A Borel-reducibility Counterpart of Shelah’s Main Gap Theorem

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Abstract

We study the Borel-reducibility of isomorphism relations of complete first order theories and show the consistency of the following: For all such theories $T$ and $T'$, if $T$ is classifiable and $T'$ is not, then the isomorphism of models of $T'$ is strictly above the isomorphism of models of $T$ with respect to Borel-reducibility. In fact, we can also ensure that a range of equivalence relations modulo various non-stationary ideals are strictly between those isomorphism relations. The isomorphism relations are considered on models of some fixed uncountable cardinality obeying certain restrictions.

1 Introduction

Throughout this article we assume that $\kappa$ is an uncountable cardinal that satisfies $\kappa^{< \kappa} = \kappa$. The generalized Baire space is the set $\kappa^\kappa$ with the bounded topology. For every $\zeta \in \kappa^{< \kappa}$, the set

$$[\zeta] = \{ \eta \in \kappa^\kappa \mid \zeta \subset \eta \}$$

is a basic open set. The open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets. The collection of $\kappa$-Borel subsets of $\kappa^\kappa$ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length $\kappa$. A $\kappa$-Borel set is any element of this collection. We usually omit the prefix “$\kappa$-”. In [Vau74] Vought studied this topology in the case $\kappa = \omega_1$ assuming CH and proved the following:

**Theorem.** A set $B \subset \omega_1^{\omega_1}$ is Borel and closed under permutations if and only if there is a sentence $\varphi$ in $L_{\omega_1^\omega}$ such that $B = \{ \eta \mid A_\eta \models \varphi \}$.

This result was generalized in [FKH14] to arbitrary $\kappa$ that satisfies $\kappa^{< \kappa} = \kappa$. Mekler and Väänänen continued the study of this topology in [MV93].

We will work with the subspace $2^\kappa$ with the relative subspace topology. A function $f : 2^\kappa \to 2^\kappa$ is Borel, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of $2^\kappa$. Let $E_1$ and $E_2$ be equivalence relations on $2^\kappa$. We say that $E_1$ is Borel reducible to $E_2$, if there is a Borel function $f : 2^\kappa \to 2^\kappa$ that satisfies $(x,y) \in E_1 \iff (f(x), f(y)) \in E_2$. We call $f$ a reduction of $E_1$ to $E_2$. This is denoted by $E_1 \leq_B E_2$ and if $f$ is continuous, then we say that $E_1$ is continuously reducible to $E_2$ and this is denoted by $E_1 \leq_c E_2$.

The following is a standard way to code structures with domain $\kappa$ with elements of $2^\kappa$. To define it, fix a countable relational vocabulary $L = \{ P_n \mid n < \omega \}$. 

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Definition 1.1. Fix a bijection $\pi : \kappa^\omega \to \kappa$. For every $\eta \in 2^\kappa$ define the $L$-structure $A_\eta$ with domain $\kappa$ as follows: For every relation $P_m$ with arity $n$, every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$ satisfies

$$(a_1, a_2, \ldots, a_n) \in P^A_m \iff \eta(m, a_1, a_2, \ldots, a_n) = 1.$$ 

Note that for every $L$-structure $A$ there exists $\eta \in 2^\kappa$ with $A = A_\eta$. For club many $\alpha < \kappa$ we can also code the $L$-structures with domain $\alpha$:

Definition 1.2. Denote by $C_\pi$ the club $\{\alpha < \kappa \mid \pi[\alpha^\omega] \subseteq \alpha\}$. For every $\eta \in 2^\kappa$ and every $\alpha \in C_\pi$ define the structure $A_\eta|\alpha$ with domain $\alpha$ as follows: For every relation $P_m$ with arity $n$, every tuple $(a_1, a_2, \ldots, a_n)$ in $\alpha^n$ satisfies

$$(a_1, a_2, \ldots, a_n) \in P^A_m|\alpha \iff \eta|\alpha(m, a_1, a_2, \ldots, a_n) = 1.$$ 

For every $\alpha \in C_\pi$ and every $X \subseteq \alpha$ we will denote the structure $A_F$ by $A_X$, where $F$ is the characteristic function of $X$. We will work with two equivalence relations on $2^\kappa$: the isomorphism relation and the equivalence modulo the non-stationary ideal.

Definition 1.3 (The isomorphism relation). Assume $T$ is a complete first order theory in a countable vocabulary. We define $\cong_T^\kappa$ as the relation

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T)\}.$$ 

We will omit the superscript “$\kappa$” in $\cong_T^\kappa$ when it is clear from the context. For every first order theory $T$ in a countable vocabulary there is an isomorphism relation associated with $T$, $\cong_T^\kappa$. For every stationary $X \subset \kappa$, we define an equivalence relation modulo the non-stationary ideal associated with $X$:

Definition 1.4. For every $X \subset \kappa$ stationary, we define $E_X$ as the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid \eta^{-1}[1] \triangle \xi^{-1}[1] \cap X \text{ is not stationary}\}$$

where $\triangle$ denotes the symmetric difference.

