ON Σ_1^1-COMPLETENESS OF QUASI-ORDERS ON κ^κ

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Abstract. We prove under $V = L$ that the inclusion modulo the non-stationary ideal is a Σ_1^1-complete quasi-order in the generalised Borel-reducibility hierarchy ($\kappa > \omega$). This improvement to known results in L has many new consequences concerning the Σ_1^1-completeness of quasi-orders and equivalence relations such as the embeddability of dense linear orders and equivalence modulo various versions of the non-stationary ideal. This serves as a partial or complete answer to several open problems stated in literature. Additionally the theorem is applied to prove a dichotomy in L: If the isomorphism of a countable first-order theory (not necessarily complete) is not Δ_1^1, then it is Σ_1^1-complete.

We also study the case $V \neq L$ and prove Σ_1^1-completeness results for weakly ineffable and weakly compact κ

§1. Introduction. We work in the setting of generalised descriptive set theory [4], GDST for short. The spaces $\kappa^\kappa = \{f : \kappa \to \kappa\}$ and $2^\kappa = \{f : \kappa \to 2\}$ are equipped with the bounded topology where the basic open sets are of the form $\{\eta \in \kappa^\kappa \mid \eta \supset p\}$, $p \in \kappa^{<\kappa}$. Borel sets are generated by κ-long unions and intersection of basic open sets. Notions of Borel-reducibility between equivalence relations and quasi-orders as well as Wadge-reducibility between sets are generalised accordingly.

In [4] a Lemma was introduced (a version of the Lemma and a detailed proof can be found in [9, Lemma 1.9 & Remark 1.10]) saying that if $V = L$, then any Σ_1^1 subset of κ^κ can be Wadge-reduced to

$$\mathrm{CLUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a } \mu\text{-club}\}, \quad \mu < \kappa \text{ regular},$$

where "μ-club" is short for unbounded set closed under increasing sequences of length μ. In [4] this was used to show that if $V = L$, then $\Sigma_1^1 = Borel^*$. In [9] the Wadge-reducibility result was strengthened by the first two authors of the present paper. It was shown (still in L) that every Σ_1^1-equivalence relation is Borel-reducible to the following equivalence relation on κ^κ:

$$(1) \qquad E_\mu^\kappa = \{(\eta, \xi) \in (\kappa^\kappa)^2 \mid \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \text{ contains a } \mu\text{-club}\}.$$

We say that E_μ^κ is Σ_1^1-*complete.*

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However, we would have wanted to show that the same equivalence relation on 2^κ:

$$E^2_\mu = \{(\eta,\xi) \in (2^\kappa)^2 \mid \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \text{ contains a } \mu\text{-club}\}. \tag{2}$$

has the same completeness property. The reason for this was that we knew many more equivalence relations to which E^2_μ can be Borel reduced than equivalence relations to which E^κ_μ can be Borel reduced. The corollaries of (1) and (2) were explored in [4, 9, 16]. In particular, the question "Is $E^\kappa_\mu \leqslant_B E^2_\mu$?" that was stated in [5, Q. 15] and re-stated in [15, Q. 3.46] was open (and it is still open in the general case). Of course if E^2_μ is Σ^1_1-complete the answer to this question is positive and in this paper we show that this is the case in L (Theorem 4.2) by first proving the same result for quasi-orders (Theorem 3.1). Borel-reducibility between quasi-orders is a natural generalisation of reducibility between equivalence relations (see Section 2 for precise definitions).

We then prove a range of new results which are all consequences of Theorem 3.1. One of these is our main result: If $V = L$ then the isomorphism relation of *any* countable first-order theory (not necessarily complete) is either Δ^1_1 or Σ^1_1-complete. This classification problem in Baire space was also studied in [11], the "Borel-reducibility counterpart of Shelah's main gap theorem". The other results are partial answers to [17, Q.'s 11.3 and 11.4] (which are re-stated as [15, Q's 3.49 and 3.50]), [5, Q. 15] and a complete answer to [15, Q. 3.47].

These questions ask about the (consistency of) reducibility between relations of the form E^λ_μ, quasi-orders of the form $\sqsubseteq_\mu$, quasi-orders of embeddability between linear orders as well as various isomorphism relations, where $\lambda \in \{2,\kappa\}$ and $\mu \in \operatorname{reg}(\kappa)$. In particular, [17, Q. 11.4] asks whether the embeddability of dense linear orders $\sqsubseteq_{\mathrm{DLO}}$ is a Σ^1_1-complete quasi-order for weakly compact κ. From those results that are described above it follows that $\sqsubseteq_{\mathrm{DLO}}$ is Σ^1_1-complete in L for all κ that are not successors of an ω-cofinal cardinal. In Section 5.1 we extend this to weakly ineffable cardinals (without the assumption $V = L$). Thus the only case in which [17, Q. 11.4] is still open is the case when $V \neq L$ and κ is a weakly compact cardinal which is not weakly ineffable. In Section 5.2 we prove that the isomorphism of DLO, $\cong_{\mathrm{DLO}}$, on κ weakly compact is Σ^1_1-complete (here again, we do not assume $V = L$). The existence of Σ^1_1-complete equivalence relations has been previously known to hold in L [10]. It is still unknown whether there exists a model of ZFC and $\kappa > \omega$ on which no isomorphism relation is Σ^1_1-complete. Given the present situation such a counterexample will have to satisfy both $V \neq L$ and κ is not weakly compact. This is a strong contrast to the classical case $\kappa = \omega$ in which the isomorphism of graphs is strictly below the universal equivalence relation induced by a Borel action of a Polish group, and does not reduce even some simple[1] Borel equivalence relations such as E_1 [14].

§2. Preliminaries and Definitions. In this section we define the notions and concepts we work with. Throughout this article we assume that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$ which is a standard assumption in GDST. In this paper, however, this assumption is mostly redundant, because we

[1]Simple in the sense that it is low in the Borel-hierarchy, namely Σ^0_2.

work either with strongly inaccessible κ or under the assumption $V = L$. For sets X and Y denote by X^Y the set of all functions from Y to X. For ordinal α denote by $X^{<\alpha}$ the set of all functions from any $\beta < \alpha$ to X. We work with the generalised Baire and Cantor spaces associated with κ these being κ^κ and 2^κ respectively, where $2 = \{0, 1\}$. The generalised Baire space κ^κ is equipped with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$\{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. The collection of *κ-Borel* subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ. A *κ-Borel set* is any element of this collection. In this paper we do not consider any other kind of Borel sets, so we always omit the prefix "κ-". The subspace $2^\kappa \subset \kappa^\kappa$ (the generalised Cantor space) is equipped with the subspace topology. We will also work in the subspaces of the form Mod^κ_T which are sets of codes for models with domain κ of a first-order countable theory T. Special cases include Mod^κ_G and $\mathrm{Mod}^\kappa_{\mathrm{DLO}}$ for graphs and dense linear orders respectively. These are Borel subspaces of 2^κ. This enables us to view the quasi-order of embeddability of models, say $\sqsubseteq_{\mathrm{DLO}}$, as a quasi-order on 2^κ. In order to precisely define this, we have to introduce some notions.

The following is a standard way to code structures with domain κ by elements of κ^κ (see e.g. [4]). Suppose $\mathcal{L} = \{P_n \mid n < \omega\}$ is a countable relational vocabulary.

Definition 2.1. Fix a bijection $\pi\colon \kappa^{<\omega} \to \kappa$. For every $\eta \in 2^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n satisfies

$$(a_1, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, \ldots, a_n)) = 1.$$

Note that for every $\mathcal{L}$-structure $\mathcal{A}$ with $\mathrm{dom}(\mathcal{A}) = \kappa$ there exists $\eta \in 2^\kappa$ with $\mathcal{A} = \mathcal{A}_\eta$. It is clear how this coding can be modified for a finite vocabulary. For club many $\alpha < \kappa$ we can also code the $\mathcal{L}$-structures with domain α:

Definition 2.2. Denote by C_π the club $\{\alpha < \kappa \mid \pi[\alpha^{<\omega}] \subseteq \alpha\}$. For every $\eta \in 2^\kappa$ and every $\alpha \in C_\pi$ define the structure $\mathcal{A}_{\eta\restriction\alpha}$ with domain α as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \ldots, a_n)$ in α^n satisfies

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\alpha}} \iff (\eta \restriction \alpha)(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Note that for every $\alpha \in C_\pi$ and every $\eta \in 2^\kappa$ the structures $\mathcal{A}_{\eta\restriction\alpha}$ and $\mathcal{A}_\eta \restriction \alpha$ are the same.

Let us denote by Mod^κ_T the subset of 2^κ consisting of those elements that code the models of a first-order countable theory (not necessarily complete). Abbreviate first-order countable theory as FOCT from now on. We will be interested in particular in $T = G$, the theory of graphs (symmetric and irreflexive) and $T = \mathrm{DLO}$, the theory of dense linear orders without end-points. We consider Mod^κ_T as a topological space endowed with the subspace topology. For more background on GDST see e.g. [4].

