Series of lectures on Generalized Descrptive Set Theory

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1 An introduction to generalized descriptive set theory

Generalized descriptive set theory is the generalization of descrptive set theory to uncountable cardinals. During this notes, κ will be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$, unless otherwise is stated.

First Session

The aim of this talk is to introduce the notions of κ-Borel class, κ-Δ_1^1 class, κ-Borel* class, and show the relation between these classes.

Definition 1.1 (The Generalized Baire space $\mathbf{B}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta \in \kappa^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in \kappa^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.2 (The Generalized Cantor space $\mathbf{C}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ endowed with the following topology. For every $\eta \in 2^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in 2^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.3 (κ-Borel class)**.** *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$. The class κ-Borel(S) of all κ-Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, unions and intersections both of length at most κ.*

Definition 1.4. *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$.*

- *$X \subset S$ is a $\Sigma_1^1(\kappa)$ set if there is a set $Y \subset S \times S$ a closed set such that $pr(Y) = \{x \in S \mid \exists y \in S\ (x, y) \in Y\} = X$.*
- *$X \subset S$ is a $\Pi_1^1(\kappa)$ set if $S \backslash X$ is a $\Sigma_1^1(\kappa)$ set.*
- *$X \subset S$ is a $\Delta_1^1(\kappa)$ set if X is a $\Sigma_1^1(\kappa)$ set and a $\Pi_1^1(\kappa)$ set.*

Definition 1.5 (κ-*Borel**-set in $\mathbf{B}(\kappa), \mathbf{C}(\kappa)$)**.** *Let $S \in \{2^\kappa, \kappa^\kappa\}$.*

1. *A subset $T \subset \kappa^{<\kappa}$ is a tree if for all $f \in T$ with $\alpha = dom(f) > 0$ and for all $\beta < \alpha$, $f \restriction \beta \in T$ and $f \restriction \beta < f$.*
2. *A tree T is a κ^+, λ-tree if does not contain chains of length λ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum in T.*
3. *A pair (T, h) is a κ-Borel*-code if T is a closed κ^+, λ-tree, $\lambda \leq \kappa$, and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.*
4. *For an element $\eta \in S$ and a κ-Borel*-code (T, h), the κ-Borel*-game $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.*

5. *A set $X \subseteq S$ is a κ-Borel*-set if there is a κ-Borel*-code (T,h) such that for all $\eta \in S$, $\eta \in X$ if and only if* **II** *has a winning strategy in the game $B^*(T,h,\eta)$.*

We will write $\mathbf{II} \uparrow B^*(T,h,\eta)$ when **II** has a winning strategy in the game $B^*(T,h,\eta)$.

Example 1.1. *Let $\mu < \kappa$ be a regular cardinal, we say that $X \subseteq \kappa$ is a μ-club if X is an unbounded set and it is closed under μ-limits.*

Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in 2^\kappa$ we say that η and ξ are $E^2_{\mu\text{-club}}$ equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club.

The relation $E^2_{\omega\text{-club}}$ is a κ-Borel set. Let us define the following Borel*-code (T,h):*

- *$T = \{f \in \kappa^{<\omega+2} \mid f$ is strictly incresing$\}$.*
- *For f not a leave, $h(f) = \cup$ if dom(f) is even and $h(f) = \cap$ if dom(f) is odd.*
- *To define $h(f)$ for a leave f, first define the set $L(g) = \{f \in \kappa^{\omega+1} \mid g \subseteq f\}$ for all $g \in T$ with domain ω, and $\gamma_g = sup_{n<\omega}(g(n))$. Let $h \restriction L(g)$ be a bijection between $L(g)$ and the set $\{N_p \times N_q \mid p, q \in \kappa^{\gamma_g+1}, p(\gamma_g) = q(\gamma_g)\}$.*

