology σ having as a prebase of neighborhoods of 0 the subgroups $\{p_n\mathbb{Z}\colon\ n > 0\}$. Then $\sigma \subseteq \tau$ is Hausdorff and $p_0\mathbb{Z}$ is σ-dense.
   (ii) $|supp_\infty(\tau_s)| > 1$. For $p \in supp(\tau_s)$, $p\mathbb{Z}$ is τ_s-open and τ_q-dense for any $q \in supp_\infty(\tau_s)$ and $q \neq p$. So $p\mathbb{Z} \notin \mathcal{C}_c(\mathbb{Z}, \tau)$, fails to be strongly closed, a contradiction.
   (iii) Now we are left with $|supp(\tau_s)| < \infty$ and $|supp_\infty(\tau_s)| \leqslant 1$. As τ_s is Hausdorff, $|supp(\tau_s)| < \infty$ yields $|supp_\infty(\tau_s)| = 1$, i.e., $supp_\infty(\tau_s) = \{p\}$ is a singleton. Now every Hausdorff group topology $\sigma \subseteq \tau$ is necessarily linear and consequently contains τ_p. Thus for every $q \neq p$ the subgroup $q \in \mathbb{Z}$ fails to be strongly open. By the definition of τ_s, this yields $supp(\tau) = supp_\infty(\tau) = \{p\}$ i.e., $\tau_s = \tau_p$. This proves $\mathcal{C}_c(\mathbb{Z}, \tau) = \{p^n\mathbb{Z}\colon\ n \in \omega\}$.

We do not know whether $m > 1$ may occur in item (a), i.e.,

Question 3.18. Let τ be a non-discrete Hausdorff topology on $\mathbb{Z}$ having only finitely many strongly open subgroups. Is it possible to have more than just one (namely, $\mathbb{Z}$) strongly open subgroups?

One can show that the answer is negative for maximally almost periodic topologies τ.

4. Relative minimality and co-minimality in products

From the very first stages of the development of the theory, minimality has been involved with products. More specifically, the fact that a direct product $G = H \times N$ fails to be minimal, even if both summands H and N are minimal (take for instance $\mathbb{Z}$ in 2-adic topology both as H and K [10]). Note that the quotient $N \cong G/H$ is certainly minimal, while H is relatively minimal (being minimal itself), so non-minimality of the group G yields that H cannot be co-minimal (see Corollary 3.2). By Theorem 3.10 this means that H is not even strongly closed. In other words, Theorem 3.10 provides a useful tool to check when a direct product $G = H \times N$ of minimal groups H, N is minimal: namely, iff H is strongly closed (or, equivalently, co-minimal) in G. This property obviously extends to semidirect products.

The goal of this section is to determine a sufficient condition for minimality of a semidirect product $M = X \leftthreetimes G$.

Definition 4.1. Let (X, τ) and (G, σ) be Hausdorff topological groups and $\alpha : G \times X \to X$ be a continuous action.

(1) X is a *G-group* if every g-translation $\alpha^g : X \to X$ is a group automorphism of X.
(2) X is a *G-minimal* group if X is a G-group and there is no strictly coarser Hausdorff group topology $\tau' \subsetneq \tau$ on X such that α remains continuous with respect to the triple (σ, τ', τ') of topologies (see [20]).
(3) The action α is *topologically exact* (*t-exact*, for short) if there is no strictly coarser (not necessarily Hausdorff) group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ', τ, τ)-continuous (see [13,15]).

Remark 4.2.

(a) Sometimes, we need to consider continuous actions $\alpha : G \times X \to X$, where G need not be Hausdorff, while X will always be assumed Hausdorff (see item (2) of the above definition). In such a case, the subgroup $\overline{\{e_G\}}$ trivially acts on X. In particular, the only continuous action of an indiscrete group G is the trivial action.
(b) Note that if X is a locally compact Hausdorff group and G is a subgroup of $Aut(X)$ endowed with the standard *Birkhoff topology* (see [8,12]) then the corresponding action is t-exact.
(c) If G is a simple minimal group, then the action is *t-exact* iff it is non-trivial. Indeed, obviously every t-exact action, with a non-trivial group G, is non-trivial. Conversely, the assumption that the action is not t-exact, along with the simplicity of G, would imply that G equipped with the indiscrete topology (the only strictly coarser group topology on G) is continuous. Then (a) would imply that the action is trivial.
(d) It is easy to see that every t-exact action is *algebraically exact* (that is, for every $g \neq e$ in G there exists $x \in X$ such that $gx \neq x$).
(e) If a topological semidirect product $(M, \gamma) := (X, \tau) \leftthreetimes_\alpha (G, \sigma)$ is minimal and $\alpha : G \times X \to X$ is algebraically exact then α is necessarily t-exact. Indeed, if G admits a strictly coarser group topology σ_1 such that the action remains continuous then this topology must be Hausdorff because the action is algebraically exact and X is Hausdorff. Then the topological semidirect product $(M, \gamma_1) := (X, \tau) \leftthreetimes_\alpha (G, \sigma_1)$ is a Hausdorff group, violating the minimality of the given group (M, γ).