For every regular cardinal $\mu < \kappa$ denote $\{\alpha < \kappa \mid cf(\alpha) = \mu\}$ by $S_\mu^\kappa$. A set $C$ is a $\mu$-club if it is unbounded and closed under $\mu$-limits, i.e. if $S_\mu^\kappa \setminus C$ is non-stationary. Accordingly, we will denote the equivalence relation $E_X$ for $X = S_\mu^\kappa$ by $E^2_{\mu}$-club. Note that $(f, g) \in E^2_{\mu}$-club if and only if the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ contains a $\mu$-club.

2 Reduction to $E_X$

Classifiable theories (superstable with NOTOP and NDOP) have a close connection to the Ehrenfeucht-Fraïssé games (EF-games for short). We will use them to study the reducibility of the isomorphism relation of classifiable theories. The following definition is from [HM15, Def 2.3]:

Definition 2.1 (The Ehrenfeucht-Fraïssé game). Fix an enumeration $\{X_\gamma\}_{\gamma < \kappa}$ of the elements of $P_\kappa(\kappa)$ and an enumeration $\{f_\gamma\}_{\gamma < \kappa}$ of all the functions with both the domain and range in $P_\kappa(\kappa)$. For every $\alpha \leq \kappa$ the game $\text{EF}^\alpha_{\omega}(A|_\alpha, B|_\alpha)$ on the restrictions $A|_\alpha$ and $B|_\alpha$ of the structures $A$ and $B$ with domain $\kappa$ is defined as follows: In the $n$-th move, first I chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$ and $X_{\beta_{n-1}} \subseteq X_{\beta_n}$. Then II chooses an ordinal $\theta_n < \alpha$ such that $\text{dom}(f_{\theta_n}), \text{ran}(f_{\theta_n}) \subset \alpha$, $\ldots$
by symmetry). By the definition of Lemma 2.3.
Assume $T$ is a classifiable theory and every two models $A$ equivalent are isomorphic. On the other hand

Remark 2. In [HM15, Lemma 2.7] it was proved that there exists a club $C_{EF}$ of $\alpha$ such that the relation defined by the game
\[
\{(A, B) \mid \text{II } \uparrow \text{EF}^\alpha(A, B, \langle A \upharpoonright \alpha, B \upharpoonright \alpha \rangle)\}
\]
is an equivalence relation.

Remark 2. Shelah proved in [She90], that if $T$ is classifiable then every two models of $T$ that are $L_{\infty, \kappa}$-equivalent are isomorphic. On the other hand $L_{\infty, \kappa}$-equivalence is equivalent to $\text{EF}^\kappa$-equivalence. So for every two models $A$ and $B$ of $T$ we have $\text{II } \uparrow \text{EF}^\alpha(A, B) \iff A \cong B$ and $\text{I } \uparrow \text{EF}^\alpha(A, B) \iff A \not\cong B$.

Lemma 2.3. Assume $T$ is a classifiable theory and $\mu < \kappa$ is a regular cardinal. If $\diamondsuit_\kappa(X)$ holds then $\cong^T_\kappa$ is continuously reducible to $E_X$.

Proof. Let $\{S_\alpha \mid \alpha < \kappa\}$ be a sequence testifying $\diamondsuit_\kappa(X)$ and define the function $F : 2^\kappa \to 2^\kappa$ by
\[
F(\eta)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in X \cap C_\mu \cap C_{EF}, \text{ II } \uparrow \text{EF}^\alpha(A_\eta \upharpoonright \alpha, A_{S_\alpha}) \text{ and } A_\eta \upharpoonright \alpha \models T \\ 0 & \text{otherwise.} \end{cases}
\]

Let us show that $F$ is a reduction of $\cong^T_\kappa$ to $E_X$, i.e. for every $\eta, \xi \in 2^\kappa$, $(\eta, \xi) \in \cong^T_\kappa$ if and only if $(F(\eta), F(\xi)) \in E_X$. Notice that when $\alpha \in C_\mu$, the structure $A_\eta \upharpoonright \alpha$ is defined and equals $A_\eta \upharpoonright \alpha$.

Consider first the direction from left to right. Suppose first that $A_\eta$ and $A_\xi$ are models of $T$ and $A_\eta \cong A_\xi$. Since $A_\eta \cong A_\xi$, we have $\text{II } \uparrow \text{EF}^\alpha(A_\eta, A_\xi)$. By Lemma 2.2 there is a club $C$ such that $\text{II } \uparrow \text{EF}^\alpha(A_\eta \upharpoonright \alpha, A_{S_\alpha})$ for every $\alpha$ in $C$. Since the set $\{\alpha < \kappa \mid A_\eta \upharpoonright \alpha \models T, A_{S_\alpha} \upharpoonright \alpha \models T\}$ contains a club, we can assume that every $\alpha \in C$ satisfies $A_\eta \upharpoonright \alpha \models T$ and $A_{S_\alpha} \upharpoonright \alpha \models T$. If $\alpha \in C$ is such that $F(\eta)(\alpha) = 1$, then $\text{II } \uparrow \text{EF}^\alpha(A_\eta \upharpoonright \alpha, A_{S_\alpha})$. Since $\text{II } \uparrow \text{EF}^\alpha(A_\eta \upharpoonright \alpha, A_{S_\alpha})$ and $\alpha \in C_{EF}$, we can conclude that $\text{II } \uparrow \text{EF}^\alpha(A_\xi \upharpoonright \alpha, A_{S_\alpha})$. Therefore for every $\alpha \in C$, $F(\eta)(\alpha) = 1$ implies $F(\xi)(\alpha) = 1$. Using the same argument it can be shown that for every $\alpha \in C$, $F(\xi)(\alpha) = 1$ implies $F(\eta)(\alpha) = 1$. Therefore $F(\eta)$ and $F(\xi)$ coincide in a club and $(F(\eta), F(\xi)) \in E_X$.

