On the Statistical Independence of Compound Pseudorandom Numbers Over Part of the Period

MORDECHAY B. LEVIN
Bar-Ilan University

This article deals with the compound methods with modulus m for generating uniform pseudorandom numbers, which have been introduced recently. Equidistribution and statistical independence properties of the generated sequences over part of the period are studied based on the discrepancy of d -tuples of successive pseudorandom numbers. It is shown that there exist parameters in compound methods such that the discrepancy over part of the period of the corresponding point sets in the d -dimensional unit cube is of an order magnitude of $O(N^{-1/2}(\log N)^{d+3})$ for all $N = 1, \dots, m$. This result is applied to the compound nonlinear, inversive and explicit inversive congruential methods.

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1. INTRODUCTION

Equidistribution and statistical independence properties of uniform pseudorandom numbers can be analyzed based on the discrepancy of certain point sets in $[0, 1]^d$. For N arbitrary points, $\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{N-1} \in [0, 1]^d$, the *discrepancy* is defined by

$$D(\mathbf{t}_0, \dots, \mathbf{t}_{N-1}) = D((\mathbf{t}_n)_{n=0}^{N-1}) = \sup_J |F_N(J) - V(J)|,$$

where the supremum is extended over all subintervals J of $[0, 1]^d$, $F_N(J)$ is N^{-1} times the number of points among $\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{N-1}$ falling into J , and $V(J)$ denotes the d -dimensional volume of J .

Let us consider pseudorandom numbers obtained by means of the *linear congruential* method

$$x_1(n) = \left\{ \frac{y_n}{p} \right\}, \quad y_{n+1} \equiv \lambda y_n \pmod{p}, \quad y_0 = c_0, \quad n = 0, 1, \dots$$

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Author's address: Department of Mathematics and Computer Science, Bar-Ilan University, 52900, Ramat-Gan, Israel, e-mail: mlevin@macs.biu.ac.il.

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and *shift-register* methods

$$x_2(n) = \sum_{i=1}^k \frac{z_{kn+i-1}}{p^i}, \quad x_3(n) = \sum_{i=1}^k \frac{z_{n+d_i}}{p^i},$$

$$z_{n+k} \equiv a_k z_{n+k-1} + \dots + a_1 z_n \pmod{p},$$

$$z_0 = c_1, \dots, z_{k-1} = c_k, \quad n = 0, 1, \dots,$$

where p is prime and $f(x) = x^k - a_k x^{k-1} - \dots - a_1$ is the characteristic polynomial of the sequence $(z_n)_{n \geq 0}$.

In a series of papers, Niederreiter (see the reviews in Niederreiter [1977; 1978; 1992]) proved that there exists a multiplier λ (a characteristic polynomial $f(x)$) such that for $i = 1, 2, 3$

$$D((x_i(n), \dots, x_i(n+d-1))_{n=0}^{N-1}) = O\left(\frac{\sqrt{m}(\log m)^{d+1}}{N}\right), \quad \text{for } N = 1, \dots, m,$$

where m is the period of the sequence of pseudorandom numbers.

This estimate is interesting for $N \gg \sqrt{m}(\log m)^{d+1}$. According to Levin [1989], we can obtain a nontrivial discrepancy estimate also for a small part of the period: there exists a multiplier λ , a characteristic polynomial $f(x)$ and initial values $c_0, c_1, \dots, c_k$ such that for $i = 1, 2, 3$

$$D((x_i(n), \dots, x_i(n+d-1))_{n=0}^{N-1}) = O(N^{-1/2}(\log N)^{d+3}), \quad \text{for } N = 1, \dots, m. \quad (1)$$

The smallest previously known discrepancy estimate over part of the period for other pseudorandom number generators satisfies

$$D((u(n), \dots, u(n+d-1))_{n=0}^{N-1}) = O\left(\frac{\sqrt{m}(\log m)^{d+1}}{N}\right), \quad \text{for } N = 1, \dots, m,$$

where m is the period of the sequence $(u_n)_{n \geq 0}$ (see the reviews and references in Eichenauer-Herrmann [1995] and Niederreiter [1992]).

Our goal is to obtain a nontrivial discrepancy estimate for a small part of the period for nonlinear pseudorandom numbers generators. In this article, we obtain a result similar to (1) for compound pseudorandom number generators that have been introduced recently (see the reviews in Drmota and Tichy [1997], Eichenauer-Herrmann [1995] and L'Ecuyer [1994]).

Let $p_1, \dots, p_r$ be arbitrary distinct primes. For $1 \leq i \leq r$, identify the set $\mathbb{Z}_{p_i} = \{0, 1, \dots, p_i - 1\}$ with the finite field of order p_i , and let $\mathbb{Z}_{p_i}^* = \mathbb{Z}_{p_i} \setminus \{0\}$ be its multiplicative group. Let $(z_n^{(i)})_{n \geq 0}$ be a sequence of elements of $\mathbb{Z}_{p_i}^d$, which is purely periodic with period length p_i . For a parameter $c_i \in \mathbb{Z}_{p_i}^*$, let $(\mathbf{x}_n^{(i)})_{n \geq 0}$ with

$$\mathbf{x}_n^{(i)} \equiv \frac{c_i z_n^{(i)} \pmod{1}}{p_i}, \quad n \geq 0, \quad (2)$$

be the corresponding stream of d -dimensional vectors of (ordinary) *pseudorandom numbers* in the interval $[0, 1)$. A sequence $(\mathbf{x}_n)_{n \geq 0}$ of d -dimensional vectors of *compound pseudorandom numbers* in the interval $[0, 1)$ is defined by

$$\mathbf{x}_n \equiv \mathbf{x}_n^{(1)} + \dots + \mathbf{x}_n^{(r)} \pmod{1}, \quad n \geq 0. \quad (3)$$

Since the primes $p_1, \dots, p_r$ are distinct, it follows by standard arguments that the sequence $(\mathbf{x}_n)_{n \geq 0}$ is purely periodic with period length $m = p_1 \cdots p_r$. Compound methods are particularly suitable for parallelized computations, since the underlying sequences $(\mathbf{x}_n^{(i)})_{n \geq 0}$ can be computed by parallel processors.

In the following, the d -dimensional vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{m-1}$ of compound pseudorandom numbers are considered and the abbreviation

$$D_{N; c_1, \dots, c_r}^{(d)} = D(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \quad (4)$$

is used for $1 \leq N \leq m = p_1 \cdots p_r$.

In the third section, we show that there exist parameters $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_r}^*$ such that

$$D_{N; c_1, \dots, c_r}^{(d)} = O(N^{-1/2}(\log N)^{d+3}), \quad \text{for } N = 1, \dots, m, \quad (5)$$

where the O -constant depends on the number of points among $\mathbf{z}_0^{(i)}, \mathbf{z}_1^{(i)}, \dots, \mathbf{z}_{p_i-1}^{(i)}$ falling into certain hyperplanes over the finite field $\mathbb{Z}_{p_i}$ ($i = 1, \dots, r$). To prove this we use the approach proposed in Levin [1989] and the average discrepancy estimate obtained in Eichenauer-Herrmann et al. [1997]. A similar result is obtained also for the other characteristic of equidistribution and statistical independence properties of uniform pseudorandom numbers, for the diaphony (see Drmota and Tichy [1997], Hellekalek and Niederreiter [1998], and Zinterhof [1976]). A heuristic algorithm of choosing parameters $(c_1, \dots, c_r)$ is proposed in this section. In the fourth section, the assertion (5) is applied to the compound nonlinear, inversive and explicit inversive congruential method. The second section contains necessary auxiliary results.

