ON $\Sigma^1_1$-COMPLETENESS OF QUASI-ORDERS ON $\kappa^\kappa$

TAPANI HYTTINEN, VADIM KULIKOV, AND MIGUEL MORENO

Abstract. We prove under $V = L$ that the inclusion modulo the non-stationary ideal is a $\Sigma^1_1$-complete quasi-order in the generalised Borel-reducibility hierarchy ($\kappa > \omega$). This improvement to known results in $L$ has many new consequences concerning the $\Sigma^1_1$-completeness of quasi-orders and equivalence relations such as the embeddability of dense linear orders and equivalence modulo various versions of the non-stationary ideal. This serves as a partial or complete answer to several open problems stated in literature. Additionally the theorem is applied to prove a dichotomy in $L$: If the isomorphism of a countable first-order theory (not necessarily complete) is not $\Delta^1_1$, then it is $\Sigma^1_1$-complete.

We also study the case $V \neq L$ and prove $\Sigma^1_1$-completeness results for weakly ineffable and weakly compact $\kappa$.

§1. Introduction. We work in the setting of generalised descriptive set theory [4], GDST for short. The spaces $\kappa^\kappa = \{f : \kappa \to \kappa\}$ and $2^\kappa = \{f : \kappa \to 2\}$ are equipped with the bounded topology where the basic open sets are of the form $\{\eta \in \kappa^\kappa \mid \eta \supset p\}$, $p \in \kappa^{<\kappa}$. Borel sets are generated by $\kappa$-long unions and intersection of basic open sets. Notions of Borel-reducibility between equivalence relations and quasi-orders as well as Wadge-reducibility between sets are generalised accordingly.

In [4] a Lemma was introduced (a version of the Lemma and a detailed proof can be found in [9, Lemma 1.9 & Remark 1.10]) saying that if $V = L$, then any $\Sigma^1_1$ subset of $\kappa^\kappa$ can be Wadge-reduced to

\[\text{CLUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a } \mu\text{-club}, \quad \mu < \kappa \text{ regular},\}\]

where "$\mu$-club" is short for unbounded set closed under increasing sequences of length $\mu$. In [4] this was used to show that if $V = L$, then $\Sigma^1_1 = \text{Borel}^*$. In [9] the Wadge-reducibility result was strengthened by the first two authors of the present paper. It was shown (still in $L$) that every $\Sigma^1_1$-equivalence relation is Borel-reducible to the following equivalence relation on $\kappa^\kappa$:

\[E^\kappa_\mu = \{(\eta, \xi) \in (\kappa^\kappa)^2 \mid \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \text{ contains a } \mu\text{-club}\}.\]

We say that $E^\kappa_\mu$ is $\Sigma^1_1$-complete.

1991 Mathematics Subject Classification. 03E15, 03C45.

Key words and phrases. Generalized Baire spaces, Classification theory, Main gap, Quasi-orders.

During the preparation of this paper Vadim Kulikov was supported by the Academy of Finland through their grant WBS 1285203.

During the preparation of this paper Miguel Moreno was supported by Domast.

1
However, we would have wanted to show that the same equivalence relation on $2^\kappa$:

$$E_\mu^2 = \{ (\eta, \xi) \in (2^\kappa)^2 \mid \{ \alpha < \kappa \mid \eta(\alpha) = \xi(\alpha) \} \text{ contains a } \mu\text{-club} \}.$$  

has the same completeness property. The reason for this was that we knew many more equivalence relations to which $E_\mu^2$ can be Borel reduced than equivalence relations to which $E_\kappa^\mu$ can be Borel reduced. The corollaries of (1) and (2) were explored in [4, 9, 16]. In particular, the question “Is $E_\mu^\kappa \leq_B E_\mu^2$?” that was stated in [5, Q. 15] and re-stated in [15, Q. 3.46] was open (and it is still open in the general case). Of course if $E_\mu^2$ is $\Sigma_1^1$-complete the answer to this question is positive and in this paper we show that this is the case in $L$ (Theorem 4.2) by first proving the same result for quasi-orders (Theorem 3.1). Borel-reducibility between quasi-orders is a natural generalisation of reducibility between equivalence relations (see Section 2 for precise definitions).

We then prove a range of new results which are all consequences of Theorem 3.1. One of these is our main result: If $V = L$ then the isomorphism relation of any countable first-order theory (not necessarily complete) is either $\Delta_1^1$ or $\Sigma_1^1$-complete. This classification problem in Baire space was also studied in [11], the “Borel-reducibility counterpart of Shelah’s main gap theorem”. The other results are partial answers to [17, Q.’s 11.3 and 11.4] (which are re-stated as [15, Q’s 3.49 and 3.50]), [5, Q. 15] and a complete answer to [15, Q. 3.47].