Let us now look at the case where $(\eta, \xi) \in \cong^T_\kappa$ and $A_\eta$ is not a model of $T$ (the case $T \models A_\xi$ follows by symmetry). By the definition of $\cong^T_\kappa$ we know that $A_\xi$ is not a model of $T$ either, so there is $\varphi \in T$ such that $A_\eta \models \neg \varphi$ and $A_\xi \models \neg \varphi$. Further, there is a club $C$ such that for every $\alpha \in C$ we have $A_\eta \upharpoonright \alpha \models \neg \varphi$ and $A_\xi \upharpoonright \alpha \models \neg \varphi$. We conclude that for every $\alpha \in C$ we have that $A_\eta \upharpoonright \alpha$ and $A_\xi \upharpoonright \alpha$ are not models of $T$, and $F(\eta)(\alpha) = F(\xi)(\alpha) = 0$, so $(F(\eta), F(\xi)) \in E_X$.

Let us now look at the direction from right to left. Suppose first that $A_\eta$ and $A_\xi$ are models of $T$, and $A_\eta \not\cong A_\xi$. 3
By Remark 2, we know that $I \uparrow \mathcal{EP}_\omega^\kappa(A_\eta, A_\xi)$. By Lemma 2.2 there is a club $C$ of $\alpha$ with
\[ I \uparrow \mathcal{EP}_\omega^\kappa(A_\eta |_\alpha, A_\xi |_\alpha), \]
$A_\xi |_\alpha \models T$ and $A_\eta |_\alpha \models T$.

Since $\{\alpha \in X \mid \eta \cap \alpha = S_\alpha\}$ is stationary by the definition of $\Diamond_\kappa(X)$, also the set
\[ \{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{\mathcal{EF}} \]
is stationary and every $\alpha$ in this set satisfies $II \uparrow \mathcal{EP}_\omega^\kappa(A_\eta |_\alpha, A_{S_\alpha})$. Therefore
\[ C \cap \{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{\mathcal{EF}} \]
is stationary and a subset of $\mathcal{F}(\eta)^{-1}\{1\} \triangle \mathcal{F}(\xi)^{-1}\{1\}$, where $\triangle$ denotes the symmetric difference. We conclude that $\mathcal{F}(\eta), \mathcal{F}(\xi) \notin E_X$.

Let us finally assume that $(\eta, \xi) \notin \equiv_T$ and $A_\eta \not\models T$ (the case $A_\xi \not\models T$ follows by symmetry). Assume towards a contradiction that $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^2_{\mu-club}$. Let $C$ be a club that testifies $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^2_{\mu-club}$, i.e. $C \cap (\mathcal{F}(\eta)^{-1}\{1\} \triangle \mathcal{F}(\xi)^{-1}\{1\}) \cap X = \emptyset$. Since $A_\eta \not\models T$, the set $\{\alpha < \kappa \mid A_\eta |_\alpha \not\models T\}$ contains a club. Hence, we can assume that for every $\alpha \in C$, $A_\eta |_\alpha \not\models T$ which implies that $\mathcal{F}(\eta)(\alpha) = 0$ and $\mathcal{F}(\xi)(\alpha) = 0$ for every $\alpha \in C$.

By the definition of $\equiv_T$, $A_\eta \models \not T$ implies $A_\xi \models T$. Therefore the set $\{\alpha < \kappa \mid A_\xi |_\alpha \models T\}$ contains a club. So there is a club $C'$ such that every $\alpha \in C'$ satisfies $A_\xi |_\alpha \models T$ and $\mathcal{F}(\xi)(\alpha) = 0$. Since $\{\alpha \in X \mid \xi \cap \alpha = S_\alpha\}$ is stationary, again by the definition of $\Diamond_\kappa(X)$, also $\{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{\mathcal{EF}}$ is stationary and every $\alpha$ in this set satisfies $II \uparrow \mathcal{EP}_\omega^\kappa(A_\eta |_\alpha, A_{S_\alpha})$. Therefore,
\[ C' \cap \{\alpha \in X \mid \xi \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{\mathcal{EF}} \neq \emptyset, \]
a contradiction.