2. AUXILIARY RESULTS

First, some further notation is necessary. For integers $d \geq 1$ and $q \geq 2$, let $C_d(q)$ be the set of all nonzero lattice points $(h_1, \dots, h_d) \in \mathbb{Z}^d$ with $-q/2 < h_j \leq q/2$ for $1 \leq j \leq d$. Define

$$r(h, q) = \begin{cases} q \sin(\pi |h|/q) & \text{for } h \in C_1(q), \\ 1 & \text{for } h = 0, \end{cases}$$

and

$$r(\mathbf{h}, q) = \prod_{j=1}^d r(h_j, q)$$

for $\mathbf{h} = (h_1, \dots, h_d) \in C_d(q)$. For real t , the abbreviation $e(t) = e^{2\pi\sqrt{-1}t}$ is used and $\mathbf{u} \cdot \mathbf{v}$ stands for the standard inner product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Subsequently, three known results are stated, which follow from Niederreiter [1992, Theorem 3.10 and p. 35], Korobov [1992, p. 13], Eichenauer-Herrmann et al. [1997, Lemma 3], and Niederreiter [1977, Lemma 2.3], respectively.

LEMMA 1. *Let $N \geq 1$ and $q \geq 2$ be integers. Let $\mathbf{t}_n = \mathbf{y}_n/q \in [0, 1)^d$ with $\mathbf{y}_n \in \{0, 1, \dots, q-1\}^d$ for $0 \leq n < N$. Then the discrepancy of the points*

$\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{N-1}$ satisfies

$$D(\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{N-1}) \leq \frac{d}{q} + \frac{1}{N} \sum_{\mathbf{h} \in C_d(q)} \frac{1}{r(\mathbf{h}, q)} \left| \sum_{n=0}^{N-1} e(\mathbf{h} \cdot \mathbf{t}_n) \right|.$$

LEMMA 2. Let $T \geq N \geq 1$ be integers. Then

$$\left| \sum_{n=0}^{N-1} e(t_n) \right| \leq \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \left| \sum_{n=1}^T e\left(t_n + \frac{nh_0}{T}\right) \right|.$$

PROOF. According to Niederreiter [1992, p. 35], $1/T |\sum_{n \in [0, N)} e(nh/T)| \leq r^{-1}(h, T)$. Now, repeating the proof of Korobov [1992, p. 13], we obtain the assertion of Lemma 2. $\square$

COROLLARY 1. Let $T \geq N \geq 1$ and $q \geq 2$ be integers, $\mathbf{t}_n = \mathbf{y}_n/q \in [0, 1)^d$ with $\mathbf{y}_n \in \{0, 1, \dots, q-1\}^d$ for $0 \leq n < N$. Then

$$D(\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{N-1}) \leq \frac{d}{q} + \frac{T}{N} \bar{D}((\mathbf{t}_n)_{n=0}^{T-1}), \quad (6)$$

where

$$\bar{D}((\mathbf{t}_n)_{n=0}^{T-1}) = \frac{1}{T} \sum_{\mathbf{h} \in C_d(q)} \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(\mathbf{h}, q)r(h_0, T)} \left| \sum_{n=0}^{T-1} e\left(\mathbf{h} \cdot \mathbf{t}_n + \frac{nh_0}{T}\right) \right|. \quad (7)$$

LEMMA 3. Let $q \geq 2, T \geq 1$ be integers. Then

$$\sum_{\substack{\mathbf{h} \in C_d(q) \\ \mathbf{h} \equiv \mathbf{0} \pmod{v}}} \frac{1}{r(\mathbf{h}, q)} < \frac{1}{v} \left(\frac{2}{\pi} \log q + \frac{7}{5} \right)^d$$

for any divisor v of q with $1 \leq v < q$, and

$$\sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \leq \frac{2}{\pi} \log T + \frac{7}{5}.$$

3. DISCREPANCY BOUNDS

3.1 Classical Discrepancy

The following quantities play a crucial role in the analysis of the average discrepancy. For $1 \leq i \leq r$ and $\mathbf{z}_n^{(i)}, \mathbf{h} \in \mathbb{Z}^d$ with $\mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}$, define

$$B_i(\mathbf{h}) = \max_{h' \in \mathbb{Z}_{p_i}} \# \{0 \leq n < p_i \mid \mathbf{h} \cdot \mathbf{z}_n^{(i)} \equiv h' \pmod{p_i}\}. \quad (8)$$

LEMMA 4. Let $1 \leq T \leq m = p_1 \cdots p_r$, $\mathbf{h} \in C_d(m)$, and $J = \{1 \leq i \leq r \mid \mathbf{h} \equiv \mathbf{0} \pmod{p_i}\}$. Then

$$\sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_r}^*} \left| \sum_{n=0}^{T-1} e\left(\sum_{i=1}^r \frac{c_i(\mathbf{h} \cdot \mathbf{z}_n^{(i)})}{p_i} + \frac{nh_0}{T}\right) \right|^2 \leq Tm \prod_{i \in J} p_i \prod_{\substack{i=1 \\ i \notin J}}^r B_i(\mathbf{h}). \quad (9)$$

PROOF. Repeating the proof of Eichenauer-Herrmann et al. [1997, Lemma 4], we obtain that the left side of (9) is at most

$$\begin{aligned}
& \sum_{k,n=0}^{T-1} \left| \prod_{i=1}^r \sum_{c \in \mathbb{Z}_{p_i}} e \left(\frac{c(\mathbf{h} \cdot \mathbf{z}_n^{(i)} - \mathbf{h} \cdot \mathbf{z}_k^{(i)})}{p_i} + \frac{h_0(n-k)}{T} \right) \right| \\
&= \sum_{k,n=0}^{T-1} \prod_{i=1}^r \left| \sum_{c \in \mathbb{Z}_{p_i}} e \left(\frac{c(\mathbf{h} \cdot \mathbf{z}_n^{(i)} - \mathbf{h} \cdot \mathbf{z}_k^{(i)})}{p_i} \right) \right| \\
&= m \sum_{k=0}^{T-1} \# \{ 0 \leq n < T \mid \mathbf{h} \cdot \mathbf{z}_n^{(i)} \equiv \mathbf{h} \cdot \mathbf{z}_k^{(i)} \pmod{p_i}, 1 \leq i \leq r \}.
\end{aligned}$$

Next, applying the final calculations of the proof of Eichenauer-Herrmann et al. [1997, Lemma 4], we get the desired result. $\square$

In the following, we use the abbreviation (see (2), (3), and (7))

$$\tilde{D}_{T;c_1,\dots,c_r}^{(d)} = \tilde{D}((\mathbf{x}_n)_{n=0}^{T-1}). \quad (10)$$

LEMMA 5. Let $1 \leq T \leq m$ and $m_J = \prod_{i \in J} p_i$ for subsets J of $\{1, \dots, r\}$. Then

$$\begin{aligned}
& \frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \tilde{D}_{T;c_1, \dots, c_r}^{(d)} \\
& \leq \frac{1}{\sqrt{T}} \prod_{i=1}^r \sqrt{\frac{p_i}{p_i - 1}} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \sqrt{m_J} \sum_{\substack{\mathbf{h} \in C_d(m) \\ \mathbf{h} \equiv \mathbf{0} \pmod{p_i}, i \in J \\ \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}, i \notin J}} \\
& \quad \times \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(\mathbf{h}, m)r(h_0, T)} \prod_{\substack{i=1 \\ i \notin J}}^r \sqrt{B_i(\mathbf{h})}. \quad (11)
\end{aligned}$$

