The equivalence modulo non-stationary ideals

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Outline

1. Classifying First-order countable Theories
2. The Main Gap in the Borel hierarchy
3. The Generalized Baire Space
4. Properties of $E^\kappa_\mu$ and $E^2_\mu$
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1 Classifying First-order countable Theories

2 The Main Gap in the Borel hierarchy

3 The Generalized Baire Space

4 Properties of $E^\kappa_\mu$ and $E^2_\mu$
Let $I(T, \alpha)$ denote the number of non-isomorphic models of $T$ with cardinality $\alpha$.

**What is the behavior of $I(T, \alpha)$?**

- **Löwenheim-Skolem Theorem:**
  \[ \exists \alpha \geq \omega \ I(T, \alpha) \neq 0 \Rightarrow \forall \beta \geq \omega \ I(T, \beta) \neq 0. \]

- **Morley’s categoricity:**
  \[ \exists \alpha > \omega \ I(T, \alpha) = 1 \Rightarrow \forall \beta > \omega \ I(T, \beta) = 1 \]

- **Shelah’s Main Gap Theorem:** Either, for every uncountable cardinal $\alpha$, $I(T, \alpha) = 2^\alpha$, or $\forall \alpha > 0 \ I(T, \aleph_\alpha) < \beth_1(\| \alpha \|)$. 

Approaches

- Shelah’s stability theory.
  Classify the models of $T$ by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.
The topology

$\kappa$ is a cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set $2^\kappa$ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{ \eta \in 2^\kappa \mid \zeta \subset \eta \}$$

is a basic open set.

The collection of Borel subsets of $2^\kappa$ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length $\kappa$. 
Reductions

A function $f : 2^\kappa \to 2^\kappa$ is Borel, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of $2^\kappa$.

Let $E_1$ and $E_2$ be equivalence relations on $2^\kappa$. We say that $E_1$ is Borel reducible to $E_2$, if there is a Borel function $f : 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leq_B E_2$. 
Coding structures

Fix a language \( \mathcal{L} = \{ P_n \mid n < \omega \} \)

Definition

Let \( \pi \) be a bijection between \( \kappa^{<\omega} \) and \( \kappa \). For every \( f \in 2^\kappa \) define the structure \( A_f \) with domain \( \kappa \) by: for every tuple \( (a_1, a_2, \ldots, a_n) \) in \( \kappa^n \)

\[
(a_1, a_2, \ldots, a_n) \in P_{m}^{A_f} \iff f(\pi(m, a_1, a_2, \ldots, a_n)) = 1
\]

Definition (The isomorphism relation)

Given \( T \) a first-order complete countable theory in a countable vocabulary, we say that \( f, g \in 2^\kappa \) are \( \cong^\kappa_T \) equivalent if

- \( A_f \models T, A_g \models T, A_f \cong A_g \)

or

- \( A_f \not\models T, A_g \not\models T \)
The complexity

We can define a partial order on the set of all first-order complete countable theories

\[ T \preceq^\kappa T' \iff \simeq^\kappa_T \preceq B \simeq^\kappa_T \]
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Shelah’s Main Gap Theorem

Theorem (Shelah)

*If T is classifiable and T’ is not, then T is less complex than T’ and their complexity are not close.*

**Question:**

Is there a Borel reducibility counterpart of the Main Gap Theorem in the space $2^\kappa$?
Countable

\[ T = Th(\mathbb{Q}, \leq). \]

\( T' \), the theory of vector space over the field of rational numbers.

By the Borel-reducibility hierarchy:

\[ T \preceq^\omega T' \]

\[ T' \not\preceq^\omega T \]

By the stability theory \( T' \) is simpler than \( T \).
Uncountable

Theorem (Shelah)
If $T$ is classifiable, then $\cong^\kappa_T$ is $\Delta^1_1$.

Theorem (S. Friedman, Hyttinen, Kulikov)
If $T$ is unstable then $\cong^\kappa_T$ is not $\Delta^1_1$.

Theorem (S. Friedman, Hyttinen, Kulikov)
If $T$ is unstable and $T'$ is classifiable, then $T \not\leq^\kappa T'$.
The Equivalence Modulo Non-stationary Ideals

Definition

For every $X \subseteq \kappa$ stationary, we define $E_X^2$ as the relation

$$E_X^2 = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X \text{ is not stationary}\}$$

where $\triangle$ denotes the symmetric difference.

When $X = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$, we will denote $E_X^2$ by $E_\lambda^2$. 
Looking above the Gap

Theorem (S. Friedman, Hyttinen, Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

- If $T$ is an unstable or superstable with OTOP, then $E^2_\lambda \leq_B \equiv^\kappa_T$.
- If $\lambda \geq 2^\omega$ and $T$ is a superstable with DOP, then $E^2_\lambda \leq_B \equiv^\kappa_T$.