Let us show that (T,h) codes $E^2_{\omega\text{-club}}$. Suppose $(\eta,\xi) \in E^2_{\omega\text{-club}}$, so there is an ω-club C such that $\forall \alpha \in C$ $\eta(\alpha) = \xi(\alpha)$. The following is a winning strategy for **II** *in the game $B^*(T,h,(\eta,\xi))$. For every even $n < \omega$, if the game is at f with dom$(f) = n$,* **II** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n) \in C$. Since C is closed under ω limits, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C$. So there is G an immediate successor of g, such that $h(G) = N_{\eta\restriction\gamma} \times N_{\xi\restriction\gamma}$. Finally if* **II** *chooses G in the ω move, then* **II** *wins.*

For the other direction, suppose $(\eta,\xi) \notin E^2_{\omega\text{-club}}$, so there is $A \subset S^\kappa_\omega$ stationary (S^κ_ω is the set of ω-cofinal ordinals below κ) such that for all $\alpha \in S$, $\eta(\alpha) \neq \xi(\alpha)$.

We will show that for every σ strategy of **II***, σ is not a winning strategy. Let σ be an strategy for* **II***, this mean that σ is a function from $\kappa^{<\omega+1} \to \kappa$. Notice that if* **II** *follows σ as a strategy, then when the game is at f, dom$(f) = n$ even,* **II** *chooses f' such that $f \subset f'$ and $f'(n) = \sigma((f(0), f(1), \ldots, f(n-1)))$. Let C be the set of closed points of σ, $C = \{\alpha < \kappa \mid \sigma(\alpha^{<\omega}) \subseteq \alpha\}$, C is unbounded and closed under ω-limits. Therefore $C \cap A \neq \emptyset$. Let γ be the least element of $C \cap A$ that is an ω-limit of elements of C, and let $\{\gamma_n\}_{n<\omega}$ be a sequence of elements of C cofinal to γ. The following is a winning strategy for* **I** *in the game $B^*(T,h,(\eta,\xi))$, if* **II** *uses σ as an strategy.*

When the game is at f with dom$(f) = n$, n odd, then **I** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n)$ is the least element of $\{\gamma_n\}_{n<\omega}$ that is bigger than $f(n-1)$. This element always exists because $\{\gamma_n\}_{n<\omega}$ is cofinal to γ and $\gamma \in C$, γ is a closed point of σ. Since* **I** *is following σ as a strategy and γ is a closed point of σ, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C \cap A$. Since $\eta(\gamma) \neq \xi(\gamma)$, there is no G immediate successor of g, such that $(\eta,\xi) \in h(G)$. So it does not matter what* **II** *chooses in the ω move,* **I** *will win.*

The previous definitions are the generalization of the notions of Borel, Δ^1_1, and Borel* from descriptive set theory, the spaces ω^ω and 2^ω. A classical result in descriptive set theory states that the Borel class, the Δ^1_1 class, and the Borel* class are the same. This doesn't hold in generalized descriptive set theory as we will see.

Theorem 1.6 ([1], Thm 17)**.** *κ-Borel$\subseteq$ κ-Borel**

Proof. Let us prove something even stronger. X is a κ-Borel set if and only if there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree.

Let us define the sets $(B_i)_{i\leq\kappa^+}$ by:

- $B_0 = \{N_p \mid p \in 2^{<\kappa}\}$, the set of basic open sets.
- If $\alpha = \beta + n$ for n an odd natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.
- If $\alpha = \beta + n$ for n an even positive natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcup \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.
- If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

We will show by induction over α that for every $X \in B_\alpha$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree.

For $\alpha = 0$. If $X \in B_0$, then $T = \{\emptyset\}$ and $h(\emptyset) = X$ is a κ-Borel*-code that codes X.