These questions ask about the (consistency of) reducibility between relations of the form $E_\mu^\kappa$, quasi-orders of the form $\sqsubseteq_\mu$, quasi-orders of embeddability between linear orders as well as various isomorphism relations, where $\lambda \in \{2, \kappa\}$ and $\mu \in \text{reg}(\kappa)$. In particular, [17, Q. 11.4] asks whether the embeddability of dense linear orders $\sqsubseteq_{\text{DLO}}$ is a $\Sigma_1^1$-complete quasi-order for weakly compact $\kappa$. From those results that are described above it follows that $\sqsubseteq_{\text{DLO}}$ is $\Sigma_1^1$-complete in $L$ for all $\kappa$ that are not successors of an $\omega$-cofinal cardinal. In Section 5.1 we extend this to weakly ineffable cardinals (without the assumption $V = L$). Thus the only case in which [17, Q. 11.4] is still open is the case when $V \neq L$ and $\kappa$ is a weakly compact cardinal which is not weakly ineffable. In Section 5.2 we prove that the isomorphism of DLO, $\cong_{\text{DLO}}$, on $\kappa$ weakly compact is $\Sigma_1^1$-complete (here again, we do not assume $V = L$). The existence of $\Sigma_1^1$-complete equivalence relations has been previously known to hold in $L$ [10]. It is still unknown whether there exists a model of ZFC and $\kappa > \omega$ on which no isomorphism relation is $\Sigma_1^1$-complete. Given the present situation such a counterexample will have to satisfy both $V \neq L$ and $\kappa$ is not weakly compact. This is a strong contrast to the classical case $\kappa = \omega$ in which the isomorphism of graphs is strictly below the universal equivalence relation induced by a Borel action of a Polish group, and does not reduce even some simple Borel equivalence relations such as $E_1^1$ [14].

§2. Preliminaries and Definitions. In this section we define the notions and concepts we work with. Throughout this article we assume that $\kappa$ is an uncountable cardinal that satisfies $\kappa^{\lt \kappa} = \kappa$ which is a standard assumption in GDST. In this paper, however, this assumption is mostly redundant, because we

\footnote{Simple in the sense that it is low in the Borel-hierarchy, namely $\Sigma_0^1$.}
work either with strongly inaccessible $\kappa$ or under the assumption $V = L$. For sets $X$ and $Y$ denote by $X^Y$ the set of all functions from $Y$ to $X$. For ordinal $\alpha$ denote by $X^{<\omega}$ the set of all functions from any $\beta < \alpha$ to $X$. We work with the generalised Baire and Cantor spaces associated with $\kappa$ these being $\kappa^\kappa$ and $2^\kappa$ respectively, where $2 = \{0, 1\}$. The generalised Baire space $\kappa^\kappa$ is equipped with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set
$$\{ \eta \in \kappa^\kappa \mid \zeta \subset \eta \}$$
is a basic open set. The open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets. The collection of $\kappa$-Borel subsets of $\kappa^\kappa$ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length $\kappa$. A $\kappa$-Borel set is any element of this collection. In this paper we do not consider any other kind of Borel sets, so we always omit the prefix “$\kappa$-”. The subspace $2^\kappa \subset \kappa^\kappa$ (the generalised Cantor space) is equipped with the subspace topology. We will also work in the subspaces of the form $\text{Mod}_2^\kappa$ which are sets of codes for models with domain $\kappa$ of a first-order countable theory $T$. Special cases include $\text{Mod}_2^n$, $\text{Mod}_2^n$ and $\text{Mod}_2^{DLO}$ for graphs and dense linear orders respectively. These are Borel subspaces of $2^\kappa$. This enables us to view the quasi-order of embeddability of models, say $\sqsubseteq_{\text{DLO}}$, as a quasi-order on $2^\kappa$. In order to precisely define this, we have to introduce some notions.

The following is a standard way to code structures with domain $\kappa$ by elements of $\kappa^\kappa$ (see e.g. [4]). Suppose $\mathcal{L} = \{P_n \mid n < \omega\}$ is a countable relational vocabulary.