Theorem (S. Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $T$ is a stable unsuperstable. Then $E^2_\omega \leq_B \equiv^\kappa_T$.
Looking below the Gap

Theorem (S. Friedman, Hyttinen, Kulikov)

If $T$ is a classifiable theory, then for all regular cardinal $\lambda < \kappa$, $E^2_{\lambda} \not\leq_B \equiv^\kappa_T$

Theorem (Hyttinen, Kulikov, M.)

Denote by $S^\kappa_\lambda$ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.

Suppose $T$ is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\equiv^\kappa_T \leq_B E^2_{\lambda}$. 

The Main Gap in the Borel hierarchy

A Generalized Borel-reducibility Counterpart

Theorem (Hyttinen, Kulikov, M.)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If $T$ is a classifiable theory and $T'$ is a stable unsuperstable theory, then $\equiv_T \leq_B E_\omega \leq_B \equiv_T$, and $E_\omega \not\leq_B \equiv_T$.

Let $H(\kappa)$ be the following property: If $T$ is classifiable and $T'$ is not, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Theorem (Hyttinen, Kulikov, M.)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. If $V = L$, then $H(\kappa)$ holds.
Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem that does not need to force diamonds?

It can be studied in two ways:

• Does it holds $E^2_\omega \leq_B \cong^\kappa_T$ for every theory $T$ non-classifiable under some cardinal assumptions that imply $\diamondsuit(S^\kappa_\omega)$?

• Is there a Borel reducibility counterpart of the Main Gap Theorem in another space?
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The generalized Baire space

Let $\kappa$ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set $\kappa^\kappa$ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{ \eta \in \kappa^\kappa \mid \zeta \subset \eta \}$$

is a basic open set.

The collection of Borel subsets of $\kappa^\kappa$ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length $\kappa$. 
Reductions in GBS

Let $E_1$ and $E_2$ be equivalence relations on $\kappa^\kappa$. We say that $E_1$ is Borel reducible to $E_2$, if there is a Borel function $f : \kappa^\kappa \to \kappa^\kappa$ that satisfies

$$(x, y) \in E_1 \iff (f(x), f(y)) \in E_2.$$

We write $E_1 \leq_B E_2$. 
Coding structures in GBS

Fix a language $\mathcal{L} = \{P_n|n < \omega\}$

Definition

Let $\pi$ be a bijection between $\kappa^{<\omega}$ and $\kappa$. For every $f \in \kappa^\kappa$ define the structure $A_f$ with domain $\kappa$ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P_{\pi}^A \iff f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given $T$ a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\equiv^T$ equivalent if

- $A_f \models T, A_g \models T, A_f \cong A_g$
  
  or

- $A_f \not\models T, A_g \not\models T$

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The Equivalence Modulo Non-stationary Ideals in GBS

We say that \( f, g \in \kappa^\kappa \) are \( E^\kappa_\lambda \) equivalent if the set \( \{ \alpha < \kappa | f(\alpha) = g(\alpha) \} \) contains an unbounded set that is closed under \( \lambda \)-limits.

Theorem (Hyttinen, M.)

Suppose \( T \) is a classifiable theory and \( \lambda < \kappa \) is a regular cardinal.
Then \( \equiv_T^\kappa \leq_B E^\kappa_\lambda \).
Orthogonal Chain Property (OCP)

Lemma (Hyttinen, M.)

If a theory $T$ has the OCP, then $T$ is not classifiable.

Theorem (Hyttinen, M.)

Suppose $T$ is a classifiable theory, $T'$ is an stable theory with the OCP, and $\kappa$ an inaccessible cardinal. Then $\equiv^\kappa_T \leq_B E^\kappa_\omega \leq_B \equiv^\kappa_{T'}$. 
The Generalized Baire Space

**Strong DOP (S-DOP)**

**Lemma**

*If a theory $T$ has the S-DOP, then $T$ is not classifiable.*

**Theorem (M.)**

*Suppose $T$ is a classifiable theory, $T'$ is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and $\kappa$ an inaccessible cardinal. Then $\simeq^\kappa_T \leq_B E^\kappa_\lambda \leq_B \simeq^\kappa_{T'}$.*
Motivation

- It is consistent that there is a generalized Borel reducibility counterpart of the Main Gap Theorem in the space $2^\kappa$.

- For $\kappa$ inaccessible, the classifiable theories are at most as complex as the theories with OCP or S-DOP.

1. For which $\lambda$ holds $E_\lambda^\kappa \leq_B E_\lambda^2$?
2. For which $\lambda$ holds $E_\omega^2 \leq_B E_\lambda^2$?
Properties of $E^\kappa_\mu$ and $E^2_\mu$

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\[ \Sigma^1_1 \]-completeness

**Theorem (Hyttinen, Kulikov)**

If \( V = L \), then \( E^\kappa_\mu \) is \( \Sigma^1_1 \)-complete for every \( \mu < \kappa \).