We can now define some central relations for this paper. A *quasi-order* is a transitive and reflexive relation.

Definition 2.3 (Relations). We will use the following relations.

Isomorphism: For a FOCT T, define

$$\cong^\kappa_T = \cong_T = \{(\eta,\xi) \in 2^\kappa \times 2^\kappa \mid \eta,\xi \in \mathrm{Mod}^\kappa_T, \mathcal{A}_\eta \cong \mathcal{A}_\xi \text{ or } \eta,\xi \notin \mathrm{Mod}^\kappa_T\}.$$

Embeddability: For a FOCT T, define the quasi-order

$$\sqsubseteq^\kappa_T = \sqsubseteq_T = \{(\eta,\xi) \in (\mathrm{Mod}^\kappa_T)^2 \mid \mathcal{A}_\eta \text{ is embeddable into } \mathcal{A}_\xi\}$$

Bi-embeddability: For a FOCT T and $\eta,\xi \in \mathrm{Mod}^\kappa_T$, let

$$\eta \approx_T \xi \iff \eta \sqsubseteq_T \xi \wedge \xi \sqsubseteq_T \eta.$$

Inclusion mod NS: For $\eta,\xi \in 2^\kappa$ and a stationary $S \subset \kappa$, we write $\eta \sqsubseteq_S \xi$ if $(\eta^{-1}\{1\}\backslash\xi^{-1}\{1\}) \cap S$ is non-stationary.

Equivalence mod NS: For every stationary $S \subset \kappa$ and $\lambda \in \{2,\kappa\}$, we define E^λ_S as the relation

$$E^\lambda_S = \{(\eta,\xi) \in \lambda^\kappa \times \lambda^\kappa \mid \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is not stationary}\}.$$

Note that $\eta E^2_S \xi$ if and only if $\eta \sqsubseteq_S \xi \wedge \xi \sqsubseteq_S \eta$.

If S is the set of all μ-cofinal ordinals, denote $E^\lambda_S = E^\lambda_\mu$ and $\sqsubseteq_S = \sqsubseteq_\mu$. If S is the set of all regular cardinals below κ, denote $S = \mathrm{reg}(\kappa) = \mathrm{reg}$ in which case $E^\lambda_S = E^\lambda_{\mathrm{reg}}$ and $\sqsubseteq_S = \sqsubseteq_{\mathrm{reg}}$. If $S = \kappa$, write $E^\lambda_S = E^\lambda_{\mathrm{NS}}$ and $\sqsubseteq_S = \sqsubseteq_{\mathrm{NS}}$

Note that if we define $F \colon 2^\kappa \to 2^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \in S \\ 1 & \text{otherwise} \end{cases}$$

for a fixed $S \subset \kappa$, we obtain:

Fact 2.4. For all stationary $S \subset S'$ we have $\sqsubseteq_S \leqslant_B \sqsubseteq_{S'}$.

A quasi-order Q on (a Borel set) $X \subset \kappa^\kappa$ is Σ^1_1, if $Q \subset X^2$ is the projection of a closed set in $X^2 \times \kappa^\kappa$ (X is equipped with subspace topology and $X^2 \times \kappa^\kappa$ with the product topology). All quasi-orders of Definition 2.3 (note that equivalence relations are quasi-orders) are Σ^1_1.

Suppose $X, Y \subset \kappa^\kappa$ are Borel. A function $f \colon X \to Y$ is *Borel*, if for every open set $A \subseteq Y$ the inverse image $f^{-1}[A]$ is a Borel subset of X with respect to the induced Borel structure on X and Y.

If Q_1 and Q_2 are quasi-orders respectively on X and Y, then we say that Q_1 is *Borel-reducible* to Q_2 if there exists a Borel map $f \colon X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 Q_1 x_2 \iff f(x_1) Q_2 f(x_2)$ and this is also denoted by $Q_1 \leqslant_B Q_2$. If f is continuous (inverse image of an open set is open), then we say that Q_1 is *continuously reducible* to Q_2. Note that equivalence relations are quasi-orders, so this gives naturally a notion of reducibility for them as well.

A quasi-order is Σ^1_1-*complete*, if every Σ^1_1 quasi-order is Borel-reducible to it. An equivalence relation is Σ^1_1-*complete* if every Σ^1_1 equivalence relation is Borel-reducible to it.

A Borel equivalence relation E on a Borel subspace $X \subset 2^\kappa$ can be extended to 2^κ by declaring all other elements equivalent to each other, but not equivalent to any of the elements in X. Similarly a quasi-order $\sqsubseteq$ on $X \subset 2^\kappa$ can be trivially extended to the whole space 2^κ. If the original equivalence relation or quasi-order was Σ^1_1-complete, then so are the extensions.

§3. Σ_1^1–completeness of $\sqsubseteq_S$ in L. This section is devoted to proving Theorem 3.1. In Section 4 a range of corollaries will be proved.

THEOREM 3.1. ($V = L, \kappa > \omega$) The quasi-order $\sqsubseteq_\mu$ is Σ_1^1–complete, for every regular $\mu < \kappa$.

As mentioned in Introduction, this is an improvement to a theorem in [9] which says that E_μ^κ is Σ_1^1-complete.

DEFINITION 3.2. We will need a version of the diamond principle.

- Let us define a class function $F_\diamond : On \to L$. For all α, $F_\diamond(\alpha)$ is a pair (X_α, C_α) where $X_\alpha, C_\alpha \subseteq \alpha$, if α is a limit ordinal, then C_α is either a club or the empty set, and $C_\alpha = \emptyset$ when α is not a limit ordinal. We let $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ be the $<_L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if α is a limit ordinal and such pair exists and otherwise we let $F_\diamond(\alpha) = (\emptyset, \emptyset)$.
- We let $C_\diamond \subseteq On$ be the class of all limit ordinals α such that for all $\beta < \alpha$, $F_\diamond \restriction \beta \in L_\alpha$. Notice that for every regular cardinal α, $C_\diamond \cap \alpha$ is a club.

DEFINITION 3.3. For a given regular cardinal α and a subset $A \subset \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ to be $(F_\diamond(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ to be the sequence of sets X_γ such that $F_\diamond(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.

REMARK 3.4. It is known that if α and μ are regular cardinals such that $\mu < \alpha$, then the sequence $(X_\gamma)_{\gamma \in S_\mu^\alpha}$ is a diamond sequence (i.e. for all $Y \subseteq \alpha$, the set $\{\gamma \in S_\mu^\alpha \mid Y \cap \gamma = X_\gamma\}$ is stationary). Notice that if $\beta \in C_\diamond$, then for all $\gamma < \beta$, $X_\gamma \in L_\beta$.

By ZF^- we mean ZFC+$(V = L)$ without the power set axiom. By $\mathrm{ZF}^\diamond$ we mean ZF^- with the following axiom:

"For all regular cardinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in \alpha}$ is such that for all $\gamma < \alpha$, $F_\diamond(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in S_\mu^\alpha}$ is a diamond sequence."

Whether or not ZF^- proves $\mathrm{ZF}^\diamond$ is irrelevant for the present argument. We denote by $\mathrm{Sk}(Y)^{L_\theta}$ the Skolem closure of Y in L_θ under the definable Skolem functions.

LEMMA 3.5. ($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, the following are equivalent for all $\eta \in 2^\kappa$:

- $\varphi(\eta, x)$
- $S \backslash A$ is non-stationary, where $S = \{\alpha \in S_\mu^\kappa \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and
$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha (L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$$
where $r(\alpha)$ is the formula "α is a regular cardinal".

REMARK 3.6. This Lemma is reminiscent of [9, Remark 1.10], but there is a big difference, because now S depends on η through the diamond-sequence. The proof in [9] is not applicable here.

PROOF. Let $\mu < \kappa$ be a regular cardinal. Suppose that $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ holds. Let θ be a cardinal large enough such that
$$L_\theta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta, x) \wedge r(\kappa).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, \eta, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Then D is a club set and $D \cap C_\diamond$ is a club. Since $H(\alpha)$ is an elementary submodel of L_θ and the Mostowski collapse $\bar{H}(\alpha)$ is equal to L_β for some $\beta > \alpha$, we have $D \cap C_\diamond \subseteq A$.

Suppose $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Let θ be a large enough cardinal such that

$$L_\theta \models \mathrm{ZF}^\diamond \wedge \neg\varphi(\eta, x) \wedge r(\kappa).$$

Let C be an unbounded set which is closed under μ-limits (a μ-club). Let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, C, \eta, x, (X_\gamma, C_\gamma)_{\gamma \in S^\kappa_\mu}\})^{L_\theta}.$$

Let

$$D = \{\alpha \in S^\kappa_\mu \mid H(\alpha) \cap \kappa = \alpha\}$$

Notice that since $H(\alpha)$ is an elementary substructure of L_θ, then $H(\alpha)$ calculates all cofinalities correctly below α. Then D is an unbounded set, closed under μ-limits. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and α_0 be the least ordinal in $(\lim_\mu D) \cap S$ (where $\lim_\mu D$ is the set of ordinals of D that are μ-cofinal limits of elements of D). Since $\alpha_0 \in \lim_\mu D$, $\alpha_0 > \mu$. By the elementarity of each $H(\alpha)$ we conclude that $\alpha_0 \in C$.