PROOF. Define the left side of (11) by σ . Applying the Cauchy-Schwarz inequality, we obtain from (2), (3), (7) and (9) that

$$\begin{aligned}
\sigma & \leq \frac{1}{T} \sum_{\mathbf{h} \in C_d(m)} \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(\mathbf{h}, m)r(h_0, T)} \\
& \quad \times \left(\frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \left| \sum_{n=0}^{T-1} e \left(\sum_{i=1}^r \frac{c_i(\mathbf{h} \cdot \mathbf{z}_n^{(i)})}{p_i} + \frac{nh_0}{T} \right) \right|^2 \right)^{1/2} \\
&= \frac{1}{T} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \sum_{\substack{\mathbf{h} \in C_d(m) \\ \mathbf{h} \equiv \mathbf{0} \pmod{p_i}, i \in J \\ \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}, i \notin J}} \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(\mathbf{h}, m)r(h_0, T)} \\
& \quad \times \left(\frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \left| \sum_{n=0}^{T-1} e \left(\sum_{i=1}^r \frac{c_i(\mathbf{h} \cdot \mathbf{z}_n^{(i)})}{p_i} + \frac{nh_0}{T} \right) \right|^2 \right)^{1/2}.
\end{aligned}$$

By Lemma 4, we have

$$\begin{aligned} \sigma &\leq \frac{1}{T} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \sum_{\substack{\mathbf{h} \in C_d(m) \\ \mathbf{h} \equiv \mathbf{0} \pmod{p_i}, i \in J \\ \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}, i \notin J}} \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(\mathbf{h}, m)r(h_0, T)} \\ &\quad \times \left(T \prod_{i=1}^r \frac{p_i}{p_i - 1} m_J \prod_{\substack{i=1 \\ i \notin J}}^r B_i(\mathbf{h}) \right)^{1/2}. \quad \square \end{aligned}$$

LEMMA 6. Let $1 \leq T \leq m$, and $B_i = \max\{B_i(\mathbf{h}) \mid \mathbf{h} \in \mathbb{Z}^d, \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}\}$ for $1 \leq i \leq r$. Then

$$\begin{aligned} &\frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \tilde{D}_{T; c_1, \dots, c_r}^{(d)} \\ &\leq \frac{1}{\sqrt{T}} \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d \left(\frac{2}{\pi} \log T + \frac{7}{5} \right). \end{aligned} \quad (12)$$

PROOF. Using Lemma 5 and the two inequalities of Lemma 3 (with $q = m$ and $v = m_J$), we get that the left side of (12) is less than

$$\frac{1}{\sqrt{T}} \prod_{i=1}^r \sqrt{\frac{p_i}{p_i - 1}} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \frac{1}{\sqrt{m_J}} \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d \prod_{\substack{i=1 \\ i \notin J}}^r \sqrt{B_i}. \quad (13)$$

According to Eichenauer-Herrmann et al. [1997, p. 211],

$$\prod_{i=1}^r \sqrt{\frac{p_i}{p_i - 1}} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \frac{1}{\sqrt{m_J}} \prod_{\substack{i=1 \\ i \notin J}}^r \sqrt{B_i} \leq \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}}. \quad (14)$$

Now, from (13), we obtain the desired result. $\square$

BASIC LEMMA Let $T_j = \lfloor m/2^j \rfloor$, $j = 0, 1, 2, \dots$. Then

$$\begin{aligned} &\frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \sum_{0 \leq j \leq \lfloor \log_2 m \rfloor - 1} \tilde{D}_{T_j; c_1, \dots, c_r}^{(d)} \\ &\quad \times \frac{\sqrt{T_j}}{((2/\pi) \log T_j + (7/5))(\log_2 T_j + 3) \log^{3/2}(\log_2 T_j + 3)} \\ &\leq 2 \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d. \end{aligned}$$

PROOF. It is easy to see that $\log_2 T_j \geq \lfloor \log_2 m \rfloor - j$, $j \in [0, \lfloor \log_2 m \rfloor]$. Hence

$$\begin{aligned} &\sum_{0 \leq j \leq \lfloor \log_2 m \rfloor - 1} \frac{1}{(\log_2 T_j + 3) \log^{3/2}(\log_2 T_j + 3)} \leq \sum_{j=1}^{\infty} \frac{1}{(j + 3) \log^{3/2}(j + 3)} \\ &\leq \int_3^{\infty} \frac{dx}{x(\log x)^{3/2}} = -2(\log x)^{-1/2} \Big|_3^{\infty} < 2. \end{aligned} \quad (15)$$

By Lemma 6, we get the desired result. $\square$

THEOREM 1. *Let $0 < \varepsilon \leq 1$. Then there exist more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ such that for $N = 1, 2, \dots, m$*

$$D_{N; c_1, \dots, c_r}^{(d)} \leq \frac{d}{m} + \frac{2\sqrt{2}}{\varepsilon} \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} N^{-1/2} \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d \times \left(\frac{2}{\pi} \log 2N + \frac{7}{5} \right) (\log_2 N + 4) \log^{3/2}(\log_2 N + 4). \quad (16)$$

PROOF. Let,

$$\Omega_\varepsilon = \left\{ (c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^* \mid \sum_{j=0}^{\lfloor \log_2 m \rfloor - 1} \frac{\sqrt{T_j}}{\tilde{D}_{T_j; c_1, \dots, c_r}^{(d)} ((2/\pi) \log T_j + (7/5)(\log_2 T_j + 3))} \times \log^{-3/2}(\log_2 T_j + 3) \leq \frac{2}{\varepsilon} \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} \right\}.$$

It follows from the Basic Lemma that

$$\#\Omega_\varepsilon \geq (1 - \varepsilon) \prod_{i=1}^r (p_i - 1). \quad (17)$$

For $N = 1$, the statement (16) is evident.

Now let $N \in (T_{j_0+1}, T_{j_0}]$ for any $j_0 \in [0, \lfloor \log_2 m \rfloor - 1]$, where $T_{j_0} = \lfloor m/2^{j_0} \rfloor$.

By (4), (6), and (10), we have that for $(c_1, \dots, c_r) \in \Omega_\varepsilon$

$$D_{N; c_1, \dots, c_r}^{(d)} \leq \frac{d}{m} + \frac{T_{j_0}}{N} \tilde{D}_{T_{j_0}; c_1, \dots, c_r}^{(d)} \leq \frac{d}{m} + \frac{2\sqrt{T_{j_0}}}{\varepsilon N} \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} \times \left(\frac{2}{\pi} \log m + \frac{7}{5} \right)^d \left(\frac{2}{\pi} \log T_{j_0} + \frac{7}{5} \right) (\log_2 T_{j_0} + 3) \log^{3/2}(\log_2 T_{j_0} + 3).$$

Bearing in mind that $2N \geq 2(\lfloor m/2^{j_0+1} \rfloor + 1) \geq m/2^{j_0} \geq T_{j_0}$, we obtain from (17) the assertion of Theorem 1. $\square$

The right side of (16) depends on the period m . We can obtain an estimate independently of the period m by applying the Erdős–Turan–Koksma inequality (see Drmota and Tichy [1997, p. 15]):

$$D((\mathbf{x}_n)_{n=0}^{N-1}) \leq \left(\frac{3}{2} \right)^d \left(\frac{2}{H+1} + \sum_{\mathbf{0} < \|\mathbf{h}\| \leq H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{N} \sum_{n=0}^{N-1} e(\mathbf{h} \cdot \mathbf{x}_n) \right| \right), \quad (18)$$

where $r(\mathbf{h}) = \prod_{i=1}^d \max(1, |h_i|)$ for $\mathbf{h} = (h_1, \dots, h_d) \in \mathbb{Z}^d$, $\|\mathbf{h}\| = \max_{1 \leq i \leq d} |h_i|$, $H \geq 1$. For the smaller constant in (18), see Cochrane [1988].