To show that $\mathcal{F}$ is continuous, let $[\eta |_\alpha]$ be a basic open set, $\xi \in \mathcal{F}^{-1}[\eta |_\alpha]$. Then $\xi \in [\xi |_\alpha]$ and $[\xi |_\alpha] \subseteq \mathcal{F}^{-1}[\eta |_\alpha]$. We conclude that $\mathcal{F}$ is continuous.

To define the reduction $\mathcal{F}$ it is not enough to use the isomorphism classes of the models $A_{S_\alpha}$, as opposed to the equivalence classes of the relation defined by the EF-game. It is possible to construct two non-isomorphic models with domain $\kappa$ such that their restrictions to any $\alpha < \kappa$ are isomorphic. For example the models $M = (\kappa, P)$ and $N = (\kappa, Q)$, with $\kappa = \lambda^+$,
\[ P = \{\alpha < \kappa \mid \alpha = \beta + 2n, n \in \mathbb{N} \text{ and } \beta \text{ a limit ordinal}\} \]
and
\[ Q = \{\alpha < \lambda \mid \alpha = \beta + 2n, n \in \mathbb{N} \text{ and } \beta \text{ a limit ordinal}\} \]
are non-isomorphic but $M |_\alpha \models N |_\alpha$ holds for every $\alpha < \kappa$.

The Borel reducibility of the isomorphism relation of classifiable theories was studied in [FHK14] and one of the main results is the following.

**Theorem 2.4.** ([FHK14, Thm 77]) If a first order theory $T$ is classifiable, then for all regular cardinals $\mu < \kappa$, $E^2_{\mu-club} \not\equiv^B \mathcal{F}$.

**Corollary 2.5.** Assume that $\Diamond_\kappa(S^\mu_\mu)$ holds for all regular $\mu < \kappa$. If a first order theory $T$ is classifiable, then for all regular cardinals $\mu < \kappa$ we have $\equiv^C_T \leq^C E^2_{\mu-club}$ and $E^2_{\mu-club} \not\equiv^B \mathcal{F}$.
3 Non-classifiable Theories

In [FHK14] the reducibility to the isomorphism of non-classifiable theories was studied. In particular the following two theorems were proved there:

**Theorem 3.1.** ([FHK14, Thm 79]) Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $T$ is a stable unsuperstable theory. Then $E^2_{\omega\text{-club}} \leq_c \cong_T$.

1. If $T$ is unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}} \leq_c \cong_T$.
2. If $\lambda \geq 2^\omega$ and $T$ is superstable with DOP, then $E^2_{\lambda\text{-club}} \leq_c \cong_T$.

**Theorem 3.2.** ([FHK14, Thm 86]) Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $T$ is a stable unsuperstable theory. Then $E^2_{\omega\text{-club}} \leq_c \cong_T$.

Clearly from Theorems 3.1 and 3.2 and Corollary 2.3 we obtain the following:

**Theorem 3.3.** Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$ and $\Diamond_k(S^\kappa_k)$ holds.

1. If $T_1$ is classifiable and $T_2$ is unstable or superstable with OTOP, then $\cong_{T_1} \leq_c \cong_{T_2}$ and $\cong_{T_2} \leq B \cong_{T_1}$.
2. If $\lambda > 2^\omega$, $T_1$ is classifiable and $T_2$ is superstable with DOP, then $\cong_{T_1} \leq_c \cong_{T_2}$ and $\cong_{T_2} \leq B \cong_{T_1}$.

**Theorem 3.4.** Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $\Diamond_k(S^\kappa_k)$ holds. If $T_1$ is classifiable and $T_2$ is stable unsuperstable, then $\cong_{T_1} \leq_c \cong_{T_2}$ and $\cong_{T_2} \leq B \cong_{T_1}$.

**Corollary 3.5.** Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$ and $\lambda^\omega = \lambda$. If $T_1$ is classifiable and $T_2$ is stable unsuperstable, then $\cong_{T_1} \leq_c \cong_{T_2}$ and $\cong_{T_2} \leq B \cong_{T_1}$.

**Proof.** In [She10] Shelah proved that if $\kappa = \lambda^+ = 2^\lambda$ and $S$ is a stationary subset of $\{ \alpha < \kappa \mid cf(\alpha) \neq cf(\lambda) \}$, then $\Diamond_k(S)$ holds. Since $\lambda^\omega = \lambda$, we have $cf(\lambda) \neq \omega$ and $\Diamond_k(S^\kappa_k)$ holds. On the other hand $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$ implies $\gamma^\omega < \kappa$ for all $\gamma < \kappa$. By Theorem 3.4 we conclude that if $T_1$ is a classifiable theory and $T_2$ is a stable unsuperstable theory, then $\cong_{T_1} \leq_c \cong_{T_2}$ and $\cong_{T_2} \leq B \cong_{T_1}$.

**Theorem 3.6.** Let $H(\kappa)$ be the following property: If $T$ is classifiable and $T'$ not, then $\cong_{T'} \leq_c \cong_{T'}$ and $\cong_{T'} \leq B \cong_{T'}$. Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+ = 2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. If $V = L$, then $H(\kappa)$ holds.
2. There is a $\kappa$-closed forcing notion $\mathbb{P}$ with the $\kappa^+$-c.c. which forces $H(\kappa)$.