Let $\bar{\beta}$ be such that $L_{\bar{\beta}}$ is equal to the Mostowski collapse of $H(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$, thus $\alpha_0 \in C_\diamond \cap \kappa$. There exists $\beta > \alpha_0$ such that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha_0, x \restriction \alpha_0) \wedge r(\alpha_0).$$

Since $\varphi(\eta, x)$ is a Σ_1-formula, β is a limit ordinal greater than $\bar{\beta}$.

Claim 3.6.1. L_β satisfies the following:

1. For all $\gamma \in S \cap \alpha_0$, γ has cofinality μ.
2. $S \cap \alpha_0$ is a stationary subset of α_0.
3. $D \cap \alpha_0$ is a μ-club subset of a_0.

Proof. 1. $H(\alpha_0)$ calculates all cofinalities correctly below α_0. Thus $L_{\bar{\beta}}$ calculates all cofinalities correctly below α_0. Since β is greater than $\bar{\beta}$, L_β calculates all cofinalities correctly below α_0. Since $S \cap \alpha_0 \subseteq S^\kappa_\mu$ in L, then $S \cap \alpha_0 \subseteq S^\kappa_\mu$ holds in L_β.

2. Since $\alpha_0 \in C_\diamond \cap \kappa$ and L_β satisfies $\mathrm{ZF}^\diamond$ and $r(\alpha_0)$, L_β satisfies that $S \cap \alpha_0$ is a stationary subset of α_0.

3. Let $\alpha < \alpha_0$ be such that $L_\beta \models \mathrm{cf}(\alpha) = \mu \ \wedge\ \bigcup(D \cap \alpha) = \alpha$, we will show that $L_\beta \models \alpha \in D \cap \alpha_0$. Since L_β calculates all cofinalities correctly below α_0, $L \models \mathrm{cf}(\alpha) = \mu \wedge \bigcup(D \cap \alpha) = \alpha$. D is a μ-club in L, thus $L \models \alpha \in D$. Since $\alpha < \alpha_0$, $L \models \alpha \in D \cap \alpha_0$. We will finish the proof by showing that $L \models \alpha \in D \cap \alpha_0$ implies $L_\beta \models \alpha \in D \cap \alpha_0$.

   Notice that $H(\alpha_0)$ is a definable subset of L_θ and D is a definable subset of L_θ. By elementarity, $D \cap \alpha_0$ is a definable subset of $H(\alpha_0)$, we conclude that $D\cap\alpha_0$ is a definable subset of $L_{\bar\beta}$ and $D\cap\alpha_0\in L_\beta$. Therefore $L_\beta\models\alpha\in D\cap\alpha_0$.

⊣

Since $L_\beta\models r(\alpha_0)$, by the previous claim we concluded that L_β satisfies "$\lim_\mu D\cap\alpha_0$ is a μ-club". Since $S\cap\alpha_0$ is a stationary subset of α_0 in L_β, we conclude that

$$L_\beta\models(\lim_\mu D\cap\alpha_0)\cap S\cap\alpha_0\neq\emptyset,$$

so

$$L\models(\lim_\mu D\cap\alpha_0)\cap S\cap\alpha_0\neq\emptyset.$$

This contradicts the minimality of α_0. ⊣

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose Q is a Σ^1_1 quasi-order on κ^κ. Let $a\colon\kappa^\kappa\to 2^{\kappa\times\kappa}$ be the map defined by

$$a(\eta)(\alpha,\beta)=1\Leftrightarrow\eta(\alpha)=\beta.$$

Let b be a continuous bijection from $2^{\kappa\times\kappa}$ to 2^κ, and $c=b\circ a$. Define Q' by

$$(\eta,\xi)\in Q'\Leftrightarrow(\eta=\xi)\vee(\eta,\xi\in ran(c)\wedge(c^{-1}(\eta),c^{-1}(\xi))\in Q)$$

So c is a continuous reduction of Q to Q', and Q' is a Σ^1_1 quasi-order because it is a continuous image of Q. We can assume, without loss of generality, that Q is a quasi-order on 2^κ.

There is a Σ_1-formula of set theory $\psi(\eta,\xi)=\psi(\eta,\xi,x)=\exists k\varphi(k,\eta,\xi,x)\vee\eta=\xi$ with $x\in 2^\kappa$, such that for all $\eta,\xi\in 2^\kappa$,

$$(\eta,\xi)\in Q\Leftrightarrow\psi(\eta,\xi),$$

we added $\eta=\xi$ to $\psi(\eta,\xi)$, to ensure that when we reflect $\psi(\eta\restriction\alpha,\xi\restriction\alpha)$ we get a reflexive relation. Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^Q(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta,\xi)$ defines a quasi-order on 2^κ. For all $\eta\in 2^\kappa$ and $\alpha<\kappa$, let

$$T_{\eta,\alpha}=\{p\in 2^\alpha\mid\exists\beta>\alpha(L_\beta\models\mathrm{ZF}^\diamond\wedge\psi(p,\eta\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^Q(\alpha))\}.$$

Let $(X_\alpha)_{\alpha\in S^\kappa_\mu}$ be the diamond sequence of Definition 3.3, and for all $\alpha\in S^\kappa_\mu$, let $\mathcal{X}_\alpha$ be the characteristic function of X_α. Define $\mathcal{F}\colon 2^\kappa\to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha)=\begin{cases}1 & \text{if } \mathcal{X}_\alpha\in T_{\eta,\alpha}\text{ and }\alpha\in S^\kappa_\mu\\ 0 & \text{otherwise}\end{cases}$$

Claim 3.6.2. If $\eta\ Q\ \xi$, then $T_{\eta,\alpha}\subseteq T_{\xi,\alpha}$ for club-many α's.

Proof. Suppose $\psi(\eta,\xi,x)=\exists k\varphi(k,\eta,\xi,x)$ holds and let k witnesses that. Let θ be a cardinal large enough such that $L_\theta\models\mathrm{ZF}^\diamond\wedge\varphi(k,\eta,\xi,x)\wedge r(\alpha)$. For all $\alpha<\kappa$ let $H(\alpha)=\mathrm{Sk}(\alpha\cup\{\kappa,k,\eta,\xi,x\})^{L_\theta}$. The set $D=\{\alpha<\kappa\mid H(\alpha)\cap\kappa=\alpha\wedge H(\alpha)\models\psi^Q(\alpha)\}$ is a club. Using the Mostowski collapse we have that

$$D'=\{\alpha<\kappa\mid\exists\beta>\alpha(L_\beta\models\mathrm{ZF}^\diamond\wedge\varphi(k\restriction\alpha,\eta\restriction\alpha,\xi\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^Q(\alpha))\}$$

contains a club. For all $\alpha\in D'$ and $p\in T_{\eta,\alpha}$ we have that

$$\exists\beta_1>\alpha(L_{\beta_1}\models\mathrm{ZF}^\diamond\wedge\psi(p,\eta\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^Q(\alpha))$$

and

$$\exists\beta_2 > \alpha(L_{\beta_2} \models \mathrm{ZF}^\diamond \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)).$$

Therefore, for $\beta = \max\{\beta_1, \beta_2\}$ we have that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p, \eta\restriction\alpha, x\restriction\alpha) \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha).$$

Since $\psi^Q(\alpha)$ holds and so transitivity holds for $\psi(\eta, \xi)$ in L_β, we conclude that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$$

so $p \in T_{\xi,\alpha}$ and $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$. This holds for all $\alpha \in D'$. ⊣

By the previous claim, we conclude that if $\eta\ Q\ \xi$, then there is a μ-club C such that for every $\alpha \in C$ it holds that $\mathcal{X}_\alpha \in T_{\eta,\alpha} \Rightarrow \mathcal{X}_\alpha \in T_{\xi,\alpha}$. Therefore $(\mathcal{F}(\eta)^{-1}\{1\} \backslash \mathcal{F}(\xi)^{-1}\{1\}) \cap C = \emptyset$, and $\mathcal{F}(\eta) \sqsubseteq^\kappa_\mu \mathcal{F}(\xi)$.