**Definition 2.1.** Fix a bijection $\pi : \kappa^{<\omega} \to \kappa$. For every $\eta \in 2^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain $\kappa$ as follows: For every relation $P_m$ with arity $n$, every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^\kappa$ satisfies
$$(a_1, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(m, a_1, \ldots, a_n) = 1.$$

Note that for every $\mathcal{L}$-structure $\mathcal{A}$ with $\text{dom}(\mathcal{A}) = \kappa$ there exists $\eta \in 2^\kappa$ with $\mathcal{A} = \mathcal{A}_\eta$. It is clear how this coding can be modified for a finite vocabulary. For club many $\alpha < \kappa$ we can also code the $\mathcal{L}$-structures with domain $\alpha$:

**Definition 2.2.** Denote by $C_\pi$ the club $\{ \alpha < \kappa \mid \pi[\alpha^{<\omega}] \subseteq \alpha \}$. For every $\eta \in 2^\kappa$ and every $\alpha \in C_\pi$ define the structure $\mathcal{A}_{\eta \upharpoonright \alpha}$ with domain $\alpha$ as follows: For every relation $P_m$ with arity $n$, every tuple $(a_1, a_2, \ldots, a_n)$ in $\alpha^n$ satisfies
$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta \upharpoonright \alpha}} \iff (\eta \upharpoonright \alpha)(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$ Note that for every $\alpha \in C_\pi$ and every $\eta \in 2^\kappa$ the structures $\mathcal{A}_{\eta \upharpoonright \alpha}$ and $\mathcal{A}_\eta \upharpoonright \alpha$ are the same.

Let us denote by $\text{Mod}_2^\kappa$ the subset of $2^\kappa$ consisting of those elements that code the models of a first-order countable theory (not necessarily complete). Abbreviate first-order countable theory as FOCT from now on. We will be interested in particular in $T = G$, the theory of graphs (symmetric and irreflexive) and $T = DLO$, the theory of dense linear orders without end-points. We consider $\text{Mod}_2^\kappa$ as a topological space endowed with the subspace topology. For more background on GDST see e.g. [4].

We can now define some central relations for this paper. A quasi-order is a transitive and reflexive relation.
Definition 2.3 (Relations). We will use the following relations.

**Isomorphism:** For a FOCT $T$, define
\[ \sim_T = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa | \eta, \xi \in \text{Mod}^\kappa_T, A_\eta \cong A_\xi \text{ or } \eta, \xi \notin \text{Mod}^\kappa_T\}. \]

**Embeddability:** For a FOCT $T$, define the quasi-order
\[ \sqsubseteq^\kappa = \sqsubseteq_T = \{(\eta, \xi) \in (\text{Mod}^\kappa_T)^2 | A_\eta \text{ is embeddable into } A_\xi\} \]

**Bi-embeddability:** For a FOCT $T$ and $\eta, \xi \in \text{Mod}^\kappa_T$, let
\[ \eta \sim_T \xi \iff \eta \sqsubseteq T \xi \land \xi \sqsubseteq T \eta. \]

**Inclusion mod NS:** For $\eta, \xi \in 2^\kappa$ and a stationary $S \subset \kappa$, we write $\eta \subseteq_s S \xi$ if $(\eta^{-1}\{1\} \setminus \xi^{-1}\{1\}) \cap S$ is non-stationary.

**Equivalence mod NS:** For every stationary $S \subset \kappa$ and $\lambda \in \{2, \kappa\}$, we define $E_S^\lambda$ as the relation
\[ E_S^\lambda = \{(\eta, \xi) \in \lambda^\kappa \times \lambda^\kappa | \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is not stationary}\}. \]

Note that $\eta E_S^\kappa\xi$ if and only if $\eta \subseteq_s S \xi \subseteq S \eta$. If $S$ is the set of all $\mu$-cofinal ordinals, denote $E_S^\lambda = E_{\mu}^\lambda$ and $\subseteq_s = \subseteq_{\mu}$. If $S$ is the set of all regular cardinals below $\kappa$, denote $S = \text{reg}(\kappa) = \text{reg}$ in which case $E_S^\lambda = E_{\text{reg}}^\lambda$ and $\subseteq_s = \subseteq_{\text{reg}}$. If $S = \kappa$, write $E_S^\lambda = E_{\text{NS}}^\lambda$ and $\subseteq_s = \subseteq_{\text{NS}}$.