Suppose $\alpha = \beta + n$ for n an even natural number and β a limit ordinal or 0 is such that for all $X \in B_\alpha$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. Suppose $X \in B_{\alpha+n+1}$, so either

$X\in B_\alpha+n$ or $X=\bigcap\mathcal{B}$ for some $\mathcal{B}\subseteq B_{\beta+n}$ with $\mid\mathcal{B}\mid=\gamma\leq\kappa$. Let $\mathcal{B}=\{X_i\}_{i<\gamma}$, by the induction hypothesis we know that there are κ-Borel*-code $\{(T_i,h_i)\}_{i<\gamma}$ such that (T_i,h_i) codes X_i and T_i is a κ^+,ω-tree, for all $i<\gamma$. Let $\mathcal{T}=\{r\}\cup\bigcup_{i<\gamma}T_i\times\{i\}$ be the tree ordered by $r<(x,j)$ for all $(x,j)\in\bigcup_{i<\gamma}T_i\times\{i\}$, and $(x,i)<(y,j)$ if and only if $i=j$ and $x<y$ in T_i. Let $T\subseteq\kappa^{<\omega}$ be a tree isomorphic to $\mathcal{T}$ and let $\mathcal{G}:T\to\mathcal{T}$ be a tree isomorphism. If $\mathcal{G}(x)\neq r$, then denote $\mathcal{G}(x)$ by $(\mathcal{G}_1(x),\mathcal{G}_2(x))$. Define h by $h(x)=\cap$ if $G(x)=r$, and $h(x)=h_{\mathcal{G}_2(x)}(\mathcal{G}_1(x))$.

Let us show that (T,h) codes X. Let $\eta\in X$, so $\eta\in X_i$ for all $i<\gamma$. If at the beginning **I** chooses x, then **II** follows the winning strategy from the game $B^*(T_{\mathcal{G}_2(x)},h_{\mathcal{G}_2(x)},\eta)$, choosing the element given by $\mathcal{G}^{-1}$. We conclude that $\mathbf{II}\uparrow B^*(T,h,\eta)$. Let $\eta\notin X$, so there is $i<\gamma$ such that $\eta\notin X_i$, so **II** has no winning strategy for the game $B^*(T_i,h_i,\eta)$. Since at the beginning **I** can choose x such that $\mathcal{G}_2(x)=i$, **II** cannot have a winning strategy for the game $B^*(T,h,\eta)$. Otherwise **II** would have a winning strategy the game $B^*(T_i,h_i,\eta)$.

The case $\alpha=\beta+n$ for n an odd natural number and β a limit ordinal or 0 is similar, just make $h(x)=\cup$ if $G(x)=r$ when constructing (T,h).

Suppose α is a limit ordinal such that for all $\beta<\alpha$, for all $X\in B_\beta$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. Let $X\in B_\alpha$, since $B_\alpha=\bigcup_{\beta<\alpha}B_\beta$ there is $\beta<\alpha$ such that $X\in B_\beta$. By the induction hypothesis, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. □

Second Session

Theorem 1.7 ([1], Thm 17). *1. κ-Borel* $\subseteq\Sigma_1^1(\kappa)$.*

2. κ-Borel$\subseteq\Sigma_1^1(\kappa)$.

3. κ-Borel$\subseteq\Delta_1^1(\kappa)$.

Proof. 1. Let X be a κ-Borel* set, there is a κ-Borel* code (T,h) such that X is coded by (T,h). Since $\kappa^{<\kappa}=\kappa$, we can code the strategies $\sigma:T\to T$ by elements of κ^κ.

Claim 1.8. *The set $Y=\{(\eta,\xi)\mid\xi$ is a code of a winning strategy for* **II** *in $B^*(T,h,\eta)\}$ is closed.*

Proof. Let (η,ξ) be an element not in Y. So ξ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$, there is $\alpha<\kappa$ such that for every $\zeta\in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$. Otherwise T would have a branch of length κ. Because of the same reason, there is $\beta<\kappa$ such that for every $f\in N_{\eta\restriction\beta}$, $\zeta\in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,f)\}$. So $N_{\eta\restriction\beta}\times N_{\xi\restriction\alpha}$ is a subset of the complement of Y. □

Since $pr(Y)=X$, we are done.