Proposition 4.3. *Let E, F and A be Hausdorff abelian topological groups and $w : E \times F \to A$ be a biadditive mapping.*

(1) *The induced action*
$$w^\nabla : F \times (A \times E) \to (A \times E), \qquad w^\nabla\big(f, (a, x)\big) = \big(a + f(x), x\big)$$
is continuous iff w is continuous.
(2) *If w is a minimal biadditive mapping then the action w^∇ is t-exact.*

Proof. See [12, Lemma 2.2]. □

In the next proposition we characterize G-minimality in terms of relative minimality (this will be a necessary condition for minimality of $M = X \leftthreetimes G$).

Proposition 4.4. *Let (G, σ) be a Hausdorff group and let (X, τ) be a Hausdorff G-group. The following are equivalent*:

(1) *X is G-minimal.*
(2) *X is relatively minimal in the topological semidirect product $M := (X \leftthreetimes G, \gamma)$.*

Proof. (1) ⇒ (2): Let $\gamma_1 \subset \gamma$ be a coarser Hausdorff group topology on M. Since $M := (X \leftthreetimes_\alpha G, \gamma_1)$ is a topological group the conjugation map
$$(M, \gamma_1) \times (M, \gamma_1) \to (M, \gamma_1), \qquad (a, b) \to aba^{-1}$$
is continuous. Then its restriction
$$(G, \gamma_1|_G) \times (X, \gamma_1 \restriction_X) \to (X, \gamma_1 \restriction_X), \qquad (g, x) \to g(x) = gxg^{-1}$$
is also continuous. Since $\gamma_1 \restriction_G \subset \gamma \restriction_G$ it follows that the action of the given group $(G, \gamma \restriction_G)$ on the Hausdorff group $(X, \gamma_1 \restriction_X)$ is continuous, too. Since $\gamma_1 \restriction_X \subset \gamma \restriction_X$ and X is G-minimal we obtain $\gamma_1 \restriction_X = \gamma \restriction_X$.

(2) ⇒ (1): Let X be not G-minimal. Then there exists a strictly coarser Hausdorff group topology $\tau_1 \subset \tau$ on X such that the action $G \times (X, \tau_1) \to (X, \tau_1)$ remains continuous. Consider the corresponding semidirect product $(M, \gamma_1) := (X, \tau_1) \leftthreetimes G$. Then γ_1 is a coarser Hausdorff group topology on $(M, \gamma) := (X, \tau) \leftthreetimes G$ such that $\gamma_1 \restriction_X$ is strictly coarser than $\gamma \restriction_X$. This means that X is not relatively minimal in M. □

Corollary 4.5. *For every normed space X the subgroup X is relatively minimal in the semidirect product $X \leftthreetimes GL(X)$, where $GL(X)$ is endowed with the uniform operator topology.*

Proof. Indeed as it follows by [13, Theorem 2.2], X is $GL(X)$-minimal. □

According to Proposition 4.4, G-minimality of X in the next theorem is a *necessary* condition for the minimality of semidirect product $X \leftthreetimes G$. Easy examples (of direct products) show that t-exactness is *not* a necessary condition. At the same time t-exactness also becomes a necessary condition provided that the corresponding action is algebraically exact (see Remark 4.2(e)).

Theorem 4.6. *Let (G, σ) be a Hausdorff group and let (X, τ) be a Hausdorff abelian G-minimal G-group. If the given action $\alpha : G \times X \to X$ is t-exact then the topological semidirect product $M := (X \leftthreetimes G, \gamma)$ is minimal.*

Proof. By Corollary 3.2 it suffices to prove that X is both relatively minimal and co-minimal in M. Let $\gamma_1 \subseteq \gamma$ be a coarser Hausdorff group topology on $X \leftthreetimes G$. First we show that the action

$$\alpha : (G, \gamma_1/X) \times (X, \gamma_1 \restriction_X) \to (X, \gamma_1 \restriction_X)$$

is continuous. This can be derived by [12, Proposition 2.6]; we give a more direct proof here. Each g-transition $(X, \gamma_1 \restriction_X) \to (X, \gamma_1 \restriction_X)$ is continuous (because it is a restriction of a continuous g-transition $(M, \gamma_1) \to (M, \gamma_1)$ for every $g \in G$). Therefore it suffices to show that α is continuous at (e_G, y) for every $y \in X$. Fix an arbitrary $y \in X$ and a $\gamma_1 \restriction_X$-neighborhood $O(y)$ of y. Choose a neighborhood U_1 of $y := (e_G, y)$ in (M, γ_1) such that $U_1 \cap X \subseteq O$. By the continuity of the map

$$(M, \gamma_1) \times (M, \gamma_1) \to (M, \gamma_1), \quad (a, b) \mapsto aba^{-1}$$

there exist neighborhoods V of the identity in (M, γ_1) and $U_2(y)$ of y in (M, γ_1) such that