Using (18) with $H = \lfloor \sqrt{T_j} \rfloor$ ($j = 0, 1, \dots, \lfloor \log_2 m \rfloor - 1$) instead of Lemma 1 and repeating the above calculation, we get

THEOREM 1'. *Let $0 < \varepsilon \leq 1$. Then there exist more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ such that*

$$D_{N; c_1, \dots, c_r}^{(d)} < \left(\frac{3}{2}\right)^d \frac{3}{\varepsilon} \prod_{i=1}^r \frac{\sqrt{p_i B_i} + 1}{\sqrt{p_i - 1}} N^{-1/2} (2 \log N + 4)^{d+2} \log^{3/2}(\log_2 N + 4),$$

(19)

for $N = 1, 2, \dots, m$.

It is easy to see that the estimate (19) is preferable to (16) for small parts of the period.

Remark 1. As in Levin [1989], we can obtain a discrepancy estimate for a large set of subsequences of a given pseudorandom sequence (see also Knuth [1981, pp. 154–160] on subsequences of a random sequence). Repeating the proof of Theorem 1, we get the following result for the arithmetic progression subsequences:

Let $a_v \geq 1$, $b_v \geq 0$ ($v = 1, \dots, L$) be integers, $a_1 \dots a_L \not\equiv 0 \pmod{p_i}$ ($i = 1, \dots, r$) and $0 < \varepsilon \leq 1$. Then there exist more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^ \times \dots \times \mathbb{Z}_{p_r}^*$ such that*

$$D((\mathbf{x}_{na_v+b_v})_{n=0}^{N-1}) < \left(\frac{3}{2}\right)^d \frac{3L}{\varepsilon} \prod_{i=1}^r \frac{\sqrt{p_i B_{vi}} + 1}{\sqrt{p_i - 1}} N^{-1/2}$$

$$\times (2 \log N + 4)^{d+2} \log^{3/2}(\log_2 N + 4), \quad \text{with } v = 1, \dots, L, \quad N = 1, \dots, m,$$

where

$$B_{vi} = \max_{\mathbf{h} \in \mathbb{Z}^d, \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}} \max_{h' \in \mathbb{Z}_{p_i}} \#\{0 \leq n < p_i \mid \mathbf{h} \cdot \mathbf{z}_{na_v+b_v} \equiv h' \pmod{p_i}\},$$

3.2 The Diaphony and the Choice of Parameters

Let $\mathbf{u}_0, \dots, \mathbf{u}_{N-1}$ be a finite sequence of points in $[0, 1)^d$. The number

$$F_N((\mathbf{u}_n)_{n=0}^{N-1}) = \left(\frac{1}{N^2} \sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^d} \frac{1}{r(\mathbf{h})^2} \left| \sum_{n=0}^{N-1} e(\mathbf{h} \cdot \mathbf{u}_n) \right|^2 \right)^{1/2} \quad (20)$$

is called the *diaphony* of the given sequence [Zinterhof 1976]. It is known that

$$F_N((\mathbf{u}_n)_{n=0}^{N-1}) = \left(\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \left(\prod_{i=1}^d g(u_{n,i} - u_{m,i}) - 1 \right) \right)^{1/2}, \quad (21)$$

where $g(u) = 1 - (\pi^2/6) + (\pi^2/2)(1 - 2\{u\})^2$ (see Drmota and Tichy [1997, p. 24] and Zinterhof [1976]).

Diaphony is closely related to discrepancy (see Lev [1995] and Stegbuchner [1979]). The computational complexity of the discrepancy of the sequence $(\mathbf{u}_n)_{n=0}^{N-1}$ is $O(N^d)$. In the case of diaphony, it is $O(dN^2)$. This renders it an interesting alternative to the discrepancy in higher dimensions.

Let $F_N^{(d)}; c_1, \dots, c_r$ be the diaphony of the sequence $(\mathbf{x}_n)_{n=0}^{N-1}$ in (2)–(3) and

$$\tilde{F}_T^{(d)}; c_1, \dots, c_r = \left(\frac{1}{T^2} \sum_{h_0 \in (-T/2, T/2]} \sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^d} \frac{1}{r(\mathbf{h})^2 r(h_0, T)} \left| \sum_{n=0}^{T-1} e\left(\mathbf{h} \cdot \mathbf{x}_n + \frac{h_0 n}{T}\right) \right|^2 \right)^{1/2}. \quad (22)$$

We have from (20) and (21) that

$$\begin{aligned} \tilde{F}_T^{(d)}; c_1, \dots, c_r &= \left(\frac{1}{T^2} \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \sum_{n=0}^{T-1} \sum_{m=0}^{T-1} \left(\prod_{i=1}^d g(x_{n,i} - x_{m,i}) - 1 \right) \right. \\ &\quad \left. \times e\left(\frac{h_0(n-m)}{T}\right) \right)^{1/2}. \end{aligned} \quad (23)$$

Using Lemma 2 and the Cauchy–Schwarz inequality, we obtain, for $1 \leq N \leq T$,

$$\begin{aligned} &\left| \sum_{n=0}^{N-1} e(\mathbf{h} \cdot \mathbf{x}_n) \right|^2 \\ &\leq \left(\sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \left| \sum_{n=0}^{T-1} e\left(\mathbf{h} \cdot \mathbf{x}_n + \frac{h_0 n}{T}\right) \right| \right)^2 \\ &\leq \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \left(\sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \left| \sum_{n=0}^{T-1} e\left(\mathbf{h} \cdot \mathbf{x}_n + \frac{h_0 n}{T}\right) \right|^2 \right). \end{aligned}$$

Hence, from (20), (22) and Lemma 3, we get

$$(N F_N^{(d)}; c_1, \dots, c_r)^2 \leq \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) (T \tilde{F}_T^{(d)}; c_1, \dots, c_r)^2 \quad \text{for } 1 \leq N \leq T. \quad (24)$$

LEMMA 7. With the notation defined above, we have for $1 \leq T \leq m$

$$\begin{aligned} &\frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \tilde{F}_T^{(d)}; c_1, \dots, c_r \\ &\leq \left(\frac{1}{T} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) 12d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1) B_i}{p_i - 1} \right)^{1/2}. \end{aligned} \quad (25)$$

PROOF. We define the left side of (25) by σ . Applying the Cauchy–Schwarz inequality, we get from (22), (2), and (3) that

$$\begin{aligned} \sigma^2 &\leq \frac{1}{T^2} \sum_{h_0 \in (-T/2, T/2]} \sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^d} \frac{1}{r(\mathbf{h})^2 r(h_0, T)} \\ &\quad \times \frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \left| \sum_{n=0}^{T-1} e\left(\sum_{i=1}^r \frac{c_i (\mathbf{h} \cdot \mathbf{z}_n^{(i)})}{p_i} + \frac{h_0 n}{T} \right) \right|^2. \end{aligned}$$