2. It follows from Theorem 1.6 and (1).

3. It follows from (2) and the fact that κ-Borel sets are closed under complement.

□

It has been proved, under the assumption $V=L$, that κ-Borel* $=\Sigma_1^1(\kappa)$. It was first proved in [1] Theorem 18, the idea of this proof is to show that the filter of ω-clubs is $\Sigma_1^1(\kappa)$-complete and κ-Borel*. This result was later improve in [2] Theorem 7 to show that the relations $E^\kappa_{\mu\text{-club}}$ is $\Sigma_1^1(\kappa)$-complete and κ-Borel*, where $\eta,\xi\in\kappa^\kappa$ are $E^\kappa_{\mu\text{-club}}$ related if the set $\{\alpha<\kappa\mid\eta(\alpha)=\xi(\alpha)\}$ contains an μ-club. Recently in [4] Theorem 3.1 these results were improve to show that the inclusion modulo the non-stationary ideal (below) is $\Sigma_1^1(\kappa)$-complete, which implies that the relations $E^2_{\mu\text{-club}}$ are $\Sigma_1^1(\kappa)$-complete. Because of its applications for future sessions, we will prove (under the assumption $(V=L)$) that the inclusion modulo the non-stationary ideal is $\Sigma_1^1(\kappa)$-complete, this will implies the consistency of κ-Borel* $=\Sigma_1^1(\kappa)$.

Definition 1.9 (Inclusion modulo non-stationaries). *For $\eta,\xi\in 2^\kappa$ and a stationary $S\subset\kappa$, we write $\eta\sqsubseteq_S\xi$ if $(\eta^{-1}\{1\}\backslash\xi^{-1}\{1\})\cap S$ is non-stationary. If $S=S^\kappa_\mu$, we denoted $\sqsubseteq_S$ by $\sqsubseteq_\mu$.*

If Q_1 and Q_2 are quasi-orders respectively on 2^κ, then we say that Q_1 is *Borel-reducible* to Q_2 if there exists a κ-Borel map $f\colon 2^\kappa\to 2^\kappa$ such that for all $\eta,\xi\in 2^\kappa$ we have $\eta Q_1\xi\iff f(\eta)Q_2f(\xi)$ and this is also denoted by $Q_1\leq_B Q_2$.

A quasi-order is Σ_1^1*-complete*, if it is $\Sigma_1^1(\kappa)$ and every $\Sigma_1^1(\kappa)$ quasi-order is Borel-reducible to it.

Theorem 1.10 ([4], Thm 3.1). $(V=L)$ *The quasi-order $\sqsubseteq_\mu$ is Σ_1^1–complete, for every regular $\mu<\kappa$.*

To prove Theorem 1.10 we need to make some preparations before we start with the proof.

Definition 1.11. • *Let us define a class function $F_\diamond\colon On \to L$. For all α, $F_\diamond(\alpha)$ is a pair (X_α, C_α) where $X_\alpha, C_\alpha \subseteq \alpha$, if α is a limit ordinal, then C_α is either a club or the empty set, and $C_\alpha = \emptyset$ when α is not a limit ordinal. We let $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ be the $<_L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if α is a limit ordinal and such pair exists and otherwise we let $F_\diamond(\alpha) = (\emptyset, \emptyset)$.*

- *We let $C_\diamond \subseteq On$ be the class of all limit ordinals α such that for all $\beta < \alpha$, $F_\diamond \restriction \beta \in L_\alpha$. Notice that for every regular cardinal α, $C_\diamond \cap \alpha$ is a club.*

Definition 1.12. *For a given regular cardinal α and a subset $A \subset \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ to be $(F_\diamond(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ to be the sequence of sets X_γ such that $F_\diamond(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.*

By ZF^- we mean ZFC+($V = L$) without the power set axiom. By $\mathrm{ZF}^\diamond$ we mean ZF^- with the following axiom:

> "For all regular cardinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in \alpha}$ is such that for all $\gamma < \alpha$, $F_\diamond(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence."