$$vU_2v^{-1} \subseteq U_1$$

for every $v \in V$. We claim that

$$\alpha(g, z) := gz \in O \quad \forall g \in pr(V)\ \forall z \in U_2 \cap X,$$

where $pr : M \to G$ is the natural projection. Indeed, if $v = (x, g) \in V$ and $z \in U_2 \cap X$ then $vzv^{-1} \in U_1$. From the normality of the subgroup X in M we have in fact $vzv^{-1} \in U_1 \cap X \subseteq O$. Since X is abelian we get $vzv^{-1} = x\alpha(g, z)x^{-1} = gz$. Therefore, $gz \in O$ for every $z \in U_2 \cap X$ and $g \in pr(V)$. This proves the $(\gamma_1/X, \gamma_1 \restriction_X, \gamma_1 \restriction_X)$-continuity of α because $pr(V)$ is a γ_1/X-neighborhood of the identity in G and $U_2 \cap X$ is a neighborhood of y in $(X, \gamma_1 \restriction_X)$.

By Proposition 4.4 we know that X is relatively minimal in M. Hence $\gamma_1 \restriction_X = \gamma \restriction_X$. We obtain that α is $(\gamma_1/X, \gamma \restriction_X, \gamma \restriction_X)$-continuous. On the other hand, the action of G on (X, τ) is t-exact (Definition 4.1). Therefore, $\gamma_1/X = \sigma = \gamma/X$. This means that X is also a co-minimal subgroup in (M, γ). By Corollary 3.2 we can conclude that (M, γ) is minimal. □

Question 4.7. Is it true that every normed space X is co-minimal in $X \leftthreetimes GL(X)$? What if X is a Hilbert space?

Observe that it is equivalent to replace "X is co-minimal in $X \leftthreetimes GL(X)$" by "$X \leftthreetimes GL(X)$ is minimal" or by "the action of $GL(X)$ on X is t-exact" (see Theorem 4.6 and Remark 4.2(e)).

Example 4.8. Let X be an abelian group and let $\alpha : \mathbb{Z}_2 \times X \to X$ be the action defined by $\alpha(\rho, x) = -x$, where ρ is the non-trivial element of $\mathbb{Z}_2$.

(a) This action is trivial iff X is a Boolean group (i.e., a group of exponent 2).
(b) For every Hausdorff group topology τ on X, the action is continuous whenever $\mathbb{Z}_2$ carries the discrete topology. In such a case X is a $\mathbb{Z}_2$-minimal group iff the topological group (X, τ) is minimal.

In particular, the semidirect product $X \leftthreetimes \mathbb{Z}_2$ is minimal whenever (X, τ) is a minimal abelian group (if the action is t-exact, one applies the above theorem, otherwise one uses the fact that a *direct* product of a minimal group and a compact group is minimal [10]).

According to [11, Example 10] there exists a (totally) minimal precompact non-abelian group X such that a certain semidirect product $X \leftthreetimes \mathbb{Z}_2$ with the two-element cyclic group $\mathbb{Z}_2$ is not minimal. Then the given action of $\mathbb{Z}_2$ on X is necessarily non-trivial, as the *direct* product of a minimal group with a compact group (in the given case $\mathbb{Z}_2$) is always minimal [10]. According to Remark 4.2(b) the action is t-exact. Since X is clearly also $\mathbb{Z}_2$-minimal, this example demonstrates that Theorem 4.6 is not true in general for non-abelian X.

Example 4.9. Let D be an infinite discrete group and let G be the wreath products of a compact group K with H, i.e., $D \rightthreetimes K^D$, where the action of D on the compact group K^D is given by the coordinatewise shift. It was proved by Schwanengel [21] that G is a locally compact minimal group (see also [8, 7.2.7]). Clearly the open subgroup $H = K^D$ is co-minimal, so strongly open (by Theorem 3.10(b)), while $G/H \cong D$ is infinite.

5. Relative minimality and co-minimality in Heisenberg type groups

Theorem 5.1. *Let $w : (E, \sigma) \times (F, \tau) \to (A, \nu)$ be a strongly minimal biadditive mapping. Then*:

(1) *A, $A \times E$ and $A \times F$ are co-minimal subgroups of the Heisenberg group $H(w)$.*
(2) *$E \times F$ is a relatively minimal subset in $H(w) = (A \times E) \rightthreetimes F$.*
(3) *The subgroups E and F are relatively minimal in $H(w)$.*

Proof. Denote by γ the given product topology on $H(w)$. Let $\gamma_1 \subseteq \gamma$ be a coarser Hausdorff group topology on $H(w)$.

(1) We have to check that $\gamma_1/A \times E = \gamma/A \times E$, $\gamma_1/A \times F = \gamma/A \times F$ and $\gamma_1/A = \gamma/A$.