For the other direction, suppose $\neg\psi(\eta, \xi, x)$ holds. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$. Since $(X_\gamma)_{\gamma\in S^\kappa_\mu}$ is a diamond sequence, S is a stationary set. By Lemma 3.5 we know that $S\backslash A$ is stationary, where

$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists\beta > \alpha(L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha))\}.$$

Since for all $\alpha \in S\backslash A$ we have that $X_\alpha = \eta^{-1}\{1\} \cap \alpha$, so $\mathcal{X}_\alpha \in T_{\eta,\alpha}$. We conclude that for all $\alpha \in S\backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$. On the other hand, for all $\alpha \in S\backslash A$ it holds that

$$\forall\beta > \alpha(L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha))$$

so

$$\forall\beta > \alpha(L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\mathcal{X}_\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha)).$$

Therefore

$$\forall\beta > \alpha(L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\mathcal{X}_\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

we conclude that $\mathcal{X}_\alpha \notin T_{\xi,\alpha}$, and $\mathcal{F}(\xi)(\alpha) = 0$. Hence, for all $\alpha \in S\backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$ and $\mathcal{F}(\xi)(\alpha) = 0$. Since $S\backslash A$ is stationary, we conclude that $\mathcal{F}(\eta)^{-1}\{1\}\backslash\mathcal{F}(\xi)^{-1}\{1\}$ is stationary and $\mathcal{F}(\eta) \not\sqsubseteq^\kappa_\mu \mathcal{F}(\xi)$. ⊣

§4. Corollaries of Theorem 3.1.

4.1. Σ^1_1-completeness of E^2_μ in L.

THEOREM 4.1 ($V = L$, $\kappa > \omega$). $\sqsubseteq_{\mathrm{NS}}$ is a Σ^1_1-complete quasi-order.

PROOF. Follows from Fact 2.4 and Theorem 3.1. ⊣

THEOREM 4.2 ($V = L$). Let μ be regular cardinal below κ, then E^2_μ is a Σ^1_1-complete equivalence relation.

PROOF. This follows from Theorem 3.1, because E^2_μ is a symmetrisation of the quasi-order $\sqsubseteq_\mu$. ⊣

The above result is not true in ZFC. It was shown in [4, Thm 56] that if κ is not a successor of a singular cardinal, then in a cofinality preserving forcing extension $E^\kappa_{\mu_1}$ and $E^\kappa_{\mu_2}$ are $\leqslant_B$-incomparable for regular cardinals $\mu_1 < \mu_2 < \kappa$.

Theorem 4.1 gives consistently a positive answer to "Given a weakly compact cardinal κ, is $\sqsubseteq_{\mathrm{NS}}$ complete?" [17, Q. 11.4]. Theorem 4.2 answers the questions "Is it consistently true that $E^2_\mu \leqslant_B E^2_\lambda$ for $\lambda < \mu$?" [4],[15, Q. 3.47] (take

$\lambda = \omega$, $\mu = \omega_1$ and $\kappa = \omega_2$), and gives consistently a positive answer to "Is E^κ_μ Borel-reducible to E^2_μ for a regular μ?" [5, Q. 15], [15, Q. 3.46].

4.2. Σ^1_1-completeness of $\sqsubseteq_{\mathrm{DLO}}$ in L. [17, Q. 11.3] asks "Given a weakly compact cardinal κ, is $\sqsubseteq_{\mathrm{DLO}}$ complete for Σ^1_1 quasi-orders? What about arbitrary regular cardinals κ?" In this section we apply Theorem 3.1 to show that the answer is positive if $V = L$. To do that we first have to establish a general theorem about $\sqsubseteq_{\mathrm{DLO}}$:

THEOREM 4.3. Suppose that for all $\lambda < \kappa$ we have $\lambda^\omega < \kappa$. Then there is a continuous reduction of $\sqsubseteq_\omega$ to $\sqsubseteq_{\mathrm{DLO}}$.

PROOF. We will first define a continuous function $G\colon \mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ with the following properties for every $A, B \subset \mathcal{P}(\kappa)$:

($G1$) if $A \sqsubseteq_\omega B$, then there exists a continuous $f\colon \kappa \to \kappa$ such that $f[G(A)] \subset G(B)$

($G2$) if $A \not\sqsubseteq^\kappa_\omega B$, then $G(A) \not\sqsubseteq^\kappa_\omega G(B)$.

CLAIM 4.3.1. A function G as above exists.

PROOF. Fix an ω-club $C \subset \kappa$ with the property that for all $\alpha < \kappa$ and all $\beta < \kappa$ there exists γ with $\beta < \gamma < \kappa$ such that $[\gamma, \gamma + \alpha] \cap C = \varnothing$, where $[\gamma, \gamma+\alpha] = \{\delta < \kappa \mid \gamma \leqslant \delta \leqslant \gamma + \alpha\}$, thus C is in a sense "sparse". For $A \subset \kappa$, let $G(A) = (A \cap C) \cup (\kappa \setminus C)$.

Let us show that then G is as needed. It is easy to see that it is continuous, because if $A \cap \alpha = A' \cap \alpha$, then clearly $G(A) \cap \alpha = G(A') \cap \alpha$ and vice versa for a club of α's. Suppose $A \setminus B$ is non-stationary. Let C' be a club such that $A \cap C' \subset B$ and let $D = C \cap C'$. Then define $f\colon \kappa \to \kappa$ inductively as follows. Let α_0 be the smallest ordinal in D, find $\gamma_0 > \alpha_0$ such that $[\gamma_0, \gamma_0 + a_0] \cap C = \varnothing$ and let $f \restriction \alpha_0$ be defined by $f(\alpha) = \gamma_0 + \alpha$ for all $\alpha \leqslant \alpha_0$. Suppose that a sequence $(\alpha_\pi)_{\pi \leqslant \pi'}$ has been defined as well as a sequence $(\gamma_\pi)_{\pi \leqslant \pi'}$ such that for all $\pi < \pi'$ we have

$$\gamma_\pi < \alpha_{\pi+1} < \gamma_{\pi+1} \tag{3}$$

and $f \restriction (\alpha_{\pi'} + 1)$ is defined. Then let $\alpha_{\pi'+1} > \gamma_{\pi'}$ to be an element of D, pick $\gamma_{\pi'+1} > \alpha_{\pi'+1}$ to be such that $[\gamma_{\pi'+1}, \gamma_{\pi'+1} + \alpha_{\pi'+1}] \cap C = \varnothing$ and define $f(\alpha) = \gamma_{\pi'+1} + \alpha$ for all $\alpha \in [\alpha_{\pi'} + 1, \alpha_{pi'+1}]$. Suppose that π' is a limit ordinal and $(\alpha_\pi)_{\pi<\pi'}$, $(\gamma_\pi)_{\pi<\pi'}$ are defined and $f(\alpha)$ is defined for all $\alpha < \sup_{\pi<\pi'} \alpha_\pi$. From (3) it follows that, $\alpha_{\pi'} = \sup_{\pi<\pi'} \alpha_\pi = \sup_{\pi<\pi'} \gamma_\pi$ and for all $\alpha < \alpha_{\pi'}$ we have $\alpha \leqslant f(\alpha) < \alpha_{\pi'}$. We also have that $\alpha_{\pi'} \in D$, because it is a limit of elements of D, so we can now define $f(\alpha_{\pi'}) = \alpha_{\pi'}$. In this way f is continuous and for every $\alpha < \kappa$ we have either $f(\alpha) \in \kappa \setminus C$ or $\alpha \in D$ and $f(\alpha) = \alpha$.

In both cases, if $\alpha \in G(A)$, then $f(\alpha) \in G(B)$ For ($G2$), assume that $A \not\sqsubseteq_\omega B$ and let $S \subset A \setminus B$ be ω-stationary. But then $S \cap C$ is ω-stationary and $S \cap C \subset G(A) \setminus G(B)$. ⊣

For every $p, q \in \kappa^{\leqslant\omega}$ define $p \prec q$ if either $p \supset q$ or there is smallest such $n < \omega$ that $p(n) \neq q(n)$ and for this n we have $p(n) < q(n)$. This defines a linear order on the set $C(\kappa^{\leqslant\omega})$ of all strictly increasing functions $p \in \kappa^{\leqslant\omega}$. Let $C^*(\kappa^{\leqslant\omega})$ be the set of all strictly increasing functions $p \in \kappa^{\leqslant\omega}$ whose range contains at least one infinite ordinal. This ensures that none of our linear orders has an end-point. Also let $\lim^*(\kappa)$ be the set of all limit ordinals below κ except ω.

Now for $A \in \mathcal{P}(\lim^*(\kappa))$ define the linear order $L(A)$ to be the set

$$\{p \in C^*(\kappa^{\leqslant\omega}) \mid \operatorname{dom} p = \omega \text{ and } \sup\operatorname{ran} p \in A \text{ and } p(0) = 0\}$$

equipped with the order $\prec$. This is a modification of a construction given by Baumgartner [2]. We will show that $A \mapsto L(G(A))$ is the desired reduction, where G is as was proved to exist in Claim 4.3.1.

If $f\colon \kappa \to \kappa$ is continuous and strictly increasing and $A \subset \lim^*(\kappa)$ any set, it is clear from the definition of $L(A)$ that

$$f[L(A)] = \{f \circ p \mid p \in L(A)\} \subset L(f[A]).$$

Thus, if $A \sqsubseteq_\omega B$ and $f\colon \kappa \to \kappa$ is continuous such that $f[G(A)] \subset G(B)$ (as guaranteed by $(G1)$), then $p \mapsto f \circ p$ defines an embedding from $L(G(A))$ to $L(G(B))$.