Lemma 1.13 ([4], Lemma 3.4). *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, the following are equivalent for all $\eta \in 2^\kappa$:*

- *$\varphi(\eta, x)$*
- *$S \backslash A$ is non-stationary, where $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and*

$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha (L_\beta \models ZF^\diamond \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$$

where $r(\alpha)$ is the formula "α is a regular cardinal".

Now we sketch the proof of 1.10.

Proof of Theorem 1.10 (sketch). Suppose Q is a Σ^1_1 quasi-order on 2^κ.

There is a Σ_1-formula of set theory $\psi(\eta, \xi) = \psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x) \vee \eta = \xi$ with $x \in 2^\kappa$, such that for all $\eta, \xi \in 2^\kappa$,

$$(\eta, \xi) \in Q \Leftrightarrow \psi(\eta, \xi),$$

we added $\eta = \xi$ to $\psi(\eta, \xi)$, to ensure that when we reflect $\psi(\eta \restriction \alpha, \xi \restriction \alpha)$ we get a reflexive relation. Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^Q(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta, \xi)$ defines a quasi-order on 2^κ. For all $\eta \in 2^\kappa$ and $\alpha < \kappa$, let

$$T_{\eta,\alpha} = \{p \in 2^\alpha \mid \exists \beta > \alpha (L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))\}.$$

Let $(X_\alpha)_{\alpha \in S^\kappa_\mu}$ be the diamond sequence of Definition 1.12, and for all $\alpha \in S^\kappa_\mu$, let $\mathcal{X}_\alpha$ be the characteristic function of X_α. Define $\mathcal{F}\colon 2^\kappa \to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \mathcal{X}_\alpha \in T_{\eta,\alpha} \text{ and } \alpha \in S^\kappa_\mu \\ 0 & \text{otherwise} \end{cases}$$

Claim 1.14. *$\mathcal{F}$ is a reduction of Q into $\sqsubseteq_\mu$.*

□

Third Session

As it was sketch above, the combinatorial properties of L are essential for the reduction shown on Theorem 1.10. There are three different variations of Lemma 1.13, each variation is used to define a Borel reduction.

Lemma 1.15 ([1]). *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, the following are equivalent for all $\eta \in 2^\kappa$:*

- *$\varphi(\eta, x)$*
- *$A = \{\alpha < \kappa \mid \exists \beta > \alpha (L_\beta \models \mathrm{ZF}^- \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$ contains a club, where $r(\alpha)$ is the formula "α is a regular cardinal".*

This variation was the one used in [1] to prove Theorem 1.15.

Lemma 1.16 ([2]). *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, and a stationary set $S \subset S^\kappa_\mu$, the following are equivalent for all $\eta \in 2^\kappa$:*

- $\varphi(\eta, x)$
- *$S \backslash A$ is non-stationary, where*

$$A = \{\alpha \in S \mid \exists \beta > \alpha (L_\beta \models \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge s(\alpha))\}$$

where $r(\alpha)$ is the formula "α is a regular cardinal", and $s(\alpha)$ states that $S \cap \alpha$ is stationary and $S \cap \alpha \subset S^\alpha_\mu$ in the sense that we required β to be large enough to witness that every element of $S \cap \alpha$ has cofinality μ.

This Lemma was the one used in [2] to show that the relations $E^\kappa_{\mu\text{-club}}$ is $\Sigma^1_1(\kappa)$-complete under the assumption $V = L$. This is different from Lemma 1.15 because of the stationary set S. At the same time, in Lemma 1.16 is different from 1.13. In Lemma 1.16 S is fixed from the beginning and it is independent from η and in Lemma 1.13 S depends on η and the diamond sequence. This is the reason why Lemma 1.16 cannot be used to prove Theorem 1.10, and the reason to use ZF$^\diamond$ and $C_\diamond$ to fix a diamond sequence $(X_\gamma)_{\gamma \in S^\kappa_\mu}$.