First we present the arguments for the first case. It is sufficient to establish the continuity of the map

$$w : (E, \gamma_1 \restriction_E) \times (F, \gamma_1/A \times E) \to (A, \gamma_1 \restriction_A). \qquad (*)$$

Indeed, if $(*)$ is continuous we can assume that $(F, \gamma_1/A \times E)$ necessarily is a Hausdorff group (because $(A, \gamma_1 \restriction_A)$ is a Hausdorff topological group and $w : E \times F \to A$ is separated). Therefore the triple $(\gamma_1 \restriction_E, \gamma_1/A \times E, \gamma_1 \restriction_A)$ is compatible. Since the given biadditive mapping is strongly minimal it will follow that the topology $\gamma_1/A \times E$ on F coincides with the given topology $\tau = \gamma/A \times E$.

We proof the continuity of the map $(*)$ at an arbitrary pair $(x_0, f_0) \in E \times F$. Let O be a neighborhood of $f_0(x_0)$ in $(A, \gamma_1 \restriction_A)$. Choose a neighborhood O' of $(f_0(x_0), 0_E, 0_F)$ in $(H(w), \gamma_1)$ such that $O' \cap A = O$. Consider the points $\bar{x}_0 := (0_A, x_0, 0_F)$, $\bar{f}_0 := (0_A, 0_E, f_0) \in H(w)$. Observe that the commutator $[\bar{f}_0, \bar{x}_0]$ is just $(f_0(x_0), 0_E, 0_F)$. Since $(H(w), \gamma_1)$ is a topological group there exist γ_1-neighborhoods U and V of $\bar{x}_0$ and $\bar{f}_0$ respectively such that $[v, u] \in O'$ for every pair $v \in V$, $u \in U$. In particular, for every $\bar{y} := (0_A, y, 0_F) \in U \cap E$ and $v := (a, x, f) \in V$ we have $[v, \bar{y}] = (f(y), 0_E, 0_F) \in O' \cap A = O$. We obtain that $f(y) \in O$ for every $f \in q_F(V)$ and $\bar{y} \in U \cap E$. This means that we have the continuity of $(*)$ at (f_0, x_0) because $q_F(V)$ is a neighborhood of f_0 in the space $(F, \gamma_1/A \times E)$ and $U \cap E$ is a neighborhood of x_0 in $(E, \gamma_1|E)$.

Quite similarly one can prove that the following map is continuous

$$w : (E, \gamma_1/A \times F) \times (F, \gamma_1 \restriction_F) \to (A, \gamma_1 \restriction_A)$$

which implies that $\gamma_1/A \times F = \gamma/A \times F$. Hence $A \times F$ is co-minimal in $H(w)$.

By the equalities $\gamma_1/A \times E = \gamma/A \times E = \tau$ in F and $\gamma_1/A \times F = \gamma/A \times F = \sigma$ in E it follows that the maps

$$q_E : \big(H(w), \gamma_1\big) \to (E, \sigma), \qquad (a, x, f) \mapsto x$$

and

$$q_F : \big(H(w), \gamma_1\big) \to (F, \tau), \qquad (a, x, f) \mapsto f$$

are continuous. Then we obtain that

$$q_{E \times F} : \big(H(w), \gamma_1\big) \to E \times F, \qquad (a, x, f) \mapsto (x, f)$$

is also continuous, where $E \times F$ is endowed with the product topology induced by the pair of topologies (σ, τ). This topology coincides with γ/A. Then $\gamma_1/A \supseteq \gamma/A$. Since $\gamma_1 \subseteq \gamma$ we have $\gamma_1/A = \gamma/A$. Thus, A is co-minimal in $H(w)$.

(2) Since A is co-minimal in $H(w)$ we have $\gamma_1/A = \gamma/A$. It follows that the projection

$$q_{E \times F} : \big(H(w), \gamma_1\big) \to (E \times F, \gamma \restriction_{E \times F}), \qquad (a, x, f) \mapsto (x, f)$$

is continuous. Since $q_{E \times F}$ is a retraction and $\gamma_1 \subseteq \gamma$ we get $\gamma_1 \restriction_{E \times F} = \gamma \restriction_{E \times F}$.

(3) directly follows from (2) and Lemma 3.4. □

The next result which strengthens [12, Theorem 2.10] can be derived from the above theorem.

Corollary 5.2. *The following conditions are equivalent*:

(1) *$H(w)$ is a minimal group.*
(2) *A is a minimal group and w is a minimal* (*equivalently*: *strongly minimal*) *biadditive mapping.*

Proof. (2) ⇒ (1): By Lemma 2.3(3) the map w is strongly minimal. Therefore, A is co-minimal in $(H(w), \gamma)$ by Theorem 5.1. But A is also relatively minimal in $H(w)$ (being a minimal group). Now Corollary 3.2 implies that $H(w)$ is minimal.