It is easy to see that

$$\sigma^2 \leq \sigma_1 + \sigma_2, \quad (26)$$

where

$$\sigma_1 = \frac{1}{T^2} \sum_{h_0 \in (-T/2, T/2]} \sum_{\mathbf{h} \in C_d(m)} \frac{1}{r(\mathbf{h})^2 r(h_0, T)} \frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \times \left| \sum_{n=0}^{T-1} e \left(\sum_{i=1}^r \frac{c_i (\mathbf{h} \cdot \mathbf{z}_n^{(i)})}{p_i} + \frac{h_0 n}{T} \right) \right|^2, \quad (27)$$

and

$$\sigma_2 = \sum_{h_0 \in (-T/2, T/2]} \sum_{\mathbf{h} \in \mathbb{Z}^d \setminus (\mathbf{0} \cup C_d(m))} \frac{1}{r(\mathbf{h})^2 r(h_0, T)}. \quad (28)$$

According to Hellekalek and Niederreiter [1998, pp. 49–51],

$$\sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^d, \mathbf{h} \equiv \mathbf{0} \pmod{\mathbf{M}}} \frac{1}{r(\mathbf{h})^2} = \left(1 + \frac{\pi^2}{3M^2} \right)^d - 1, \quad M = 1, 2, \dots,$$

and

$$\sum_{\mathbf{h} \in \mathbb{Z}^d} \frac{1}{r(\mathbf{h})^2} - \sum_{\mathbf{h} \in \mathbf{0} \cup C_d(M)} \frac{1}{r(\mathbf{h})^2} < \frac{12((1 + \pi^2/3)^d - 1)}{\pi^2(M - 1)}, \quad M = 2, 3, \dots$$

By the mean-value theorem of calculus, we obtain bounds

$$\sum_{\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^d, \mathbf{h} \equiv \mathbf{0} \pmod{\mathbf{M}}} \frac{1}{r(\mathbf{h})^2} \leq d \frac{\pi^2}{3M^2} \left(1 + \frac{\pi^2}{3} \right)^{d-1}, \quad M = 1, 2, \dots, \quad (29)$$

and

$$\sum_{\mathbf{h} \in \mathbb{Z}^d} \frac{1}{r(\mathbf{h})^2} - \sum_{\mathbf{h} \in \mathbf{0} \cup C_d(M)} \frac{1}{r(\mathbf{h})^2} < \frac{4d(1 + \pi^2/3)^{d-1}}{M - 1}, \quad M = 2, 3, \dots$$

It follows from (28) and Lemma 3 that

$$\sigma_2 \leq \frac{8d(1 + \pi^2/3)^{d-1}}{m} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right). \quad (30)$$

Let $J = \{1 \leq i \leq r \mid \mathbf{h} \equiv \mathbf{0} \pmod{p_i}\}$, and $m_J = \prod_{i \in J} p_i$.

Applying Lemma 4, we get from (27) that

$$\begin{aligned} \sigma_1 &\leq \frac{1}{T^2} \sum_{h_0 \in (-T/2, T/2]} \sum_{\mathbf{h} \in C_d(m)} \frac{1}{r(\mathbf{h})^2 r(h_0, T)} T \prod_{i=1}^r \frac{p_i}{p_i - 1} \prod_{i \in J} p_i \prod_{\substack{i=1 \\ i \notin J}}^r B_i(\mathbf{h}) \\ &= \frac{1}{T} \sum_{h_0 \in (-T/2, T/2]} \prod_{i=1}^r \frac{p_i}{p_i - 1} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} m_J \sum_{\substack{\mathbf{h} \in C_d(m) \\ \mathbf{h} \equiv \mathbf{0} \pmod{p_i}, i \in J \\ \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}, i \notin J}} \frac{1}{r(\mathbf{h})^2 r(h_0, T)} \prod_{\substack{i=1 \\ i \notin J}}^r B_i(\mathbf{h}). \end{aligned}$$

Using Lemma 3 and (29) with $M = m_J$, we have that

$$\sigma_1 \leq \frac{1}{T} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) \prod_{i=1}^r \frac{p_i}{p_i - 1} \sum_{\substack{J \subset \{1, \dots, r\} \\ \#J < r}} \frac{d\pi^2}{3m_J} \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r B_i,$$

where $B_i = \max\{B_i(\mathbf{h}) \mid \mathbf{h} \in \mathbb{Z}^d, \mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}\}$ for $1 \leq i \leq r$.

Bearing in mind that

$$\sum_{J \subset \{1, \dots, r\}, \#J < r} \frac{1}{m_J} < \prod_{i=1}^r \left(1 + \frac{1}{p_i}\right),$$

we get

$$\sigma_1 \leq \frac{1}{T} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) 4d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1}.$$

Now, by (26) and (30), we obtain that

$$\sigma^2 \leq \frac{1}{T} \left(\frac{2}{\pi} \log T + \frac{7}{5} \right) 12d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1}.$$

This is the desired result. $\square$

We obtain from (15)

COROLLARY 2. *Let $T_j = \lfloor m/2^j \rfloor$, $j = 0, 1, 2, \dots$. Then*

$$\begin{aligned} & \frac{1}{\prod_{i=1}^r (p_i - 1)} \sum_{(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*} \sum_{\mathbf{0} \leq j \leq \lfloor \log_2 m \rfloor - 1} \tilde{F}_{T_j; c_1, \dots, c_r}^{(d)} \\ & \quad \times \frac{\sqrt{T_j}}{((2/\pi) \log T_j + (7/5))^{1/2} (\log_2 T_j + 3) \log^{3/2}(\log_2 T_j + 3)} \\ & \leq 2 \left(12d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1} \right)^{1/2}. \end{aligned}$$

THEOREM 2. *Let $0 < \varepsilon \leq 1$. Then there exist more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ such that for $N = 1, 2, \dots, m$*

$$\begin{aligned} F_{N; c_1, \dots, c_r}^{(d)} & \leq \frac{4}{\varepsilon} \left(6d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1} \right)^{1/2} \\ & \quad \times N^{-1/2} \left(\frac{2}{\pi} \log 2N + \frac{7}{5} \right) (\log_2 N + 4) \log^{3/2}(\log_2 N + 4). \end{aligned} \quad (31)$$

PROOF. Let

$$\begin{aligned} \Omega_\varepsilon & = \left\{ (c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^* \mid \sum_{\mathbf{0} \leq j \leq \lfloor \log_2 m \rfloor - 1} \right. \\ & \quad \times \tilde{F}_{T_j; c_1, \dots, c_r}^{(d)} \frac{\sqrt{T_j}}{((2/\pi) \log T_j + (7/5))^{1/2} (\log_2 T_j + 3)^{-1} \log^{-3/2}(\log_2 T_j + 3)} \\ & \quad \left. \leq \frac{2}{\varepsilon} \left(12d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1} \right)^{1/2} \right\}. \end{aligned}$$

It follows from Corollary 2 that

$$\#\Omega_\varepsilon \geq (1 - \varepsilon) \prod_{i=1}^r (p_i - 1).$$

For $N = 1$, the statement (31) is evident.

Let $N \in (T_{j_0+1}, T_{j_0}]$ for any $j_0 \in [0, \lceil \log_2 m \rceil - 1]$, where $T_{j_0} = \lfloor m/2^{j_0} \rfloor$.