**Proof.** 1. This follows from Theorems 3.3 and 3.4.

2. Let $\mathbb{P} = \{ f : X \to 2 \mid X \subseteq \kappa, |X| < \kappa \}$ with the order $p \leq q$ if $q \subseteq p$. It is known that $\mathbb{P}$ has the $\kappa^+$-c.c. [Kun11, Lemma IV.7.5] and is $\kappa$-closed [Kun11, Lemma IV.7.14]. It is also known that $\mathbb{P}$ preserves cofinalities, cardinalities and subsets of $\kappa$ of size less than $\kappa$ [Kun11, Thm IV.7.9, Lemma IV.7.15]. Therefore, in $V[G]$, $\kappa$ satisfies $\kappa = \kappa^{<\kappa} = \lambda^+ > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. It is known that $\mathbb{P}$ satisfies $\mathbb{P} \models \Diamond_k(S^\kappa_k)$ for every regular cardinal $\mu < \kappa$. Therefore, by Theorems 3.3 and 3.4 $H(\kappa)$ holds in $V[G]$.

**Definition 3.7.** A tree $T$ is a $\kappa^+$, $\kappa$-tree if it does not contain chains of length $\kappa$ and its cardinality is less than $\kappa^+$. It is closed if every chain has a unique supremum.
2. A pair \((T, h)\) is a Borel\(^*\)-code if \(T\) is a closed \(\kappa^+, \kappa\)-tree and \(h\) is a function with domain \(T\) such that if \(x \in T\) is a leaf, then \(h(x)\) is a basic open set and otherwise \(h(x) \in \{\cup, \cap\}\).

3. For an element \(\eta \in 2^\kappa\) and a Borel\(^*\)-code \((T, h)\), the Borel\(^*\)-game \(B^*(T, h, \eta)\) is played as follows. There are two players, I and II. The game starts from the root of \(T\). At each move, if the game is at node \(x \in T\) and \(h(x) = \cap\), then I chooses an immediate successor \(y\) of \(x\) and the game continues from this \(y\). If \(h(x) = \cup\), then II makes the choice. At limits the game continues from the (unique) supremum of the previous moves by Player I. Finally, if \(h(x)\) is a basic open set, then the game ends, and II wins if and only if \(\eta \in h(x)\).

4. A set \(X \subseteq 2^\kappa\) is a Borel\(^*\)-set if there is a Borel\(^*\)-code \((T, h)\) such that for all \(\eta \in 2^\kappa\), \(\eta \in X\) if and only if II has a winning strategy in the game \(B^*(T, h, \eta)\).

Note that a strategy in a game \(B^*(T, h, \eta)\) can be seen as a function \(\sigma : \kappa^\kappa \rightarrow \kappa\), because every \(\kappa^+ \kappa\)-tree can be seen as a downward closed subtree of \(\kappa^\kappa\).

**Theorem 3.8.** Suppose that \(\kappa = \kappa^\kappa = \lambda^+, 2^\lambda > 2^\omega\) and \(\lambda^\lambda = \lambda\). Then the following statements are consistent.

1. If \(T_1\) is classifiable and \(T_2\) is not, then there is an embedding of \((\mathcal{P}(\kappa), \subseteq)\) to \((B^*(T_1, T_2), \leq_B)\), where \(B^*(T_1, T_2)\) is the set of all Borel\(^*\)-equivalence relations strictly between \(\cong_{T_1}\) and \(\cong_{T_2}\).

2. If \(T_1\) is classifiable and \(T_2\) is unstable or superstable, then

\[
\cong_{T_1} \leq_c E_{\lambda\text{-club}} \leq c \cong_{T_2} \leq E_{\lambda\text{-club}} \wedge E_{\lambda\text{-club}} \not\subseteq B \cong_{T_1} E_{\lambda\text{-club}} \not\subseteq B \cong_{T_1}.
\]

**Proof.** We will start the proof with two claims.

**Claim 3.9.** If \(\Diamond_\kappa(S)\) holds in \(V\) and \(Q\) is \(\kappa\)-closed, then \(\Diamond_\kappa(S)\) holds in every \(Q\)-generic extension.

**Proof.** Let us proceed by contradiction. Suppose \((S_\alpha)_{\alpha \in S}\) is a \(\Diamond_\kappa(S)\)-sequence in \(V\) but not in \(V[G]\), for some generic \(G\). Fix the names \(\hat{\mathcal{C}}, \hat{X} \in V^Q\) and \(p \in G\), such that:

\[
p \models (\hat{\mathcal{C}} \subseteq \kappa) \wedge (\hat{X} \subseteq \kappa) \wedge \forall \alpha \in \hat{\mathcal{C}}[\hat{S}_\alpha \neq \hat{X} \cap \alpha]).
\]

Working in \(V\), we choose by recursion \(p_\alpha, \beta_\alpha, \theta_\alpha\) and \(\delta_\alpha\) such that:

1. \(p_\alpha \in Q\), \(p_0 = p\) and \(p_\alpha \geq p_\gamma\) if \(\alpha \leq \gamma\).
2. \(\beta_\alpha \leq \beta_\gamma\) if \(\alpha \leq \gamma\).
3. \(\beta_\alpha \leq \theta_\alpha, \delta_\alpha < \beta_{\alpha+1}\).
4. If \(\gamma\) is a limit ordinal, then \(\beta_\gamma = \delta_\gamma = \bigcup_{\alpha < \gamma} \beta_\alpha\).
5. \(p_{\alpha+1} \models (\delta_\alpha \in \hat{\mathcal{C}} \wedge \hat{X} \cap \hat{\beta}_\alpha = \hat{\delta}_\alpha)\).