Proof of Theorem 1.13. Let $\mu < \kappa$ be a regular cardinal. Suppose that $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ holds. Let θ be a cardinal large enough such that

$$L_\theta \models \text{ZF}^\diamond \wedge \varphi(\eta, x) \wedge r(\kappa).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = \text{Sk}(\alpha \cup \{\kappa, \eta, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Then D is a club set and $D \cap C_\diamond$ is a club. Since $H(\alpha)$ is an elementary submodel of L_θ and the Mostowski collapse $\bar{H}(\alpha)$ is equal to L_β for some $\beta > \alpha$, we have $D \cap C_\diamond \subseteq A$.

Suppose $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Let θ be a large enough cardinal such that

$$L_\theta \models \text{ZF}^\diamond \wedge \neg\varphi(\eta, x) \wedge r(\kappa).$$

Let C be an unbounded set which is closed under μ-limits (a μ-club). Let

$$H(\alpha) = \text{Sk}(\alpha \cup \{\kappa, C, \eta, x, (X_\gamma, C_\gamma)_{\gamma \in S^\kappa_\mu}\})^{L_\theta}.$$

Let

$$D = \{\alpha \in S^\kappa_\mu \mid H(\alpha) \cap \kappa = \alpha\}$$

Notice that since $H(\alpha)$ is an elementary substructure of L_θ, then $H(\alpha)$ calculates all cofinalities correctly below α. Then D is an unbounded set, closed under μ-limits. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and α_0 be the least ordinal in $(\lim_\mu D) \cap S$ (where $\lim_\mu D$ is the set of ordinals of D that are μ-cofinal limits of elements of D). Since $\alpha_0 \in \lim_\mu D$, $\alpha_0 > \mu$. By the elementarity of each $H(\alpha)$ we conclude that $\alpha_0 \in C$.

Let $\bar{\beta}$ be such that $L_{\bar{\beta}}$ is equal to the Mostowski collapse of $H(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$, thus $\alpha_0 \in C_\diamond \cap \kappa$. There exists $\beta > \alpha_0$ such that

$$L_\beta \models \text{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha_0, x \restriction \alpha_0) \wedge r(\alpha_0).$$

Since $\varphi(\eta, x)$ is a Σ_1-formula, β is a limit ordinal greater than $\bar{\beta}$.

Claim 1.17. *L_β satisfies the following:*

1. *For all $\gamma \in S \cap \alpha_0$, γ has cofinality μ.*
2. *$S \cap \alpha_0$ is a stationary subset of α_0.*
3. *$D \cap \alpha_0$ is a μ-club subset of a_0.*

Proof. 1. $H(\alpha_0)$ calculates all cofinalities correctly below α_0. Thus $L_{\bar{\beta}}$ calculates all cofinalities correctly below α_0. Since β is greater than $\bar{\beta}$, L_β calculates all cofinalities correctly below α_0. Since $S \cap \alpha_0 \subseteq S^\kappa_\mu$ in L, then $S \cap \alpha_0 \subseteq S^\kappa_\mu$ holds in L_β.

2. Since $\alpha_0 \in C_\diamond \cap \kappa$ and L_β satisfies ZF$^\diamond$ and $r(\alpha_0)$, L_β satisfies that $S \cap \alpha_0$ is a stationary subset of α_0.

3. Let $\alpha < \alpha_0$ be such that $L_\beta \models cf(\alpha) = \mu \ \wedge \ \bigcup(D \cap \alpha) = \alpha$, we will show that $L_\beta \models \alpha \in D \cap \alpha_0$. Since L_β calculates all cofinalities correctly below α_0, $L \models cf(\alpha) = \mu \wedge \bigcup(D \cap \alpha) = \alpha$. D is a μ-club in L, thus $L \models \alpha \in D$. Since $\alpha < \alpha_0$, $L \models \alpha \in D \cap \alpha_0$. We will finish the proof by showing that $L \models \alpha \in D \cap \alpha_0$ implies $L_\beta \models \alpha \in D \cap \alpha_0$.