By (24), we have that for $(c_1, \dots, c_r) \in \Omega_\varepsilon$

$$\begin{aligned} F_{N; c_1, \dots, c_r}^{(d)} &\leq \left(\frac{2}{\pi} \log T_{j_0} + \frac{7}{5} \right)^{1/2} \frac{T_{j_0}}{N} F_{T_{j_0}; c_1, \dots, c_r}^{(d)} \\ &\leq \frac{2}{\varepsilon} \left(\frac{2}{\pi} \log T_{j_0} + \frac{7}{5} \right) (\log_2 T_{j_0} + 3) \log^{3/2}(\log_2 T_{j_0} + 3) \frac{\sqrt{T_{j_0}}}{N} \\ &\quad \times \left(12d \left(1 + \frac{\pi^2}{3} \right)^{d-1} \prod_{i=1}^r \frac{(p_i + 1)B_i}{p_i - 1} \right)^{1/2}. \end{aligned}$$

Taking into account that $2N \geq T_{j_0}$, we obtain the assertion of Theorem 2. $\square$

We choose the parameters $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$, minimizing the expression

$$\sum_{j=0}^{\lceil \log_2 m \rceil} \sqrt{T_j} (\log_2 T_j + 3)^{-3} F_{T_j; c_1, \dots, c_r}^{(d)}, \quad \text{with } T_j = \lfloor m/2^j \rfloor, j = 0, 1, 2, \dots$$

According to (23), we need for this $O(dm^4(\log m)^3)$ arithmetic operations.

It should be noted that the period m is usually more than 10^9 and it is not realistic to perform $dm^4(\log m)^3$ arithmetic operations with conventional computers. To reduce the number of operations, we propose the following heuristic procedure:

Let $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{J}_{\mathbf{x}} = [0, x_1) \times \dots \times [0, x_d)$, and

$$D^2((\mathbf{t}_n)_{n=0}^{N-1}) = \left(\int_{[0,1]^d} (F_N(\mathbf{J}_{\mathbf{x}}) - x_1 \dots x_d)^2 d\mathbf{x} \right)^{1/2}$$

is the L_2 -discrepancy of the sequence $(\mathbf{t}_n)_{n=0}^{N-1}$ in the d -dimensional unit cube $[0, 1]^d$ (see the definition of the discrepancy in the introduction). According to Heinrich [1996] and Drmota and Tichy [1997, p. 372] there is an algorithm, which computes L_2 -discrepancy of the sequence $(\mathbf{t}_n)_{n=0}^{N-1}$ with $O(N(\log N)^d)$ arithmetic operations.

Now let $D_{N; c_1, \dots, c_r}^{(1; d+1)}$ ($D_{N; c_1, \dots, c_r}^{(2; d+1)}$) be the discrepancy (L_2 - discrepancy) of the $(d+1)$ -dimensional sequence $(\mathbf{x}_n, n/N)_{n=0}^{N-1}$, where $(\mathbf{x}_n)_{n \geq 0}$ (3) is the sequence of d -dimensional vectors of compound pseudorandom numbers.

Using the well-known relation between discrepancies of d -dimensional sequences $(\mathbf{t}_n)_{n=0}^{N-1}$ and $d+1$ -dimensional sequences $(\mathbf{t}_n, n/T)_{n=0}^{T-1}$ (see, for example, Levin [1989, p. 101]), we obtain

$$D_{N; c_1, \dots, c_r}^{(d)} \leq \frac{TD_{T; c_1, \dots, c_r}^{(1; d+1)}}{N} \quad \text{for } 1 \leq N \leq T.$$

We use this inequality instead of Corollary 1 to get a slightly different variant of the proof of Theorem 1'. To obtain the discrepancy estimate as in (19) (or as in (16)) it is enough to minimize $M_{\epsilon; c_1, \dots, c_r}^{(1;d+1)}$ over $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$, with

$$M_{\epsilon; c_1, \dots, c_r}^{(i;d+1)} = \sum_{0 \leq j \leq \lfloor \log_2 m \rfloor - 1} D_{T_j; c_1, \dots, c_r}^{(i;d+1)} \frac{\sqrt{T_j}}{(\log_2 T_j + 3)^{d+2} (\log(\log_2 T_j + 3))^{3/2}},$$

where $i = 1, 2$, $T_j = \lfloor m/2^j \rfloor$ and $j = 1, 2, \dots$. It is easy to see that $M_{\epsilon; c_1, \dots, c_r}^{(2;d+1)} \leq M_{\epsilon; c_1, \dots, c_r}^{(1;d+1)}$. Bearing in mind that the L_2 -discrepancy $D_{N; c_1, \dots, c_r}^{(2;d+1)}$ is closely related to the discrepancy $D_{N; c_1, \dots, c_r}^{(1;d+1)}$, we get that if $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ minimizes $M_{\epsilon; c_1, \dots, c_r}^{(2;d+1)}$, then $M_{\epsilon; c_1, \dots, c_r}^{(1;d+1)}$ should not be too large relative to the minimum of $M_{\epsilon; c_1, \dots, c_r}^{(1;d+1)}$ over $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$. Thus, we propose (heuristically) to minimize $M_{\epsilon; c_1, \dots, c_r}^{(2;d+1)}$ over $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ instead of minimizing the $M_{\epsilon; c_1, \dots, c_r}^{(1;d+1)}$. According to Heinrich [1996] and Drmota and Tichy [1997, p. 372], we need $O(m(\log m)^{d+2})$ arithmetic operations to compute $M_{\epsilon; c_1, \dots, c_r}^{(2;d+1)}$, and $O(m^2(\log m)^{d+2})$ arithmetic operations to perform the full minimization. This minimization evidently admits a parallelization. To reduce the number of operations further we propose using a random search procedure (see Niederreiter [1992, chap. 6]) instead of the full minimization. Alternatively, it is possible to use the partial minimization over $(b_1, \dots, b_r) \in \mathbb{Z}_{q_1}^* \times \dots \times \mathbb{Z}_{q_r}^*$, taking $c_i = \lfloor p_i \{b_i/q_i\} \rfloor$, $i = 1, \dots, r$, where integers q_i are small relatively p_i , $i = 1, \dots, r$. In this way, we need $O(qm(\log m)^{d+2})$ arithmetic operations for choosing the parameters $c_1, \dots, c_r$, where $q = q_1 \dots q_r$. We propose to use this procedure for not too large dimension d and period m (say $d \leq 5$, $m \leq 10^9$).

4. APPLICATION

4.1 The Compound Nonlinear Congruential Method

The nonlinear congruential method is introduced in Eichenauer et al. [1988]. The compound version of this method is proposed in Eichenauer-Herrmann [1994a]. The following k -dimensional variant of this method is investigated in Eichenauer-Herrmann and Niederreiter [1997]. For the average-case analysis, see Eichenauer-Herrmann and Larcher [1996] and Eichenauer-Herrmann and Niederreiter [1997]. For $i \in \{1, \dots, r\}$, $j \in \{1, \dots, k\}$, let $g_{ij}: \mathbb{Z} \rightarrow \mathbb{Z}_{p_i}$ be a monic permutation polynomial of $\mathbb{Z}_{p_i}$ with degree d_{ij} as a polynomial over $\mathbb{Z}_{p_i}$ and with $g_{ij}(0) = 0$, where $3 \leq d_{ij} \leq p_i - 2$ is assumed in order to avoid uninteresting cases. Let $(y_{nj}^{(i)})_{n \geq 0}$ with

$$y_{nj}^{(i)} \equiv g_{ij}(n) + b_i \pmod{p_i}, \quad n \geq 0$$

be the corresponding (ordinary) *nonlinear congruential sequence* of elements of $\mathbb{Z}_{p_i}$ and let $(x_{nj}^{(i)})_{n \geq 0}$ with

$$x_{nj}^{(i)} = c_i z_{nj}^{(i)}, \quad z_{nj}^{(i)} = \frac{y_{nj}^{(i)}}{p_i}, \quad c_i \not\equiv 0 \pmod{p_i}, \quad n \geq 0$$

be the corresponding stream of (ordinary) *nonlinear congruential pseudorandom numbers*.