We will show how to choose them such that 1-5 are satisfied. First, for the successor step assume that for some \(\alpha < \kappa\) we have chosen \(p_{\alpha+1}, \beta_\alpha, \theta_\alpha\) and \(\delta_\alpha\). We choose any ordinal satisfying 3 as \(\beta_{\alpha+1}\). Since \(p_{\alpha+1} \models (\hat{\mathcal{C}} \subseteq \kappa)\), there exists \(q \in Q\) stronger than \(p_{\alpha+1}\) and \(\delta < \kappa\) such that \(q \models (\delta \in \hat{\mathcal{C}} \wedge \hat{\beta}_\alpha \leq \hat{\delta})\). Now set \(\delta_{\alpha+1} = \delta\). Since \(Q\) is \(\kappa\)-closed, there exists \(Y \in \mathcal{P}(\hat{\beta}_{\alpha+1})\) and \(r \in Q\) stronger than \(q\) such that \(r \models \hat{X} \cap \hat{\beta}_{\alpha+1} = \hat{Y}\). By \(\Diamond_\kappa(S)\) in \(V\), the set \(\{\gamma < \kappa \mid Y = S_\gamma\}\) is stationary, so we can choose the least ordinal \(\theta_{\alpha+1} \geq \beta_{\alpha+1}\) such that \(r \models \hat{X} \cap \hat{\beta}_{\alpha+1} = \hat{S}_{\theta_{\alpha+1}}\). Clearly \(r = p_{\alpha+2}\) satisfies 1 and 5.
For the limit step, assume that for some limit ordinal \( \alpha < \kappa \) we have chosen \( p_\gamma, \beta_\gamma, \theta_\gamma \) and \( \delta_\gamma \) for every \( \gamma < \alpha \). Note that by 4 we know how to choose \( \beta_\alpha \) and \( \delta_\alpha \). Since \( Q \) is \( \kappa \)-closed, there exists \( p_\alpha \) that satisfies 1. We choose \( \theta_\alpha \) as in the successor case with \( q = p_\alpha \) and \( p_{\alpha+1} \) as the condition \( r \) used to choose \( \theta_\alpha \).

Define \( A, B \) and \( C_\delta \) by \( B = \bigcup_{\alpha < \kappa} S_{\theta_\alpha} \) and \( C_\delta = \{ \alpha \in S \mid B \cap \alpha = S_{\theta_\alpha} \} \). Let \( \delta_\alpha \subseteq A \bigcap C_\delta \). Then by 1, 2 and 5, for every \( \gamma > \alpha \) we have \( p_{\gamma+1} \models (\delta_{\theta_\alpha} = \delta_{\theta_\alpha} \cap \beta_{\alpha}) \). Therefore, \( S_{\theta_\alpha} = B \bigcap \beta_{\alpha} \) and \( \delta_{\alpha} \subseteq A \bigcap C_\delta \) and so by 4 we have \( S_{\theta_\alpha} = B \bigcap \delta_{\alpha} = S_{\theta_\alpha} \). But now by 5 we get \( p_{\alpha+1} \models (\delta_{\theta_\alpha} \in \bigcap X \bigcap \delta_{\alpha} = S_{\theta_\alpha}) \) which is a contradiction.

**Claim 3.10.** For all stationary \( X \subseteq \kappa \), the relation \( E_X \) is a Borel*-set.

*Proof.* The idea is to code the club-game into the Borel*-game: in the club-game the players pick ordinals one after another and if the limit is in a predefined set \( A \), then the second player wins. Define \( T_X \) as the tree whose elements are all the increasing elements of \( \kappa^{<\lambda} \), ordered by end-extension. For every element of \( T_X \) that is not a leaf, define

\[
H_X(x) = \bigcup_{\gamma < \alpha} x^{-} \cap H_X(x^{-}) = \bigcap_{\alpha \notin \gamma} x^{-}
\]

and for every leaf \( b \) define \( H_X(b) \) by:

\[
(\eta, \xi) \in H_X(b) \quad \iff \quad \text{for every } \alpha \in \text{ran}(\gamma) \cap X(\eta(a) = \xi(a))
\]

where \( \alpha \in \text{lim}(\text{ran}(b)) \) if \( \sup(\text{ran}(b)) = \alpha \).