   Notice that $H(\alpha_0)$ is a definable subset of L_θ and D is a definable subset of L_θ. By elementarity, $D \cap \alpha_0$ is a definable subset of $H(\alpha_0)$, we conclude that $D \cap \alpha_0$ is a definable subset of $L_{\bar{\beta}}$ and $D \cap \alpha_0 \in L_\beta$. Therefore $L_\beta \models \alpha \in D \cap \alpha_0$.

□

□

Fourth Session

Let us continue with the proof of Theorem 1.10.

Proof.

Claim 1.18. *If $\eta \ Q \ \xi$, then $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$ for club-many α's.*

Proof. Suppose $\psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x)$ holds and let k witnesses that. Let θ be a cardinal large enough such that $L_\theta \models \mathrm{ZF}^\diamond \wedge \varphi(k, \eta, \xi, x) \wedge r(\alpha)$. For all $\alpha < \kappa$ let $H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, k, \eta, \xi, x\})^{L_\theta}$. The set $D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha \wedge H(\alpha) \models \psi^Q(\alpha)\}$ is a club. Using the Mostowski collapse we have that

$$D' = \{\alpha < \kappa \mid \exists \beta > \alpha (L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(k \restriction \alpha, \eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))\}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta,\alpha}$ we have that

$$\exists \beta_1 > \alpha (L_{\beta_1} \models \mathrm{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

and

$$\exists \beta_2 > \alpha (L_{\beta_2} \models \mathrm{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)).$$

Therefore, for $\beta = \max\{\beta_1, \beta_2\}$ we have that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha).$$

Since $\psi^Q(\alpha)$ holds and so transitivity holds for $\psi(\eta, \xi)$ in L_β, we conclude that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$$

so $p \in T_{\xi,\alpha}$ and $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$. This holds for all $\alpha \in D'$. □

By the previous claim, we conclude that if $\eta \ Q \ \xi$, then there is a μ-club C such that for every $\alpha \in C$ it holds that $\mathcal{X}_\alpha \in T_{\eta,\alpha} \Rightarrow \mathcal{X}_\alpha \in T_{\xi,\alpha}$. Therefore $(\mathcal{F}(\eta)^{-1}\{1\} \backslash \mathcal{F}(\xi)^{-1}\{1\}) \cap C = \emptyset$, and $\mathcal{F}(\eta) \ \sqsubseteq_\mu \ \mathcal{F}(\xi)$.

For the other direction, suppose $\neg\psi(\eta, \xi, x)$ holds. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$. Since $(X_\gamma)_{\gamma \in S^\kappa_\mu}$ is a diamond sequence, S is a stationary set. By Lemma 1.13 we know that $S \backslash A$ is stationary, where

$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha (L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}.$$

Since for all $\alpha \in S \backslash A$ we have that $X_\alpha = \eta^{-1}\{1\} \cap \alpha$, so $\mathcal{X}_\alpha \in T_{\eta,\alpha}$. We conclude that for all $\alpha \in S \backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$. On the other hand, for all $\alpha \in S \backslash A$ it holds that

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))$$

so

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\mathcal{X}_\alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha)).$$

Therefore

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\mathcal{X}_\alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

we conclude that $\mathcal{X}_\alpha \notin T_{\xi,\alpha}$, and $\mathcal{F}(\xi)(\alpha) = 0$. Hence, for all $\alpha \in S \backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$ and $\mathcal{F}(\xi)(\alpha) = 0$. Since $S \backslash A$ is stationary, we conclude that $\mathcal{F}(\eta)^{-1}\{1\} \backslash \mathcal{F}(\xi)^{-1}\{1\}$ is stationary and $\mathcal{F}(\eta) \ \not\sqsubseteq_\mu \ \mathcal{F}(\xi)$.

□

Theorem 1.19 ([1], Thm 18)**.** *($V = L$) κ-Borel* $= \Sigma^1_1(\kappa)$.*

Proof. Because of Example 1.1 and the previous Lemma, it is enough to prove the following Claim.