Now in the compound approach, the corresponding stream of k -dimensional *nonlinear congruential pseudorandom vectors* in $[0, 1]^k$ is defined by $\mathbf{x}_n = (x_{n1}, \dots, x_{nk})$, with $x_{nj} \equiv x_{nj}^{(1)} + \dots + x_{nj}^{(r)} \pmod{1}$, $n \geq 0$. Since the primes $p_1, \dots, p_r$ are distinct and $(g_{ij}), (i = 1, \dots, r; j = 1, \dots, k)$ are permutation polynomials, the sequence $(\mathbf{x}_n)_{n \geq 0}$ is purely periodic with period length $m = p_1 \cdots p_r$.

In the following, mutual statistical independence properties of the stream $(\mathbf{x}_n)_{n \geq 0}$ of uniform pseudorandom vectors are studied based on the equidistribution behavior of the s -tuples

$$(\mathbf{x}_n, \dots, \mathbf{x}_{n+s-1}) \in [0, 1]^{ks}, \quad n \geq 0,$$

which can be analyzed by the discrepancy of the corresponding point sets in $[0, 1]^{ks}$.

THEOREM 3. *For $i \in \{1, \dots, r\}$ let the polynomials $(g_{ij}(sz + v - 1))$ ($1 \leq j \leq k, 1 \leq v \leq s$) be linearly independent over $\mathbb{Z}_{p_i}$; $\varepsilon \in (0, 1]$, $d_i = \max_{1 \leq j \leq k} d_{ij}$. Then there exist more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ such that*

$$D_{N; c_1, \dots, c_r}^{(ks)} < \left(\frac{3}{2}\right)^{ks} \frac{3}{\varepsilon} \prod_{i=1}^r (\sqrt{d_i + 1} + 0.5) N^{-1/2} (2 \log N + 4)^{ks+2} \\ \times \log^{3/2}(\log_2 N + 4), \quad N = 1, 2, \dots, m.$$

PROOF. For $i \in \{1, \dots, r\}$ and $\mathbf{h} \in \mathbb{Z}^{ks}$, let

$$Q^{(i)}(\mathbf{h}, z) \equiv \sum_{j=1}^k \sum_{v=1}^s h_{ij} g_{ij}(sz + v - 1) \pmod{p_i}.$$

We have from the condition of the theorem that for $\mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}$, $1 \leq \deg(Q^{(i)}(\mathbf{h}, z)) \leq d_i$. Hence

$$B_i = \max_{1 \leq h' \leq p_i} \#\{0 \leq n < p_i \mid Q^{(i)}(\mathbf{h}, z) \equiv h' \pmod{p_i}\} \leq d_i.$$

Taking into account that $3 \leq d_i \leq p_i - 2$ ($i = 1, \dots, r$), we have that

$$\frac{\sqrt{p_i B_i + 1}}{\sqrt{p_i - 1}} \leq \sqrt{d_i + 1} + 0.5.$$

Now, applying Theorem 1', we get the assertion of Theorem 3. $\square$

4.2 The Compound Inversive Method

The inversive congruential method is introduced in Eichenauer and Lehn [1986]. The compound version of this method is analyzed in Eichenauer-Herrmann [1994b]. For the average-case analysis, see Eichenauer-Herrmann and Emmerich [1996]. Let $p_1, \dots, p_r \geq 5$ be distinct primes. For $1 \leq i \leq r$, let $a_i \in \mathbb{Z}_{p_i}^* = \mathbb{Z}_{p_i} \setminus \{0\}$ and let $(z_n^{(i)})_{n \geq 0}$ be a sequence in $\mathbb{Z}_{p_i}$ with

$$z_{n+1}^{(i)} \equiv a_i (z_n^{(i)})^{-1} + 1 \pmod{p_i}, \quad n \geq 0,$$

where z^{-1} denotes the multiplicative inverse of z in $\mathbb{Z}_{p_i}^*$ and $0^{-1} = 0$. Obviously, the sequence $(z_n^{(i)})_{n \geq 0}$ is always purely periodic and p_i is the maximum possible period length. Let $\mathbb{M}_{p_i}$ be the set of all $a_i \in \mathbb{Z}_{p_i}^*$ which belong to sequences with period length p_i . The set $\mathbb{M}_{p_i}$ is always nonvoid and its elements can be characterized by properties of the polynomial $x^2 - x - a_i \in \mathbb{Z}_{p_i}[x]$ (cf. Niederreiter [1992]). In the following, let $a_i \in \mathbb{M}_{p_i}$, $c_i \in \mathbb{Z}_{p_i}^*$ and

$$x_n^{(i)} = \frac{c_i z_n^{(i)}}{p_i} \in [0, 1), \quad n \geq 0,$$

be the corresponding stream of (ordinary) *inversive congruential pseudorandom numbers*.

Now, a sequence $(x_n)_{n \geq 0}$ of *compound inversive congruential pseudorandom numbers* in the interval $[0, 1)$ is defined by

$$x_n \equiv x_n^{(1)} + \dots + x_n^{(r)} \pmod{1}, \quad n \geq 0.$$

Since the primes $p_1, \dots, p_r$ are distinct, the sequence $(x_n)_{n \geq 0}$ is purely periodic with period length $m = p_1 \cdots p_r$ and $x_0, x_1, \dots, x_{m-1}$ runs through all rationals in $[0, 1)$ with denominator m .

In the following, nonoverlapping s -tuples of k -dimensional vectors $(\mathbf{x}_{sn}, \dots, \mathbf{x}_{sn+s-1})$ with $\mathbf{x}_n = (x_{kn}, x_{kn+1}, \dots, x_{kn+k-1}) \in [0, 1)^k$, $n \geq 0$, of compound inversive congruential pseudorandom numbers are considered.

THEOREM 4. *Let $1 \leq ks < \min(p_1, \dots, p_r)$ and $\varepsilon \in (0, 1]$. Then there are more than $(1 - \varepsilon) \prod_{i=1}^r (p_i - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ with*

$$D_{N; c_1, \dots, c_r}^{(ks)} < \left(\frac{3}{2}\right)^{ks} \frac{3}{\varepsilon} (\sqrt{2ks+1} + 0.5)^r N^{-1/2} (2 \log N + 4)^{ks+2} \\ \times \log^{3/2}(\log_2 N + 4), \quad N = 1, 2, \dots, m.$$

PROOF. For $1 \leq i \leq r$, $1 \leq s_1 < p_i$, $h_0 \in \mathbb{Z}_{p_i}$, and $\mathbf{h} \in \mathbb{Z}^{s_1}$, $\mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}$, it follows from Eichenauer-Herrmann and Emmerich [1996], Lemma 4 that

$$\#\{0 \leq n < p_i \mid \mathbf{h} \mathbf{z}_n^{(i)} \equiv h_0 \pmod{p_i}\} \leq 2s_1 - 1.$$

Hence, for $1 \leq i \leq r$ and for any $h' \in \mathbb{Z}_{p_i}$, $\mathbf{h}_v \in \mathbb{Z}^k$ ($v = 0, \dots, s-1$) with $(\mathbf{h}_0, \dots, \mathbf{h}_{s-1}) \not\equiv \mathbf{0} \pmod{p_i}$,

$$\#\left\{0 \leq n < p_i \mid \sum_{0 \leq v \leq s-1} \mathbf{h}_v \cdot \mathbf{z}_{n+v}^{(i)} \equiv h' \pmod{p_i}\right\} \leq 2ks - 1.$$

Therefore, Theorem 1' can be applied with $B_i \leq 2ks - 1$.