**Corollary**

If \( V = L \) and \( T \) is a theory with the OCP or the S-DOP, then \( \cong^\kappa_T \) is \( \Sigma^1_1 \)-complete.
Borel*-codes

For every regular cardinal $\gamma < \kappa$ define the following Borel*-code. Define $T_\gamma$ as the tree whose elements are all the increasing elements of $\kappa^{\leq \gamma}$, ordered by end-extension. For every element of $T_\gamma$ that is not a leaf, define

$$H_\gamma(x) = \begin{cases} \bigcup & \text{if } x \text{ has an immediate predecessor } x^- \text{ and } H_x(x^-) = \cap \\ \cap & \text{otherwise} \end{cases}$$

and for every leaf $b$ define $H_\gamma(b)$ by:

$$(\eta, \xi) \in H_\gamma(b) \iff \alpha = \sup(\text{ran}(b))(\eta(\alpha) = \xi(\alpha)).$$

Let us denote by $T_\gamma \upharpoonright \alpha$ the subtree of $T_\gamma \cap \alpha^{\leq \gamma}$ with

$$\{ b \in T_\gamma \upharpoonright \alpha \mid b \text{ a leaf} \} = \{ b \in T_\gamma \mid b \text{ a leaf} \} \cap \{ b \in T_\gamma \cap \alpha^{\leq \gamma} \mid b \text{ a leaf} \}$$

and $H_\gamma^\alpha$ is $H_\gamma$ restricted to $\{ b \in T_\gamma \upharpoonright \alpha \mid b \text{ a leaf} \}$. 
**Borel*-reflection**

**Definition**

For every $\gamma < \lambda < \kappa$ regular cardinals, we say that $S^\kappa_\gamma$ Borel*-reflects to $S^\kappa_\lambda$ if the following holds for every $\eta, \xi \in \kappa^\kappa$:

$$II \uparrow B^* (T_\gamma, H_\gamma, (\eta, \xi)) \iff II \uparrow B^* (T_\gamma \upharpoonright \alpha, H_\gamma^\alpha, (\eta, \xi))$$

for $\lambda$-club many $\alpha$'s in $S^\kappa_\lambda$.

**Lemma**

If $S^\kappa_\gamma$ Borel*-reflects to $S^\kappa_\lambda$, then $E^\kappa_\gamma \leq_B E^\kappa_\lambda$. 

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\(\Diamond\)-reflection

**Definition**

Let \(X, Y\) be subsets of \(\kappa\) and suppose \(Y\) consists of ordinals of uncountable cofinality. We say that \(X \Diamond\) reflects to \(Y\) if there exists a sequence \(\{D_\alpha\}_{\alpha \in Y}\) such that:

- \(D_\alpha \subset \alpha\) is stationary in \(\alpha\).
- if \(Z \subset X\) is stationary, then \(\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}\) is stationary.

**Theorem (S. Friedman, Hyttinen, Kulikov)**

If \(X \Diamond\)-reflects to \(Y\), then \(E_2^X \leq_B E_2^Y\).
Lemma

Suppose $\lambda^{<\lambda} = \lambda$ and $\gamma < \lambda$ regular cardinals. If $S^\kappa_\gamma \diamond$-reflects to $S^\kappa_\lambda$, then $S^\kappa_\gamma$ Borel*-reflects to $S^\kappa_\lambda$.

Lemma

Suppose that $\kappa$ is a weakly compact cardinal and that $V = L$. Then there is a forcing extension where $\lambda^{++} = \kappa$ and

1. $E^2_\lambda \leq_B E^2_{\lambda^+}$.
2. $E^K_\lambda \leq_B E^K_{\lambda^+}$. 

Lemma

The following is consistent:
There are $\lambda^+$ many disjoint stationary subsets of $S^{\lambda^{++}}_\lambda$ such that $S^{\lambda^{++}}_\lambda$ $\Diamond$-reflects to $S_\gamma$ for every $\gamma < \lambda^+$.

Corollary

The following is consistent:

$$E^{\lambda^{++}}_\lambda \leq_B E^2_{\lambda^+}.$$
Questions

1. For which $\lambda$ and $\kappa$ holds $E_\lambda^{\kappa} \leq_B E_\lambda^2$?

2. Is it consistent that $E_\lambda^2 \leq_B E_\omega^2$?
Properties of $E^\kappa_\mu$ and $E^2_\mu$

References


- T. Hyttinen and V. Kulikov, *On $\Sigma^1_1$-complete equivalence relations on the generalized Baire space*, Mathematical Logic Quarterly 61, 66-81. 2015.

Properties of $E^k_\mu$ and $E^2_\mu$

References