Let us assume there is a winning strategy \( \sigma \) for Player II in the game \( B^+(T_X, H_X, (\eta, \xi)) \) and let us conclude that \( (\eta, \xi) \in E_X \). Clearly by the definition of \( H_X \) we know that \( \eta \) and \( \xi \) coincide in the set \( B = \{ \alpha < \kappa \mid | \sigma[\text{dom}(\sigma) \cap \alpha^{<\lambda}] \subseteq \lambda^{<\lambda} \cap X \}. \) Since \( \lambda^{<\lambda} = \lambda \), we know that \( B' = \{ \alpha < \kappa \mid \sigma[\text{dom}(\sigma) \cap \alpha^{<\lambda}] \subseteq \lambda^{<\lambda} \} \) is closed and unbounded. Therefore, there exists a club that doesn’t intersect \( (\eta^{-1}[1] \Delta \xi^{-1}[1]) \cap X \).

For the other direction, assume that \( (\eta^{-1}[1] \Delta \xi^{-1}[1]) \cap X \) is not stationary and denote by \( C \) the club that does not intersect \( (\eta^{-1}[1] \Delta \xi^{-1}[1]) \cap X \). The second player has a winning strategy for the game \( B^+(T_X, H_X, (\eta, \xi)) \): she makes sure that, if \( b \) is the leaf in which the game ends and \( A \subseteq \text{ran}(b) \) is such that \( \sup(\text{ran}(b)) \in X \), then \( \sup(\text{ran}(b)) \in C \). This can be done by always choosing elements \( f \in \kappa^{<\lambda} \) such that \( \sup(\text{ran}(f)) \in C \).

Let \( P \) be \( \{ f : X \rightarrow 2 \mid X \subseteq \kappa, |X| < \kappa \} \) with the order \( p \leq q \) if \( q \subseteq p \). It is known that in any \( P \)-generic extension, \( V[G] \), \( \Diamond_x \) holds for every \( S \in V \), \( S \) a stationary subset of \( \kappa \).

1. In [FHK14, Thm 52] the following was proved under the assumption \( \kappa = \lambda^+ \) and GCH:

   For every \( \mu < \kappa \) there is a \( \kappa \)-closed forcing notion \( Q \) with the \( \kappa^+ \)-c.c. which forces that there are stationary sets \( K(A) \subseteq S_\mu^x \) for each \( A \subseteq \kappa \) such that \( E_{K(A)} \notin B \) if and only if \( A \notin B \).

In [FHK14, Thm 52] the proof starts by taking \( (S_\alpha)_\alpha^{<\kappa} \), \( x \) pairwise disjoint stationary subsets of \( \text{lim}(S_\mu^x) = \{ \alpha \in S^x_\mu \mid \alpha \text{ is a limit ordinal in } S^x_\mu \} \), and defining \( K(A) = \bigcup_{\alpha \in A} S_\alpha \). \( Q \) is an iterated forcing that satisfies: For every name \( \sigma \) of a function \( f : 2^x \rightarrow 2^x \), exists \( \beta < \kappa \) such that, \( P_\beta \models \sigma \) is not a reduction.

With a small modification on the iteration, it is possible to construct \( Q \) a \( \kappa \)-closed forcing with the \( \kappa^+ \)-c.c. that forces
(*) For \( \mu \in \{ \omega, \lambda \} \) and \( A \subsetneq \kappa \), there are stationary sets \( K(\mu, A) \subsetneq S^s_\mu \) for which \( E_{K(\mu, A)} \not\subseteq B \) \( E_{K(\mu, B)} \) if and only if \( A \not\subseteq B \).

Assume without loss of generality that GCH holds in \( V \). Let \( G \) be a \( \mathbb{P}_* \mathbb{Q} \)-generic. It is enough to prove that for every \( A \subsetneq \kappa \) in \( V[G] \) the following holds:

(a) If \( T_2 \) is unstable, or superstable with OTOP or with DOP, then \( E_{K(\lambda, A)} \in B^*(T_1, T_2) \).

(b) If \( T_2 \) is stable unsuperstable, then \( E_{K(\omega, A)} \in B^*(T_1, T_2) \).

In both cases the proof is the same; we will only consider (a).

Working in \( V[G] \), let \( T_2 \) be as in (a). Since \( Q \) is \( \kappa \)-closed, we have \( V[G] \models \diamondsuit_\kappa(S) \) for every stationary \( S \subset \kappa, S \in V \). Since \( \mathbb{P} \) and \( Q \) are \( \kappa \)-closed and have the \( \kappa^+ \)-c.c., we have \( \kappa = \kappa^{< \kappa} = \lambda^+, 2^\lambda > 2^\omega \) and \( \lambda^{< \lambda} = \lambda \). By Lemma 2.3, Theorems 3.1 and 3.4, we have that \( \equiv_{T_1}^\kappa \equiv_{T_2}^\kappa \equiv_{K(\lambda, A)}^\kappa \) holds for every \( A \subsetneq \kappa \). The argument in the proof of Theorem 2.4 can be used to prove that \( E_{K(\lambda, A)} \not\subseteq B \) \( \equiv_{T_1}^\kappa \) holds for every \( A \subsetneq \kappa \).