Bearing in mind that for $p \geq 5$ and $B \in [0, 2(p-1)]$,

$$\frac{\sqrt{pB+1}}{\sqrt{p-1}} \leq \sqrt{B+2} + 0.5,$$

we get the assertion of Theorem 4. $\square$

4.3 The Compound Explicit Inversive Method

The explicit inversive congruential method was introduced in Eichenauer-Herrmann [1993b]. The compound version of this method was studied in Eichenauer-Herrmann [1993a]. The following k -dimensional variant of the explicit inversive congruential method was proposed in Niederreiter [1994]. For the average-case analysis, see Eichenauer-Herrmann et al. [1997].

For $1 \leq i \leq r$, let $a_{i1}, \dots, a_{ik} \in \mathbb{Z}_{p_i}^*$ and $b_{i1}, \dots, b_{ik} \in \mathbb{Z}_{p_i}$ be arbitrary fixed parameters. Let $z_{nj}^{(i)} \in \mathbb{Z}_{p_i}$ be defined by

$$z_{nj}^{(i)} \equiv (a_{ij}n + b_{ij})^{-1} \pmod{p_i}, \quad n \geq 0, \quad 1 \leq j \leq k,$$

where z^{-1} denotes the multiplicative inverse of $z \in \mathbb{Z}_{p_i}^*$ and $0^{-1} = 0$. It is obvious that the sequence $(\mathbf{z}_n^{(i)})_{n \geq 0}$ of elements of $\mathbb{Z}_{p_i}^k$ given by

$$\mathbf{z}_n^{(i)} = (z_{n1}^{(i)}, \dots, z_{nk}^{(i)}), \quad n \geq 0,$$

is purely periodic, with period length p_i . Now, in the compound approach, the corresponding stream of k -dimensional vectors of (ordinary) *explicit inversive congruential pseudorandom numbers* is defined by

$$\mathbf{x}_n^{(i)} \equiv \frac{c_i \mathbf{z}_n^{(i)}}{p_i} \pmod{1}, \quad n \geq 0,$$

where $c_i \in \mathbb{Z}_{p_i}^*$ is an additional parameter.

Using Theorem 1', we obtain the following discrepancy estimate for the distribution of ks -dimensional sequence $(\mathbf{x}_n, \dots, \mathbf{x}_{n+s-1}) \in [0, 1)^{ks}$:

THEOREM 5. *Let $1 \leq ks \leq \min\{p_1, \dots, p_r\}$. For $1 \leq i \leq r$, let $(a_{ij}^{-1}b_{ij} + v) \in \mathbb{Z}_{p_i}$ ($j = 1, \dots, k$, $v = 0, \dots, s-1$) be distinct elements and $\varepsilon \in (0, 1]$. Then there are more than $(1 - \varepsilon)(p_1 - 1) \cdots (p_r - 1)$ values of $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_r}^*$ with*

$$D_N^{(ks)}(\mathbf{c}_1, \dots, \mathbf{c}_r) < \left(\frac{3}{2}\right)^{ks} \frac{3}{\varepsilon} \prod_{i=1}^r \frac{\sqrt{2ksp_i} + 1}{\sqrt{p_i - 1}} N^{-1/2} (2 \log N + 4)^{ks+2} \log^{3/2}(\log_2 N + 4)$$

for all $N \in [1, m]$.

PROOF. For $1 \leq i \leq r$, $1 \leq s_1 < p_i$, $h_0 \in \mathbb{Z}_{p_i}$, and $\mathbf{h} \in \mathbb{Z}^{s_1}$, $\mathbf{h} \not\equiv \mathbf{0} \pmod{p_i}$, it follows from Niederreiter [1994, Theorem 1] and Eichenauer-Herrmann et al. [1997, p. 215], that

$$\#\{0 \leq n < p_i \mid \mathbf{h} \mathbf{z}_n^{(i)} \equiv h_0 \pmod{p_i}\} \leq 2s_1.$$

Hence, for $1 \leq i \leq r$ and for any $h' \in \mathbb{Z}_{p_i}$, $\mathbf{h}_v \in \mathbb{Z}^k$ ($v = 0, \dots, s-1$) with $(\mathbf{h}_0, \dots, \mathbf{h}_{s-1}) \not\equiv \mathbf{0} \pmod{p_i}$,

$$\#\left\{0 \leq n < p_i \mid \sum_{0 \leq v \leq s-1} \mathbf{h}_v \cdot \mathbf{z}_{n+v}^{(i)} \equiv h' \pmod{p_i}\right\} \leq 2ks.$$

Therefore, Theorem 1' can be applied with $B_i \leq 2ks$, which yields the desired result. $\square$

Remark 2. In Levin [2000], similar results are obtained for the *digital inversive* pseudorandom number generators [Eichenauer-Herrmann and Niederreiter 1994].

Remark 3. In accordance with the law of the iterated logarithm for the discrepancy of N true random points $\mathbf{z}_n \in [0, 1]^d$ ($n = 1, \dots, N$) [Kiefer 1961],

$$D(\mathbf{z}_1, \dots, \mathbf{z}_N) = O(N^{-1/2}(\log \log N)^{1/2}) \quad \lambda \text{ a.e.} \quad (32)$$

It is obvious that for large d , the upper bounds (16) are very far from the order of magnitude (32). Using the generators of pseudorandom numbers based on low discrepancy sequences [Levin 1999], we obtain a discrepancy estimate with an order of magnitude (32) for the large d .

5. CONCLUSION

Let the number of prime factors r be fixed. Then Theorem 1' shows that there exist parameters $(c_1, \dots, c_r)$ such that the discrepancy $D_{N; c_1, \dots, c_r}^{(d)}$ over part of the period of d -tuples of successive pseudorandom numbers is of order of magnitude $O(N^{-1/2}(\log N)^{d+3})$ for all $N \in [1, m]$. This fits well with the asymptotic behavior of the discrepancy of N true random points from $[0, 1]^d$ according to the law of the iterated logarithm, provided that the degree d is not large. In Section 3.2, we propose the heuristic algorithm for the choice of such parameters $(c_1, \dots, c_r) \in \mathbb{Z}_{p_1}^* \times \dots \times \mathbb{Z}_{p_r}^*$ for not too large d and m (say $d \leq 5$, $m \leq 10^9$). The considered pseudorandom numbers can be used also in simulation methods for the case when the number of calculations is not known in advance.

Theorem 2 is the analog of Theorem 1', where we use the diaphony instead of the discrepancy to characterize the uniformity and statistical independence of the successive pseudorandom numbers.

Theorem 3, 4, and 5 are immediate consequences of Theorem 1' for the compound nonlinear, inversive and explicit inversive congruential method.

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