Claim 1.20. *Assume $f: 2^\kappa \to 2^\kappa$ is a Borel function and $B \subset 2^\kappa$ is Borel*. Then $f^{-1}[B]$ is Borel*.*

Let (T_B, H_B) be a Borel*-code for B. Define the Borel*-code (T_A, H_A) by letting $T_B = T_A$ and $H_A(b) = f^{-1}[H_B(b)]$ for every branch b of T_B. Let A be the Borel*-set coded by (T_A, H_A). Clearly, $\mathbf{II} \uparrow B^*(T_B, H_B, \eta)$ if and only if $\mathbf{II} \uparrow B^*(T_A, H_A, f^{-1}(\eta))$, so $f^{-1}[B] = A$. □

Theorem 1.21 ([7], Corollary 34)**.** *Suppose A and B are disjoint $\Sigma_1^1(\kappa)$ sets. There are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals.*

Theorem 1.22 ([1], Theorem 17)**.** $\Delta_1^1(\kappa) \subseteq \kappa$*-Borel**

Lemma 1.23 ([3], Corollary 3.2)**.** *It is consistently that $\Delta_1^1(\kappa) \subsetneq \kappa$-Borel* $\subsetneq \Sigma_1^1(\kappa)$.*

Definition 1.24. *Fix a bijection $\pi: \kappa^{<\omega} \to \kappa$. For every $\eta \in \kappa^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \dots, a_n)$ in κ^n satisfies*

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, a_2, \dots, a_n)) \geq 0.$$

Theorem 1.25 ([1], Theorem 18)**.**

1. κ-Borel$\subsetneq \Delta_1^1(\kappa)$

2. $\Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$

Proof. 1. Let $\xi \mapsto (T_\xi, h_\xi)$ be a continuous coding of the κ-Borel*-codes with T a $\kappa^+\omega$-tree, such that for all $\kappa^+\omega$-tree, T, and h, there is ξ such that $T_\xi, h_\xi = (T, h)$.

Claim 1.26. *The set $B = \{(\eta, \xi) \mid \eta$ is in the set coded by $(T_\xi, h_\xi)\}$ is $\Sigma_1^1(\kappa)$ and is not κ-Borel, otherwise $D = \{\eta \mid (\eta, \eta) \notin B\}$ would be Borel (Hint: use the set $C = \{(\eta, \xi, \sigma) \mid \sigma$ is a winning strategy for $\mathbf{II}$ in $B^*(T_\xi, h_\xi, \eta)\}$).*

2.

Claim 1.27. *There is $A \subseteq 2^\kappa \times 2^\kappa$ such that if $B \subseteq 2^\kappa$ is a $\Sigma_1^1(\kappa)$ set, then there is $\eta \in 2^\kappa$ such that $B = \{\xi \mid (\xi, \eta) \in A\}$ (Hint: the construction used in the classical case works too).*

The set $D = \{\eta \mid (\eta, \eta) \in A\}$ is $\Sigma_1^1(\kappa)$ but not $\Pi_1^1(\kappa)$.

□

Question 1.28. *Is it consistent that $\Delta_1^1 = \kappa$-Borel*?*

Definition 1.29 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T$ as the relation*

$$\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

Theorem 1.30 ([1], Theorem 70)**.** *If T is a classifiable theory, then $\cong_T$ is Δ_1^1.*

Theorem 1.31 ([1], Theorem 87)**.** *Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable countable theory. Then $E^2_{\omega\text{-}club} \leq_c \cong^\kappa_T$.* □

Theorem 1.32 ([1], Theorem 79)**.** *Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.*

1. If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-}club} \leq_c \cong^\kappa_T$.

2. If $\lambda \geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-}club} \leq_c \cong^\kappa_T$. □

Corollary 1.33. *($V = L$) Suppose that κ is the successor of a regular uncountable cardinal. If T is a non-classifiable countable theory, then $\cong_T$ is a Σ_1^1-complete relation.*

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