The other direction is essentially a simplification of the proof of Baumgartner Theorem 5.3(ii) [2]. If $A \not\sqsubseteq_\omega B$, then by $(G2)$ also $G(A) \not\sqsubseteq_\omega G(B)$ and so $G(A) \setminus G(B)$ is ω-stationary. So it is sufficient to show that that for *any* unbounded $A, B \subset \lim^*(\kappa)$, if $A \setminus B$ is ω-stationary, then $L(A)$ cannot be embedded into $L(B)$. Notice that $L(A) = L(A \cap \lim^*(\kappa))$.

So suppose that $A \setminus B$ is stationary and assume towards a contradiction that $h\colon L(A) \to L(B)$ preserves the ordering $\prec$. For any $X \subset C^*(\kappa^{\leqslant\omega})$, let $T(X) = \{p \in C^*(\kappa^{\leqslant\omega}) \mid \exists q \in X (p \subset q)\}$. Note that for every strictly increasing $p \in \kappa^{<\omega}$ with $p(0) = 0$, we have $p \in T(L(A))$ and $p \in T(L(B))$. For $g \in T(L(B))$, let

$$\operatorname{Right}(g) = \{f \in L(A) \mid h(f) = g \text{ or } g \prec h(f)\},$$
$$\operatorname{Left}(g) = \{f \in L(A) \mid h(f) \prec g\}.$$

Let

$$\rho(g) = \{f' \in T(\operatorname{Right}(g)) \mid \text{ for all } g' \in T(\operatorname{Right}(g)), \text{ if } g' \prec f', \text{ then } f' \subset g'\},$$
$$\lambda(g) = \{f' \in T(\operatorname{Left}(g)) \mid \text{ for all } g' \in T(\operatorname{Left}(g)), \text{ if } f' \prec g', \text{ then } g' \subset f'\}.$$

Note that $\rho(g)$ and $\lambda(g)$ are linearly ordered by $\subset$. Now let $C \subset \lim^*(\kappa)$ be the set of all $\alpha \in \lim^*(\kappa)$ satisfying

1. for all $f \in L(A)$, $\sup\operatorname{ran}(f) < \alpha \iff \sup\operatorname{ran}(h(f)) < \alpha$,
2. $A \cap \alpha$ is unbounded in α,
3. if $g \in T(L(B))$ and $\sup\operatorname{ran}(g) < \alpha$, then $\sup\{\sup\operatorname{ran}(f) \mid f \in \rho(g)\} < \alpha$ and $\sup\{\sup\operatorname{ran}(f) \mid f \in \lambda(g)\} < \alpha$,
4. if $g \in T(L(B))$, $f \in T(\operatorname{Left}(g))$, $\sup\operatorname{ran}(g), \sup\operatorname{ran}(f) < \alpha$, and there exists $f' \in \operatorname{Left}(g)$ such that $f \prec f'$ and $f' \not\subset f$, then there exists such an f' with $\sup\operatorname{ran}(f') < \alpha$,
5. if $g \in T(L(B))$, $f \in T(\operatorname{Right}(g))$, $\sup\operatorname{ran}(g), \sup\operatorname{ran}(f) < \alpha$, and there exists $f' \in \operatorname{Right}(g)$ such that $f \prec f'$ and $f' \not\subset f$, then there exists such an f' with $\sup\operatorname{ran}(f') < \alpha$,

Also assume w.l.o.g. that $C \subseteq \lim^*(\kappa)$. Our cardinality assumption on κ guarantees that C is a club. We will show that $C \cap A \subset B$ which is a contradiction. Let $\alpha \in C \cap A$ and let $f \in L(A)$ be such that $\sup\operatorname{ran}(f) = \alpha$. We will show that $\sup\operatorname{ran}(h(f)) = \alpha$ and so $h(f) \in L(B)$ and $\alpha \in B$. Suppose not. If

$\sup\operatorname{ran}(h(f)) < \alpha$, then by (i), $\sup\operatorname{ran}(f) < \alpha$ which is a contradiction. So we can assume that $\sup\operatorname{ran}(h(f)) > \alpha$. Because we assumed that $p(0) = 0$ for all functions in question, there is $n_0 < \omega$ such that $h(f)(n_0) < \alpha \leqslant h(f)(n_0 + 1)$. Let

$$g = h(f)\restriction(n_0 + 1). \qquad (*)$$

In particular $\sup\operatorname{ran}(g) < \alpha$ (∗∗). For every $m < \omega$, pick $\alpha_m \in A$ such that $f(m) < \alpha_m < \alpha$. Such α_m exists by (ii). Now for each m fix f_m with $\sup\operatorname{ran}(f_m) = \alpha_m$ and $f_m \supset f \restriction (m + 1)$. We have two cases: either (A) $\sup\{m < \omega \mid f_m \in \operatorname{Left}(g)\} = \omega$ or (B) $\sup\{m < \omega \mid f_m \in \operatorname{Right}(g)\} = \omega$. We will show that both (A) and (B) lead to a contradiction.

Let us start with (A) and suppose that there are infinitely many $m < \omega$ with $f_m \in \operatorname{Left}(g)$.

Claim 4.3.2. For all $m < \omega$ we have $f\restriction(m + 1) \in \lambda(g)$.

Proof. For every m, there is $m' > m$ such that $f_{m'} \in \operatorname{Left}(g)$ and since $f \restriction (m + 1) \subset f \restriction (m' + 1) \subset f_{m'}$, we have that $f \restriction (m + 1) \in T(\operatorname{Left}(g))$. Suppose that $f\restriction(m + 1) \notin \lambda(g)$ for some m. Then by the definition of $\lambda(g)$, there exists $g'' \in T(\operatorname{Left}(g))$ such that $f\restriction(m + 1) \prec g''$, but $g'' \not\subset f\restriction(m + 1)$, so there exists $n < m + 1$ with $g''(n) > f(n)$ and n is the smallest such that $g''(n) \neq f(n)$. This g'' can be extended to g' in $\operatorname{Left}(g)$ and by (iv) we can assume that $\sup\operatorname{ran}(g') < \alpha$. The number n witnesses that $f \prec g'$ and so we must have $h(f) \prec h(g')$. The latter implies that for the first $n' < \omega$ with $h(f)(n') \neq h(g')(n')$ we have $h(g')(n') > h(f)(n')$. If $n' > n_0$ (n_0 is defined at (∗) above) then $\sup\operatorname{ran}(h(g')) \geq h(g')(n') > h(f)(n') \geq \alpha$, a contradiction. So $n' \leqslant n_0$ and $h(g)(n') > h(f)(n') = g(n')$, so we have $g \prec h(g')$. But this implies that $g' \in \operatorname{Right}(g)$ which is a contradiction again. This proves the claim. ⊣

Now $\{f \restriction (m + 1) \mid m < \omega\} \subset \lambda(g)$ and since $\sup\operatorname{ran}(f) = \alpha$ we have $\sup\{\sup\operatorname{ran}(k) \mid k \in \lambda(g)\} = \alpha$ contradicting (iii) above. This shows that (A) leads to a contradiction.

Assume (B) i.e. suppose that there are infinitely many $m < \omega$ with $f_m \in \operatorname{Right}(g)$.

Claim 4.3.3. For all $m < \omega$ we have $f\restriction(m + 1) \in \rho(g)$.

Proof. For every m, there is $m' > m$ such that $f_{m'} \in \operatorname{Right}(g)$ and since $f \restriction (m + 1) \subset f \restriction (m' + 1) \subset f_{m'}$, we have that $f \restriction (m + 1) \in T(\operatorname{Right}(g))$. Suppose that $f\restriction(m + 1) \notin \rho(g)$ for some m. Then by the definition of $\rho(g)$, there exists $g'' \in T(\operatorname{Right}(g))$ such that $g'' \prec f\restriction(m + 1)$, but $f\restriction(m + 1) \not\subset g''$, so there exists $n < m + 1$ with $g''(n) < f(n)$ and n is the smallest such that $g''(n) \neq f(n)$. This g'' can be extended to g' in $\operatorname{Right}(g)$ and by (v) we can assume that $\sup\operatorname{ran}(g') < \alpha$. The number n witnesses that $g' \prec f$ and so we must have $h(g') \prec h(f)$. The latter implies that for the first $n' < \omega$ with $h(f)(n') \neq h(g')(n')$ we have $h(g')(n') < h(f)(n')$. If $n' > n_0$ (n_0 is defined at (∗) above), then $g \subset h(g')$ and $h(g') \prec g$ which is a contradiction with $g' \in \operatorname{Right}(g)$.