(1) $\Rightarrow$ (2): Observe that A is a closed central subgroup of the minimal group $H(w)$, so A is minimal [8]. This can be checked also directly as follows. Assume for a contradiction that $\nu_1 \subsetneq \nu$ is a strictly coarser Hausdorff group topology on A then (τ, σ, ν_1) is a compatible triple. The corresponding Heisenberg group (see Lemma 2.3(2)) has a strictly coarser Hausdorff group topology on $H(w)$. This contradicts the minimality of $H(w)$. Analogously one may check that w is a minimal biadditive mapping. □

Theorem 5.3. *Let X be a normed space and $H(X) = (\mathbb{R} \times X) \leftthreetimes X^*$ be the corresponding Heisenberg group. Then*

(1) *$X \times X^*$ is a relatively minimal subset in the Heisenberg group $H(X)$.*
(2) *$\mathbb{R}$, $\mathbb{R} \times X$ and $\mathbb{R} \times X^*$ are co-minimal subgroups in $H(X)$.*

Proof. Use Proposition 2.4(2) and Theorem 5.1. □

Note that $H(X)$ itself is not minimal, as its center $Z(H(X)) \cong \mathbb{R}$ is not minimal [22].

Theorem 5.4. *Let G be a locally compact abelian group. Denote by $H(G) = (\mathbb{T} \times G) \leftthreetimes G^*$ the Heisenberg group of the canonical biadditive mapping $\Delta : G \times G^* \to \mathbb{T}$. Then*

(1) *$\mathbb{T}$ is co-minimal in $H(G)$.*
(2) *$H(G)$ is minimal.*

Proof. (1) Use Proposition 2.4(1) and Theorem 5.1(1) to show that $\mathbb{T}$ is co-minimal in $H(G)$.
(2) Clearly, $\mathbb{T}$ is also relatively minimal (being compact) in $H(G)$. Therefore $H(G)$ is minimal by Corollary 3.2. □

The minimal abelian groups are precompact by the well-known theorem of Prodanov and Stoyanov. This motivated the question, raised by the first named author (see [4, Question 3.5]), of whether this remains true for nilpotent groups. The following corollary negatively answers this question.

Corollary 5.5. *There exists a two-step nilpotent non-abelian minimal (locally compact) group which is not precompact.*

Proof. Take for example $G := H(\mathbb{T}) = (\mathbb{T} \times \mathbb{T}) \leftthreetimes \mathbb{Z}$. □

Corollary 5.6.

(1) *Let A be a unital topological ring such that A, as a topological group, is minimal. Then the Heisenberg group $H(w) = (A \times A) \leftthreetimes A$ of the multiplication mapping $w : A \times A \to A$ is minimal.*
(2) *In particular, the Heisenberg group $H(w) = (\mathbb{Z} \times \mathbb{Z}) \leftthreetimes \mathbb{Z}$ of the mapping $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p) \to (\mathbb{Z}, \tau_p)$ is a minimal two-step nilpotent precompact group for every p-adic topology τ_p.*

Proof. (1) Use Corollary 5.2 and Proposition 2.4(3).
(2) directly follows from (1). □

Note that Corollary 5.6(2) has been proved first in [4, Example 3.6] by other arguments.

In [4, Question 3.7] the first named author posed the following.

Question 5.7. Prove or a disprove that a precompact nilpotent group G is minimal if and only if the center $Z(G)$ is minimal.

The following example negatively solves this question.

Example 5.8. There exists a precompact ring topology σ on $\mathbb{Z}$ strictly finer than τ_2 (e.g., the profinite topology of $\mathbb{Z}$). Then the group $G := ((\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \sigma)) \leftthreetimes (\mathbb{Z}, \sigma)$ is the desired precompact non-minimal group with minimal center $Z(G) = (\mathbb{Z}, \tau_2)$. Indeed, the triple (τ_2, τ_2, τ_2) is compatible and the corresponding Hausdorff group topology is strictly coarser than the original topology generated by the triple (σ, σ, τ_2).

Remark 5.9. Some interesting applications of generalized Heisenberg groups and in particular of Corollary 5.2 can be found in [9,19,5].

6. Relative minimality criteria

According to a theorem of Uspenskij [23], every topological group X embeds into some minimal group G. Therefore, X is both relatively minimal and co-minimal in G. (Note that the group G with these properties must be necessarily minimal by Corollary 3.2.) This can be substantially improved by using Theorem 1.3:

Proposition 6.1. *Every topological group X is a* closed *relatively minimal subgroup and a co-minimal subgroup into some group G.*

Proof. By Theorem 1.3 every Hausdorff topological group X is a group retract of a minimal group G. □

We shall see in Corollary 6.7 that this proposition is no more true if we replace *closed* by *dense*. Indeed, if X is abelian, then the existence of a group G such that X is relatively minimal and dense in G implies that the group X is precompact.

The next corollary introduces the interesting class of groups that have minimal (two-sided) completion.

Corollary 6.2. *For a topological group X The following are equivalent*:

(i) *X is dense and relatively minimal in some group G*;
(ii) *X is relatively minimal in its (two-sided) completion $\widetilde{X}$*;
(iii) *$\widetilde{X}$ is minimal.*

The precompact groups X have the properties described in the above corollary. We shall see below (Corollary 6.7) that for abelian X these properties imply precompactness. A subgroup H of a topological group G is said to be essential if it non-trivially meets every closed non-trivial subgroup of G.