To show that \( \equiv_{T_2}^\kappa \not\subseteq B \) \( E_{K(\lambda, A)} \) holds for every \( A \subsetneq \kappa \), assume towards a contradiction that there exists \( B \subsetneq \kappa \) such that \( \equiv_{T_2}^\kappa \subseteq E_{K(\lambda, B)} \). But then \( E_{K(\lambda, A)} \subseteq B \) \( E_{K(\lambda, B)} \) holds for every \( A \subsetneq \kappa \) and by (a), \( A \subseteq B \) for every \( A \subsetneq \kappa \). So \( B = \kappa \) which is a contradiction.

2. In [HK12, Thm 3.1] it is proved (under the assumptions \( 2^\kappa = \kappa^+ \) and \( \kappa = \kappa^{< \kappa} > \omega \)) that there is a generic extension in which \( \equiv_{DLO}^\kappa \) is not a Borel\( ^* \)-set. The forcing is constructed using the following claim [HK12, Claim 3.1.5]:

For each \( (t, h) \) there exists a \( \kappa^+ \)-c.c. \( \kappa \)-closed forcing \( \mathbb{R}(t, h) \) such that in any \( \mathbb{R}(t, h) \)-generic extension \( \equiv_{DLO}^\kappa \) is not a Borel\( ^* \)-set.

The forcing in [HK12, Thm 3.1] works for every theory \( T \) that is unstable, or \( T \) non-classifiable and superstable (not only DLO, see [HK12] and [HT91]). Therefore, this claim can be generalized to:

For each \( (t, h) \) there exists a \( \kappa^+ \)-c.c. \( \kappa \)-closed forcing \( \mathbb{R}(t, h) \) such that in any \( \mathbb{R}(t, h) \)-generic extension, \( \equiv_T^\kappa \) is not a Borel\( ^* \)-set, for all \( T \) unstable, or \( T \) non-classifiable and superstable.

By iterating this forcing (as in [HK12, Thm 3.1]), we construct a forcing \( Q \) \( \kappa \)-closed, \( \kappa^+ \)-c.c. that forces \( \equiv_T^\kappa \) is not a Borel\( ^* \)-set, for all \( T \) unstable, or \( T \) non-classifiable and superstable.

Assume without loss of generality that \( 2^\kappa = \kappa^+ \) holds in \( V \). Let \( G \) be a \( \mathbb{P}_* \mathbb{Q} \)-generic. Since \( Q \) is \( \kappa \)-closed, \( V[G] \models \diamondsuit_\kappa(S) \) for every stationary \( S \subset \kappa, S \in V \). Since \( \mathbb{P} \) and \( Q \) are \( \kappa \)-closed and have the \( \kappa^+ \)-c.c., we have \( \kappa = \kappa^{< \kappa} = \lambda^+, 2^\lambda > 2^\omega \) and \( \lambda^{< \lambda} = \lambda \). Working in \( V[G] \), let \( T_2 \) be unstable, or non-classifiable and superstable. By Lemma 2.3, Theorems 3.3 and 3.4 we finally have that \( \equiv_{T_1}^\kappa \leq_E E_{\lambda^+\text{-club}}^2 \leq E_{\lambda^+\text{-club}}^2 \not\subseteq B \) \( \equiv_{T_2}^\kappa \) holds.

Since \( 2^\kappa \times 2^\kappa \) is homeomorphic to \( 2^\kappa \), in order to finish the proof, it is enough to show that if \( f : 2^\kappa \to 2^\kappa \) is Borel, then for all Borel\( ^* \)-sets \( A \), the set \( f^{-1}[A] \) is Borel\( ^* \). This is because if \( f \) were the reduction \( \equiv_{T_2}^\kappa \subseteq E_{\lambda^+\text{-club}}^2 \) we would have \( (f \times f)^{-1}[E_{\lambda^+\text{-club}}^2] = \equiv_{T_2}^\kappa \) and since \( E_{\lambda^+\text{-club}}^2 \) is Borel\( ^* \), this would yield the latter Borel\( ^* \) as well.

Claim 3.11. Assume \( f : 2^\kappa \to 2^\kappa \) is a Borel function and \( B \subset 2^\kappa \) is Borel\( ^* \). Then \( f^{-1}[B] \) is Borel\( ^* \).

Proof. Let \( (T_B, H_B) \) be a Borel\( ^* \)-code for \( B \). Define the Borel\( ^* \)-code \( (T_A, H_A) \) by letting \( T_B = T_A \) and \( H_A(b) = f^{-1}[H_B(b)] \) for every branch \( b \) of \( T_B \). Let \( A \) be the Borel\( ^* \)-set coded by \( (T_A, H_A) \). Clearly, \( \Pi \uparrow B^*(T_B, H_B, \eta) \) if and only if \( \Pi \uparrow B^*(T_A, H_A, f^{-1}(\eta)) \), so \( f^{-1}[B] = A \).
We end this paper with an open question:

**Question 3.12.** Is it provable in ZFC that $\sim_T \leq_B \sim_{T'}$ (note the strict inequality) for all complete first-order theories $T$ and $T'$, $T$ classifiable and $T'$ not? How much can the cardinality assumptions on $\kappa$ be relaxed?

**References**