So $n' \leqslant n_0$ and so $h(g)(n') < h(f)(n') = g(n')$, so we have $g \prec h(g')$ and again this implies that $g' \in \operatorname{Left}(g)$, contradiction. This proves the claim. ⊣

Now $\{f\restriction(m+1) \mid m < \omega\} \subset \rho(g)$ and since $\sup \operatorname{ran}(f) = \alpha$ we have $\sup\{\sup \operatorname{ran}(k) \mid k \in \rho(g)\} \geq \alpha$ contradicting (iii) above. This shows that (B) leads to a contradiction too. ⊣

THEOREM 4.4 ($V = L$). If $\kappa > \omega$ is a regular cardinal which is not the successor of an ω-cofinal cardinal, then $\sqsubseteq_{\mathrm{DLO}}$ is Σ_1^1-complete.

PROOF. By Theorem 3.1 it is sufficient to reduce $\sqsubseteq_\omega$ to $\sqsubseteq_{\mathrm{DLO}}$. But since $V = L$ every cardinal $\kappa > \omega$ which is not the successor of an ω-cofinal cardinal satisfies the assumption of Theorem 4.3. ⊣

4.3. Dichotomy for countable first-order theories in L. In [11] it was proved that if $V = L$, κ is a successor of a uncountable regular cardinal λ, then $\cong_{T_1} \leqslant_c \cong_{T_2}$ and $\cong_{T_2} \not\leqslant_B \cong_{T_1}$ holds for all T_1 classifiable and T_2 non-classifiable. This result can be improved using Theorem 4.2 together with some results from [4]:

THEOREM 4.5. ([4, Thm 86]) Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable complete countable theory. Then $E^2_\omega \leqslant_c \cong_T$. ⊣

COROLLARY 4.6 ($V = L$). Suppose that κ is regular and not the successor of an ω-cofinal cardinal and T is a stable unsuperstable complete countable theory. Then $\cong_T$ is a Σ_1^1-complete relation.

PROOF. Follows from Theorems 4.5 and 4.2 and GCH in L. ⊣

THEOREM 4.7. ([4, Thm 79]) Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

1. If T is complete unstable or superstable with OTOP, then $E^2_\lambda \leqslant_c \cong_T$.
2. If $\lambda \geq 2^\omega$ and T is complete superstable with DOP, then $E^2_\lambda \leqslant_c \cong_T$. ⊣

COROLLARY 4.8 ($V = L$). Suppose that κ is the successor of a regular uncountable cardinal λ. If T is a non-classifiable complete countable theory, then $\cong_T$ is a Σ_1^1-complete relation.

PROOF. Follows from Theorems 4.2, 4.5, and 4.7. ⊣

By using yet another Theorem from [4] we obtain the following dichotomy in L. The class of Δ_1^1 sets consists of sets A such that both A and the complement of A are Σ_1^1 [4].

THEOREM 4.9 ($V = L$). Suppose that κ is the successor of a regular uncountable cardinal λ. If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:

- $\cong_T$ is Δ_1^1.
- $\cong_T$ is Σ_1^1-complete.

PROOF. For this proof it is useful to bare in mind how the isomorphism relation of a theory is defined, Definition 2.3. Sometimes in literature it is defined differently, but these are mutually Borel bi-reducible.

It has been shown [4, Thm 70] that if a complete theory T is classifiable, then $\cong_T$ is Δ_1^1. So for a complete countable theory T the result follows from Corollary 4.8. Suppose T is not a complete theory. Let $\mathcal{L}$ be the vocabulary of T and $\{T_\alpha\}_{\alpha<2^\omega}$ be the set of all the complete theories in $\mathcal{L}$ that extend T.

Notice that $\cong_T = \bigcap_{\alpha<2^\omega} \cong_{T_\alpha}$, therefore if $\cong_{T_\alpha}$ is a Δ^1_1 equivalence relation for all $\alpha < \kappa$, then so is $\cong_T$ since $2^\omega < \kappa$.

Suppose T' is a complete countable theory in $\mathcal{L}$ that extends T such that $\cong_{T'}$ is not a Δ^1_1 equivalence relation. Then T' is a non-classifiable countable theory. By Corollary 4.8 $\cong_{T'}$ is a Σ^1_1-complete equivalence relation. We will show that $\cong_{T'} \leqslant_B \cong_T$ which finishes the proof. Define $\mathcal{F} : \kappa^\kappa \to \kappa^\kappa$ by

$$\mathcal{F}(\eta) = \begin{cases} \eta & \text{if } \mathcal{A}_\eta \models T' \\ \xi & \text{otherwise.} \end{cases}$$

where ξ is a fixed element of κ^κ such that $\mathcal{A}_\xi \not\models T'$. Since T' extends T, $\eta \cong_{T'} \zeta \Leftrightarrow \mathcal{F}(\eta) \cong_T \mathcal{F}(\zeta)$. To show that $\mathcal{F}$ is Borel, note that

$$\mathcal{F}^{-1}([\eta \restriction \alpha]) = \begin{cases} [\eta \restriction \alpha] \backslash \{\zeta \mid \mathcal{A}_\zeta \not\models T'\} & \text{if } \xi \notin [\eta \restriction \alpha] \\ \{\zeta \mid \mathcal{A}_\zeta \not\models T'\} \cup [\eta \restriction \alpha] & \text{if } \xi \in [\eta \restriction \alpha]. \end{cases}$$

Since $[\eta \restriction \alpha]$ is a basic open set and $\{\zeta \mid \mathcal{A}_\zeta \not\models T'\}$ is a Borel set, $[\eta \restriction \alpha] \backslash \{\zeta \mid \mathcal{A}_\zeta \not\models T'\}$ and $[\eta \restriction \alpha] \cup \{\zeta \mid \mathcal{A}_\zeta \not\models T'\}$ are Borel sets. ⊣

The dichotomy of Theorem 4.9 is not provable in ZFC. In [12, 13] it was shown, assuming κ is a successor and $\kappa \in I[\kappa]$, that there is a stable unsuperstable countable theory T in a countable vocabulary such that $\cong_T$ is Borel* (a generalisation of Borel sets to non-well-founded trees [4, 7]). Because of this, $\cong_T$ cannot be a Σ^1_1-complete equivalence relation, unless Borel$^* = \Sigma^1_1$ and the fairly mild combinatorial assumptions mentioned above still hold. In L it holds that Borel$^* = \Sigma^1_1$ [9], but there is a model of ZFC in which $\Delta^1_1 \subsetneq$ Borel$^* \subsetneq \Sigma^1_1$ [10]. In this model E^2_ω is not Δ^1_1 and we still have $\kappa \in I[\kappa]$, so by Theorem 4.5 $\cong_T$ is neither Δ^1_1 nor Σ^1_1-complete.

§5. The case $V \neq L$.

5.1. Σ^1_1-completeness of $\sqsubseteq_{\mathrm{NS}}$ for weakly ineffable κ. In Section 4 we answered in L the questions [15, Q. 3.47], [17, Q.'s 11.3 and 11.4] and [5, Q. 15]. We used Theorem 4.2 as the starting point. But what if $V \neq L$? In this section we provide further partial answers to [17, Q.'s 11.3 and 11.4] outside of L. Recall that these questions ask "Given a weakly compact cardinal κ, are $\sqsubseteq_{\mathrm{NS}}$ and $\sqsubseteq_{\mathrm{DLO}}$ complete for Σ^1_1 quasi-orders?" We will use the following theorem:

THEOREM 5.1. ([17, Cor 10.24]) If κ is weakly compact, then both the quasi-order of embeddability and the equivalence relation of bi-embeddability of graphs, $\sqsubseteq_G$ and $\approx_G$ respectively, are Σ^1_1-complete.

DEFINITION 5.2 (Weakly compact diamond). Let $\kappa > \omega$ be a cardinal. The *weakly compact ideal* is generated by the sets of the form $\{\alpha < \kappa \mid \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \neg\varphi\}$ where $U \subset V_\kappa$ and φ is a Π^1_1-sentence such that $\langle V_\kappa, \in, U\rangle \models \varphi$. A set $A \subset \kappa$ is said to be *weakly compact*, if it does not belong to the weakly compact ideal. Note that κ is weakly compact if and only if there exists $A \subset \kappa$ which is weakly compact, i.e. the weakly compact ideal is proper. For weakly compact $S \subset \kappa$, the *S-weakly compact diamond*, $\mathrm{WC}_\kappa(S)$, is the statement that there exists a sequence $(A_\alpha)_{\alpha<\kappa}$ such that for every $A \subset S$ the set

$$\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$$

is weakly compact. We denote $\mathrm{WC}_\kappa = \mathrm{WC}_\kappa(\kappa)$.

Weakly compact diamond was originally introduced in [18] and thoroughly analysed in [8]. In [1] it was used to study the reducibility properties of E^κ_{reg}. It has been sometimes called the *dual diamond*.

Fact 5.3. If κ is weakly ineffable (same as almost ineffable), then WC_κ holds. See [8] for proofs and references.

The proof of Lemma 5.4 can be found in [1] in complete detail.