Lemma 6.3. *Let H be a dense essential subgroup of a topological group $(G, \mathcal{T})$. Then*:

(a) *A subgroup X of H is relatively minimal in G if and only if X is relatively minimal in H.*
(b) *The following conditions are equivalent*:
  (b_1) *H is minimal*;
  (b_2) *G is minimal*;
  (b_3) *H is relatively minimal in G.*

Proof. (a) Let $\mathcal{T}' = \mathcal{T}\restriction_H$. Consider a coarser Hausdorff group topology σ on H. Then the identity $i : (H, \mathcal{T}') \to (H, \sigma)$ can be extended to a continuous homomorphism $j : (G, \mathcal{T}) \to H_1$, where H_1 is the two-sided completion of (H, σ). Then $N = \ker j$ is a closed normal subgroup of G. Since $N \cap H$ is trivial, the essentiality of H yields $N = \{e\}$. Hence j is injective, and consequently the restriction of j to X is open onto the image as X is relatively minimal in G, i.e., $\sigma\restriction_X = \mathcal{T}'\restriction_X$. This proves that X is relatively minimal in H as well.

(b) Obviously, (b_2) ⇒ (b_3). By (a) applied to $X = H$, (b_3) ⇒ (b_1). Finally, (b_1) ⇒ (b_2) by Lemma 3.4(5). □

As a corollary we obtain an immediate proof of the celebrated minimality criterion due to Stephenson, Banaschewski and Prodanov:

Corollary 6.4 *(Minimality criterion). A dense subgroup H of a topological group G is minimal iff G is minimal and H is essential in G.*

Proof. Suppose that the subgroup H is minimal. Then Lemma 3.4(5) can be applied to H to conclude that G is minimal. We prove that H is essential in G. If N is a closed normal subgroup of G then $N \cap H = \{e\}$ would imply that the restriction of the canonical homomorphism $f : G \to G/N$ to H is injective, hence it must be an embedding. This is possible only if $N = \{e\}$. Indeed, let $a \in N$. Since H is dense in G there exists a net $h_\alpha \in H$ such that $a = \lim h_\alpha$. Then $e = f(a) = \lim f(h_\alpha)$ in $f(H) \cong H$, so $a = e$ as $f\restriction_H : H \to f(H)$ is a homeomorphism by the minimality of H.

If G is minimal and H is a dense essential subgroup in G then we obtain the minimality of H by Lemma 6.3(b). □

The following theorem holds:

Theorem 6.5. *Let X be a central subgroup of a topological group (G, τ) and let H denote the closure of X in G. Then the following are equivalent*:

(a) *X is relatively minimal in G*;
(b) *X is relatively minimal in H*;
(c) *H is a minimal topological group.*

Proof. (c) → (a) is obvious.

(a) → (b) Let $\sigma \leqslant \tau\restriction_H$ be a Hausdorff group topology on H. We show that there exists a Hausdorff group topology $\Sigma \leqslant \tau$ on G such that $\Sigma\restriction_H = \sigma$. Then by the relative minimality of X in G we conclude that $\Sigma\restriction_X = \sigma\restriction_X = \tau\restriction_X$. As a typical basic neighborhood of 1 in Σ take a product UV, where U is a τ-neighborhood of 1 in G and V is a σ-neighborhood of 1 in H. Since H is central, one can easily show that this defines a group topology on G. To see that it is Hausdorff take any $x \neq 1$ in G. If $x \notin H$, then by the closedness of H there exists a τ-neighborhood U of 1 such that $x \notin UH \in \Sigma$. If $x \in H$, then find a σ-neighborhood V of 1 in H such that $x \notin VV$. By $\sigma \leqslant \tau\restriction_H$ there exists a τ-neighborhood U of 1 such that $U \cap H \subseteq V$. Then $x \notin UV$. Finally, $\Sigma \leqslant \tau$ and $\Sigma\restriction_H = \sigma$ are obvious.

(b) → (c) follows from Lemma 3.4(4). □

As a corollary one gets:

Corollary 6.6. *A closed central subgroup X of a topological group G is relatively minimal in G iff X is minimal. In particular, the center $Z(G)$ of G is relatively minimal iff $Z(G)$ is minimal.*

One cannot omit "central" above. Indeed, $\mathbb{R}$ is relatively minimal in a non-abelian topological group (namely, in the Heisenberg group $H(\mathbb{R}) = (\mathbb{R} \times \mathbb{R}) \leftthreetimes \mathbb{R}$). That group is nilpotent (but cannot be abelian). By the way there is another natural group G containing $\mathbb{R}$ that makes it relatively minimal namely the minimal matrix group first considered by Dierolf and Schwanengel [3] (it is isomorphic to the semidirect product of $\mathbb{R}$ with the multiplicative group $\mathbb{R}_+ := (0, \infty)$). This group is no more nilpotent, but it is meta-abelian (step two soluble). For further comments in this direction see Section 3.2.