Lemma 5.4. Let κ be a weakly compact cardinal. The weakly compact diamond WC_κ implies the following principle WC^*_κ. There exists a sequence $\langle f_\alpha\rangle_{\alpha\in\mathrm{reg}(\kappa)}$ such that

- $f_\alpha\colon \alpha\to\alpha$,
- for all $g\in\kappa^\kappa$ and stationary $Z\subset\kappa$ the set
$$\{\alpha\in\mathrm{reg}(\kappa)\mid g\restriction\alpha = f_\alpha\wedge\alpha\cap Z\text{ is stationary}\}$$
is stationary. ⊣

Following this result, we will introduce the following principle WC^*_G. Let us denote by $G_{<\kappa}$ the set of all graphs with domain $\alpha<\kappa$. There exists a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ such that

- $f_\alpha\in(G_{<\kappa})^\alpha$,
- if (S,g) is a pair such that $S\subseteq\kappa$ is stationary and $g\in(G_{<\kappa})^\kappa$, the set
$$\{\alpha\in\mathrm{reg}(\kappa)\mid g\restriction\alpha = f_\alpha\wedge S\cap\alpha\text{ is stationary}\}$$
is stationary.

Fact 5.5. If WC^*_κ holds, then WC^*_G holds.

Proof. Let $\langle\bar f_\alpha\rangle_{\alpha<\kappa}$ be a sequence that witnesses WC^*_κ. Let $\{\mathcal{A}_\beta\}_{\beta<\kappa}$ be an enumeration of the elements of $G_{<\kappa}$, and for every $\alpha<\kappa$, let $G_{<\alpha}=\{\mathcal{A}_\beta\}_{\beta<\alpha}$. Construct the sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ by $f_\alpha(\beta)=\mathcal{A}_{\bar f_\alpha(\beta)}$.

To show that $\langle f_\alpha\rangle_{\alpha<\kappa}$ witnesses WC^*_G, let $g\in(G_{<\kappa})^\kappa$ be any function and $S\subseteq\kappa$ be a stationary. There is a function $\bar g\colon\kappa\to\kappa$ such that $g(\alpha)=\mathcal{A}_{\bar g(\alpha)}$. Because of WC^*_κ we know that the set
$$\{\alpha\in\mathrm{reg}(\kappa)\mid \bar g\restriction\alpha=\bar f_\alpha\wedge Z\cap\alpha\text{ is stationary}\}$$
is stationary. By the way $\langle f_\alpha\rangle_{\alpha<\kappa}$ and $\bar g$ were defined, we conclude that the set
$$\{\alpha\in\mathrm{reg}(\kappa)\mid g\restriction\alpha=f_\alpha\wedge Z\cap\alpha\text{ is stationary}\}$$
is stationary. ⊣

Theorem 5.6. If κ is weakly compact and WC^*_G holds, then $\sqsubseteq_{\mathrm{reg}}$ as well as $\sqsubseteq_{\mathrm{NS}}$ are Σ^1_1-complete.

Proof. The claim for $\sqsubseteq_{\mathrm{NS}}$ follows from Fact 2.4 once we prove the claim for $\sqsubseteq_{\mathrm{reg}}$. By Theorem 5.1 it is enough to show that $\sqsubseteq_G\leqslant_B\sqsubseteq_{\mathrm{reg}}$. For all $K,H\in G_{<\kappa}$ we write $K\sqsubseteq H$ if K is embeddable to H. Let us denote by Q the quasi-order $((G_{<\kappa})^\kappa,\leqslant_Q)$, where $f\leqslant_Q g$ holds if there is a club C such that for all $\alpha\in C$, $f(\alpha)\sqsubseteq g(\alpha)$ holds.

Let H be the graph with domain 2 and no edges. Define $F : Mod^\kappa_G \to (G_{<\kappa})^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} \mathcal{A}_{\eta\restriction\alpha} & \text{if } \alpha \in C_\pi \\ H & \text{otherwise.} \end{cases}$$

where C_π is as in Definition 2.2.

Claim 5.6.1. $\eta \sqsubseteq_G \xi$ if and only if $F(\eta) \leqslant_Q F(\xi)$.

Proof. Let us show that if $\eta \sqsubseteq_G \xi$, then $F(\eta) \leqslant_Q F(\xi)$. Suppose $\eta \sqsubseteq_G \xi$, then there is $f : \kappa \to \kappa$ an embedding of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$. Let D be the set of closed points of f, D is a club. Therefore $f \restriction \alpha$ is an embedding of $\mathcal{A}_{\eta\restriction\alpha}$ to $\mathcal{A}_{\xi\restriction\alpha}$, for all $\alpha \in D \cap C_\pi$. We conclude that $F(\eta) \leqslant_Q F(\xi)$. Let us show that if $(\eta, \xi) \notin \sqsubseteq_G$, then $F(\eta) \not\leqslant_Q F(\xi)$. Suppose $(\eta, \xi) \notin \sqsubseteq_G$. The property

There is no embedding of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$ $\wedge$ κ is regular $\wedge$ C_π is unbounded

is a Π^1_1-property of the structure $(V_\kappa, \in, A)$, where $A = (\eta \times \{0\}) \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\})$. Since κ is weakly compact, then there is stationary many γ's such that $C_\pi \cap \gamma$ is unbounded, $\gamma \in C_\pi$, γ is regular, and *there is no embedding of $\mathcal{A}_{\eta\restriction\gamma}$ to $\mathcal{A}_{\xi\restriction\gamma}$*. We conclude that there are stationary many γ's such that $F(\eta)(\gamma) \not\sqsubseteq F(\xi)(\gamma)$, hence $F(\eta) \not\leqslant_Q F(\xi)$. ⊣

Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence that witnesses WC^*_G. For all $\alpha \in \mathrm{reg}(\kappa)$ define the relation $\leqslant^\alpha_Q$ on $(G_{<\kappa})^\alpha$ by: $f \leqslant^\alpha_Q g$ if there is a club $C \subseteq \alpha$ such that for all $\beta \in C$, $f(\beta) \sqsubseteq g(\beta)$ holds. Notice that since the intersection of two clubs is a club, then $\leqslant^\alpha_Q$ is a quasi-order. Define the map $\mathcal{F} : (G_{<\kappa})^\kappa \to 2^\kappa$ by

$$\mathcal{F}(f)(\alpha) = \begin{cases} 0 & \text{if } f \restriction \alpha \leqslant^\alpha_Q f_\alpha \\ 1 & \text{otherwise} \end{cases}$$

Claim 5.6.2. $f \leqslant_Q g$ if and only if $\mathcal{F}(f) \sqsubseteq_{\mathrm{reg}} \mathcal{F}(g)$.

Proof. Let us show that if $f \leqslant_Q g$, then $\mathcal{F}(f) \sqsubseteq_{\mathrm{reg}} \mathcal{F}(g)$. Suppose $f \leqslant_Q g$, then there is a club $C \subset \kappa$ such that for all $\alpha \in C$, $f(\alpha) \sqsubseteq g(\alpha)$. Therefore, for all $\alpha \in C \cap \mathrm{reg}(\kappa)$ it holds that $f \restriction \alpha \leqslant^\alpha_Q g \restriction \alpha$. Now if $\alpha \in C \cap \mathrm{reg}(\kappa)$ is such that $\mathcal{F}(g)(\alpha) = 0$, then $g \restriction \alpha \leqslant^\alpha_Q f_\alpha$, so $f \restriction \alpha \leqslant^\alpha_Q f_\alpha$ and $\mathcal{F}(f)(\alpha) = 0$. We conclude that $(\mathcal{F}(f)^{-1}[1] \backslash \mathcal{F}(g)^{-1}[1]) \cap \mathrm{reg}(\kappa)$ is non-stationary. Hence $\mathcal{F}(f) \sqsubseteq_{\mathrm{reg}} \mathcal{F}(g)$. Let us show that if $f \not\leqslant_Q g$, then $\mathcal{F}(f) \not\sqsubseteq_{\mathrm{reg}} \mathcal{F}(g)$. Suppose that $f \not\leqslant_Q g$, then there is a stationary set $S \subseteq \kappa$ such that for all $\alpha \in S$, $f(\alpha) \not\sqsubseteq g(\alpha)$. Because of WC^*_G we know that the set

$$A = \{\alpha \in \mathrm{reg}(\kappa) \mid g \restriction \alpha = f_\alpha \wedge S \cap \alpha \text{ is stationary}\}$$

is a stationary set. Therefore, for all $\alpha \in A$, $\mathcal{F}(g)(\alpha) = 0$, and for all $\beta \in S \cap \alpha$, $f(\beta) \not\sqsubseteq g(\beta)$. Since for all $\alpha \in A$, $g \restriction \alpha = f_\alpha$, and $S \cap \alpha$ is stationary, we conclude that $f \restriction \alpha \not\leqslant^\alpha_Q f_\alpha$ holds for all $\alpha \in A$. Hence, for all $\alpha \in A$, $\mathcal{F}(g)(\alpha) = 0$ and $\mathcal{F}(f)(\alpha) = 1$. We conclude that $A \subseteq (\mathcal{F}(f)^{-1}[1] \backslash \mathcal{F}(g)^{-1}[1]) \cap \mathrm{reg}(\kappa)$, and since A is stationary, $\mathcal{F}(f) \not\sqsubseteq_{\mathrm{reg}} \mathcal{F}(g)$. ⊣

Clearly $\mathcal{F} \circ F : Mod^\kappa_G \to 2^\kappa$ is a Borel reduction of $\sqsubseteq_G$ to $\sqsubseteq_{\mathrm{reg}}$. ⊣

Theorem 5.7. If κ is weakly ineffable, then $\sqsubseteq_{\mathrm{NS}}$ is Σ^1_1-complete.

Proof. Follows from Fact 5.3, Lemma 5.4, Fact 5.5, and Theorem 5.6. ⊣

Thus, the only case concerning [17, Q. 11.4] that is still open is the case where $V \neq L$ and κ is a weakly compact, but not weakly ineffable cardinal. For example the first weakly compact is such [3, Lemma 1.12]. For successor cardinals, we know from [6] that it can be forced the relation E^2_{NS} to be a Δ^1_1 equivalence relation. So it is consistently true that $\sqsubseteq_{\mathrm{NS}}$ is not Σ^1_1-complete.

5.2. Σ^1_1-completeness of $\cong_{\mathrm{DLO}}$ for weakly compact κ. In this section we prove:

THEOREM 5.8. Suppose that κ is weakly compact. Then the isomorphism relation on dense linear orders is Σ^1_1-complete.

Note that the isomorphism of linear orders is reducible to graph isomorphism, so $\cong_G$ is also Σ^1_1-complete for weakly compact κ. Before proving this, we first prove the following:

LEMMA 5.9. If κ is weakly compact, then the bi-embeddability of graphs $\approx_G$ is reducible to E^κ_{reg} (Definition 2.3).

PROOF. Let C_π be the club as in Definition 2.2 and for all $\alpha \in C_\pi$ define the relation $\approx^\alpha_G$ as follows. For all $\eta, \xi \in Mod^\kappa_G$, let $\eta \approx^\alpha_G \xi$, if $\mathcal{A}_{\eta\restriction\alpha}$ is embeddable in $\mathcal{A}_{\xi\restriction\alpha}$ and $\mathcal{A}_{\eta\restriction\alpha}$ is embeddable in $\mathcal{A}_{\xi\restriction\alpha}$.

There are at most κ many equivalence classes of $\approx^\alpha_G$, so let $g_\alpha\colon Mod^\kappa_G \to \kappa$ be a function with the property that for all $\eta, \xi \in Mod^\kappa_G$ we have $g_\alpha(\eta) = g_\alpha(\xi)$ if and only if $\eta \approx^\alpha_G \xi$.

Define the reduction $\mathcal{F}\colon Mod^\kappa_G \to \kappa^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} g_\alpha(\eta) & \text{if } \alpha \in C_\pi \\ 0 & \text{otherwise} \end{cases}$$

Let us show that if $\eta \approx_G \xi$, then $(\eta, \xi) \in E^\kappa_{\mathrm{reg}}$. Suppose that $\eta \approx_G \xi$. Then there are embeddings $F_1\colon \kappa \to \kappa$ and $F_2\colon \kappa \to \kappa$ from $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$, and from $\mathcal{A}_\xi$ to $\mathcal{A}_\eta$ respectively. Let D_1 and D_2 be the sets of closed points of F_1 and F_2 respectively. These are closed unbounded sets in κ. Then for all $\alpha \in D_1 \cap D_2 \cap C_\pi$, $\mathcal{A}_{\eta\restriction\alpha}$ and $\mathcal{A}_{\xi\restriction\alpha}$ are bi-embeddable. Hence for all $\alpha \in D_1 \cap D_2 \cap C_\pi$, $\mathcal{F}(\eta)(\alpha) = \mathcal{F}(\xi)(\alpha)$. We conclude that $(\eta, \xi) \in E^\kappa_{\mathrm{reg}}$.

Let us show that if $(\eta, \xi) \notin \approx_G$, then η and ξ are not E^κ_{reg}-equivalent. Suppose that $(\eta, \xi) \notin \approx_G$, without loss of generality, suppose that there is no embedding of $\mathcal{A}_\eta$ into $\mathcal{A}_\xi$. The property

There is no embedding of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$ $\wedge$ κ is regular $\wedge$ C_π is unbounded

is a Π^1_1-property of the structure $(V_\kappa, \in, A)$, where $A = (\eta \times \{0\}) \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\})$. Since κ is weakly compact, there are stationary many ordinals $\gamma < \kappa$ such that $C_\pi \cap \gamma$ is unbounded, $\gamma \in C_\pi$, γ is regular, and *there is no embedding of $\mathcal{A}_{\eta\restriction\gamma}$ to $\mathcal{A}_{\xi\restriction\gamma}$*. We conclude that there are stationary many points γ with $\mathcal{F}(\eta)(\gamma) \neq \mathcal{F}(\xi)(\gamma)$, hence η and ξ are not E^κ_{reg}-equivalent. ⊣

COROLLARY 5.10. If κ is weakly compact, then E^κ_{reg} is Σ^1_1-complete.

PROOF. Follows from Theorem 5.1 and Lemma 5.9. ⊣

Now we can prove Theorem 5.8:

PROOF OF THEOREM 5.8. By [1, Thm 3.9] we have $E^\kappa_{\mathrm{reg}} \leqslant_c \cong_{\mathrm{DLO}}$, so the result follows from Corollary 5.10 ⊣

REFERENCES

[1] D. Aspero, T. Hyttinen, V. Kulikov, and M. Moreno, *Reducibility of equivalence relations arising from non-stationary ideals under large cardinal assumptions*, ***Notre Dame Journal of Symbolic Logic***, to appear.

[2] James E Baumgartner, *A new class of order types*, ***Annals of Mathematical Logic***, vol. 9 (1976), no. 3, pp. 187–222.

[3] Harvey M Friedman, *Subtle cardinals and linear orderings*, ***Annals of Pure and Applied Logic***, vol. 107 (2001), no. 1-3, pp. 1–34.

[4] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov, *Generalized descriptive set theory and classification theory*, ***Memoirs of the American Mathematical Society***, vol. 230 (2014), no. 1081.

[5] ———, *On Borel reducibility in generalised Baire space*, ***Fundamenta Mathematicae***, (2015), no. 3, pp. 285–298.

[6] Sy-David Friedman, Liu zhen Wu, and Lyubomyr Zdomskyy, Δ_1-*-definability of the non-stationary ideal at successor cardinals*, ***Fundamenta Mathematicae***, vol. 229 (2015), no. 3, pp. 231–254.

[7] A Halko, *Negligible subsets of the generalized Baire space math formula*, ***Ph.D. thesis***, Ph. D. thesis, University of Helsinki, 1996.

[8] Alex Hellsten et al., ***Diamonds on large cardinals***, vol. 134, Suomalainen Tiedeakatemia, 2003, Thesis.

[9] Tapani Hyttinen and Vadim Kulikov, *On* Σ_1^1*-complete equivalence relations on the generalized Baire space*, ***Mathematical Logic Quarterly***, vol. 61 (2015), no. 1-2, pp. 66–81.

[10] ———, *Borel* sets in the generalized Baire space and infinitary languages*, ***Jaakko hintikka on knowledge and game-theoretical semantics*** (H. van Ditmarsch and G. Sandu, editors), Springer, 2018, pp. 395–412.

[11] Tapani Hyttinen, Vadim Kulikov, and Miguel Moreno, *A generalized Borel-reducibility counterpart of Shelah's main gap theorem*, ***Archive for Mathematical Logic***, vol. 56 (2017), no. 3-4, pp. 175–185.

[12] Tapani Hyttinen and Saharon Shelah, *Constructing strongly equivalent nonisomorphic models for unstable theories, part A*, ***The Journal of Symbolic Logic***, vol. 59 (1994), pp. 984–996.

[13] Tapani Hyttinen and Saharon Shelah, *Constructing strongly equivalent nonisomorphic models for unsuperstable theories. Part B*, ***The Journal of Symbolic Logic***, vol. 60 (1995), no. 4, pp. 1260–1272.

[14] Alexander Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 10 (1997), no. 1, pp. 215–242.

[15] Yurii Khomskii, Giorgio Laguzzi, Benedikt Löwe, and Ilya Sharankou, *Questions on generalised Baire spaces*, ***Mathematical Logic Quarterly***, vol. 62 (2016), no. 4-5, pp. 439–456.

[16] Miguel Moreno, *The isomorphism relation of theories with S-DOP*, ***Preprint, arXiv:1803.08070***.

[17] Luca Motto Ros, *The descriptive set-theoretical complexity of the embeddability relation on models of large size*, ***Annals of Pure and Applied Logic***, vol. 164 (2013), no. 12, pp. 1454–1492.

[18] Wenzhi Sun, *Stationary cardinals*, ***Archive for Mathematical Logic***, vol. 32 (1993), no. 6, pp. 429–442.

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