Corollary 6.7. *For a topological abelian group X the following are equivalent*:

(a) *X is relatively minimal in some abelian topological group G*;
(b) *X is relatively minimal in some topological group G containing X as a central subgroup*;
(c) *X is relatively minimal in some topological group G containing X as a dense subgroup*;
(d) *X is precompact.*

Corollary 6.8. *A complete abelian group X is relatively minimal in some abelian group G iff X is compact.*

This corollary shows again that the reals $\mathbb{R}$ cannot be relatively minimal in any abelian topological group G.

We give now a complete characterization of the co-minimal subgroups and the strongly closed subgroups of the precompact groups. According to Lemma 3.4, we can limit the criterion for co-minimality to closed subgroups.

Theorem 6.9. *Let G be a precompact group with compact completion K and let H be a closed subgroup of G.*

(a) *H is co-minimal iff $H = G \cap N\overline{H}^K$ for every closed normal subgroup N of K with*

$$N \cap G = \{1\}. \tag{1}$$

(b) *H is co-minimal iff $N \leqslant \overline{H}^K$ for every closed normal subgroup N of K with* (1).

Proof. Let us see first the role of the restraint (1) for a closed normal subgroup N of K. Denote by $q: K \to K/N$ the canonical homomorphism. Then (1) yields that the restriction of q to G is injective so it induces a coarser Hausdorff group topology σ on G. Vice versa, every coarser Hausdorff group topology σ on G can be obtained in this way w.r.t. an appropriate closed normal subgroup N of K with (1). Indeed, it suffices to take the extension to the completions of the continuous identity map $G \to (G, \sigma)$. If K_1 denotes the completion of (G, σ), then this extension $h: K \to K_1$ has dense image, so must be surjective (by the compactness of K). Now $N = \ker h$ is the closed normal subgroup N of K with (1).

Next we need a property, established first by Grant in the case when L is a closed normal subgroup of K. We give a proof for the sake of completeness.

Claim. *Let K be a topological group, let G be a dense subgroup of K and let L be a closed subgroup of K such that $L \cap G$ is dense in L. Then the restriction of the open map $f: K \to K/L$ to G gives an open map $G \to f(G)$.*

Proof. Let U be an open neighborhood of e in G. Pick an open symmetric neighborhood V of e in G such that $V^2 \subseteq U$. Then $W = \overline{V}^K$ is a neighborhood of e in K. It suffices to check that $f(W) \cap f(G) \subseteq f(U)$.

Let $f(g) \in f(W) \cap f(G)$, where $g \in G$. Then there exists $w \in W$ such that $f(g) = f(w)$, i.e., $gL = wL$. Then $g = wl$ for some $l \in L$. By the density of $L \cap G$ in L, we can write $l = \lim h_\alpha$ with $h_\alpha \in L \cap G$. Let $w = \lim_\alpha v_\alpha$, with $v_\alpha \in V$. Then $\lim_\alpha g h_\alpha^{-1} v_\alpha^{-1} = e$ in G. Hence there exists α such that $g h_\alpha^{-1} v_\alpha^{-1} \in V$. Therefore, $g \in V^2(L \cap G) \subseteq UL$. Hence $f(g) \in f(U)$. □

To continue the proof of the theorem denote by L the closure $\overline{H}^K$ of H in K and by $f : K \to K/L$ the open canonical map. By the claim, the restriction of f to G is open. Therefore, the image $f(G)$ with the topology induced by K/L is topologically isomorphic to the quotient homogeneous space $(G/H, \tau/H)$.

To prove (a) assume that σ is a coarser Hausdorff group topology on G. Then (G, σ) is isomorphic to a subgroup of the quotient group K/N for some closed normal subgroup N of K with (1). Since the projection $q : K \to K/N$ is a closed map, $q(L) = NL/N$ is a closed subgroup of K/N. Moreover, $q(H)$ is dense in NL/N, so $NL/N = \overline{q(H)}$. Hence, the σ-closure of H in G is $G \cap q^{-1}(q(L)) = G \cap NL$. Therefore, H is σ-closed iff $H = G \cap NL$.

(b) Assume first that $N \leqslant L$ for every closed normal subgroup N of K with (1). To prove that H is co-minimal assume that σ is a coarser Hausdorff group topology on G. Then (G, σ) is isomorphic to a subgroup of the quotient group K/N for some closed normal subgroup N of K with (1). Let $q : K \to K/N$ be the projection. By our hypothesis, $N \leqslant L$ we conclude that there exists a continuous map $\eta : K/N \to K/L$ such that $\eta \circ q = f$. So by the claim the quotient topology σ/H on G/H, can be obtained also as the topology induced on $\eta(q(G))$ by K/L. Since the image $\eta(q(G))$ coincides with $f(G)$, this proves that $\sigma/H = \tau/H$ on G/H.

Now assume that H is co-minimal. Consider a closed normal subgroup N of K with (1). We have to prove that $N \leqslant L$. To this end consider the projection $q : K \to K/N$ and the coarser topology $\sigma \subseteq \tau$ on G induced by the injective map $q : G \to K/N$. Then $\sigma/H = \tau/H$ since H is co-minimal. Moreover, for the same reason, H must be strongly closed (Theorem 3.10), so $G \cap NL = H$ by (a).

Being K/N the completion of (G, σ), the closure of $H = q(H)$ in K/N is precisely $q(L) = (L \cdot N)/N$ by the compactness of L. Therefore, arguing as above we conclude that $(G/H, \sigma/H)$ isomorphic to the image of $q(G)$ under the quotient map

$$\xi : K/N \to (K/N)/q(L) = (K/N)/\big((L \cdot N)/N\big) \cong K/(L \cdot N). \tag{2}$$

Since $\xi \circ q = \psi \circ f$, where $\psi : K/L \to K/(L \cdot N)$ is the canonical map, we conclude that the restriction of ψ to $f(G) = (G/H, \tau)$ sends this group isomorphically to $\psi(f(G)) = \xi(q(G)) = (G/H, \sigma/H)$. The restriction $\psi \restriction_{f(G)}$ is open by our hypothesis $\sigma/H = \tau/H$. To check $N \leqslant L$ take any $a \in N$ and let $a = \lim g_\alpha$ with $g_\alpha \in G$. Then for $x_\alpha = f(g_\alpha H) \in G/H$ one has $x_\alpha \to a.H$ in $(G/H, \tau/H)$. By the continuity of ψ and $a \in N$ one gets $\psi(x_\alpha) \to e.NL$ in K/NL, i.e, $\psi(x_\alpha) \to e.H$ in $(G/H, \sigma/H)$. Since $\psi \restriction_{f(G)}$ is open, we conclude that also $x_\alpha \to e.H \in G/H$ in $(G/H, \tau/H)$. By the uniqueness of limits in the Hausdorff space K/L we have $a.L = e.L$, so $a \in L$. □

As a corollary it gives another property: *if a precompact group G has a compact co-minimal subgroup, then it is minimal* (this follows from Corollary 3.2 without the assumption "precompact" for G and "compact" for H). Indeed, apply to G and its compact subgroup H the previous theorem and note that H remains closed also in the completion of G. Hence every closed normal subgroup N of the completion satisfying (1) must be contained in $H \leqslant G$, and consequently $N = \{1\}$. Now the minimality criterion implies that G is minimal.

Now we can give some examples of groups where only the dense subgroups are co-minimal.

Example 6.10. Let G be a dense torsion-free subgroup of $\mathbb{T}^n$. Then the co-minimal subgroups of G are dense. Indeed, let H be a closed co-minimal subgroup of G and let L be its closure in $\mathbb{T}^n$. Let p be a prime and $k \in \mathbb{N}$. Then for every element $x \in \mathbb{T}^n$ of order p^k the finite cyclic subgroup N generated by x trivially meets G, so by the above criterion, $N \leqslant L$. Therefore, L contains all elements of $\mathbb{T}^n$ of order p^k. This holds true for all k. So L contains the subgroup $t_p(\mathbb{T}^n)$ of all p-torsion elements of $\mathbb{T}^n$. Since the latter is dense, this implies $L = \mathbb{T}^n$, i.e., $H = G$. The same argument works with the relaxed condition $t_p(G) = 0$, instead of asking G to be torsion-free.

Both items of the example below can be obtained also from Example 3.17.

Example 6.11. Let us see examples of groups with few (actually, the minimum number of) co-minimal subgroups. Clearly such groups have no proper dense subgroups.

(a) Let $G = \mathbb{Z}^\#$ (the integers with the Bohr topology). Then every subgroup of G is closed. To see that no proper subgroup $H = n\mathbb{Z}$ $(1 < n)$ can be co-minimal it suffices to note that the Bohr compactification $b\mathbb{Z}$ (i.e. the completion of G) is isomorphic to $c(b\mathbb{Z}) \times \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the compact group of p-adic integers and $c(b\mathbb{Z})$ is connected component of $b\mathbb{Z}$. Now pick any prime $p|n$. Then

$$nb\mathbb{Z} \leqslant c(b\mathbb{Z}) \times p\mathbb{Z}_p \times \prod_{q \neq p} \mathbb{Z}_q$$

(since $c(b\mathbb{Z})$ is a divisible connected compact group). In particular, $nb\mathbb{Z} \not\supseteq \mathbb{Z}_p$ while $G \cap \mathbb{Z}_p = \{0\}$. Since $nb\mathbb{Z}$ is the closure of $H = n\mathbb{Z}$, Theorem 6.9 implies that H is not co-minimal.

(b) Let ν be the profinite topology of $\mathbb{Z}$. Then the completion of $G = (\mathbb{Z}, \nu)$ is isomorphic to the product $\prod_p \mathbb{Z}_p$ and a similar argument as above shows that no proper subgroup $H = n\mathbb{Z}$ $(1 < n)$ of G can be co-minimal.

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