Note that if we define $F: 2^\kappa \to 2^\kappa$ by
\[ F(\eta)(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \in S \\ 1 & \text{otherwise} \end{cases} \]
for a fixed $S \subset \kappa$, we obtain:

**Fact 2.4.** For all stationary $S \subset S'$ we have $\subseteq_s \leq_B \subseteq_{S'}$.

A quasi-order $Q$ on (a Borel set) $X \subset \kappa^n$ is $\Sigma^1_1$, if $Q \subset X^2$ is the projection of a closed set in $X^2 \times \kappa^n$ ($X$ is equipped with subspace topology and $X^2 \times \kappa^n$ with the product topology). All quasi-orders of Definition 2.3 (note that equivalence relations are quasi-orders) are $\Sigma^1_1$.

Suppose $X, Y \subset \kappa^n$ are Borel. A function $f: X \to Y$ is **Borel**, if for every open set $A \subset Y$ the inverse image $f^{-1}[A]$ is a Borel subset of $X$ with respect to the induced Borel structure on $X$ and $Y$.

If $Q_1$ and $Q_2$ are quasi-orders respectively on $X$ and $Y$, then we say that $Q_1$ is **Borel-reducible to** $Q_2$ if there exists a Borel map $f: X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 Q_1 x_2 \iff f(x_1) Q_2 f(x_2)$ and this is also denoted by $Q_1 \leq_B Q_2$. If $f$ is continuous (inverse image of an open set is open), then we say that $Q_1$ is **continuously reducible to** $Q_2$. Note that equivalence relations are quasi-orders, so this gives naturally a notion of reducibility for them as well.

A quasi-order is $\Sigma^1_1$-**complete**, if every $\Sigma^1_1$ quasi-order is Borel-reducible to it. An equivalence relation is $\Sigma^1_1$-**complete** if every $\Sigma^1_1$ equivalence relation is Borel-reducible to it.

A Borel equivalence relation $E$ on a Borel subspace $X \subset 2^\kappa$ can be extended to $2^\kappa$ by declaring all other elements equivalent to each other, but not equivalent to any of the elements in $X$. Similarly a quasi-order $\sqsubseteq$ on $X \subset 2^\kappa$ can be trivially extended to the whole space $2^\kappa$. If the original equivalence relation or quasi-order was $\Sigma^1_1$-complete, then so are the extensions.
§3. $\Sigma_1^1$-completeness of $\subseteq_S$ in $L$. This section is devoted to proving Theorem 3.1. In Section 4 a range of corollaries will be proved.

Theorem 3.1. $(V = L, \kappa > \omega)$ The quasi-order $\subseteq_{\mu}$ is $\Sigma_1^1$-complete, for every regular $\mu < \kappa$.

As mentioned in Introduction, this is an improvement to a theorem in [9] which says that $E^+_{\kappa}$ is $\Sigma_1^1$-complete.

Definition 3.2. We will need a version of the diamond principle.

- Let us define a class function $F_0: \mathrm{On} \to L$. For all $\alpha$, $F_0(\alpha)$ is a pair $(X_\alpha, C_\alpha)$ where $X_\alpha, C_\alpha \subseteq \alpha$, if $\alpha$ is a limit ordinal, then $C_\alpha$ is either a club or the empty set, and $C_\alpha = \emptyset$ when $\alpha$ is not a limit ordinal. We let $F_0(\alpha) = (X_\alpha, C_\alpha)$ be the $<L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if $\alpha$ is a limit ordinal and such pair exists and otherwise we let $F_0(\alpha) = (\emptyset, \emptyset)$.
- We let $C_\phi \subseteq \mathrm{On}$ be the class of all limit ordinals $\alpha$ such that for all $\beta < \alpha$, $F_\phi \subseteq \alpha \in L_{\beta}$. Notice that for every regular cardinal $\alpha$, $C_\phi \cap \alpha$ is a club.

Definition 3.3. For a given regular cardinal $\alpha$ and a subset $A \subseteq \alpha$, we define the sequence $(\langle X_\gamma, C_\gamma \rangle)_{\gamma \in A}$ to be $(F_0(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ to be the sequence of sets $X_\gamma$ such that $F_0(\gamma) = (X_\gamma, C_\gamma)$ for some $C_\gamma$.

Remark 3.4. It is known that if $\alpha$ and $\mu$ are regular cardinals such that $\mu < \alpha$, then the sequence $(X_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence (i.e. for all $Y \subseteq \alpha$, the set $\{ \gamma \in S^\alpha_\mu \mid Y \cap \gamma = X_\gamma \}$ is stationary). Notice that if $\beta \in C_\phi$, then for all $\gamma < \beta$, $X_\gamma \in L_\beta$.

By $\text{ZF}^-$ we mean $\text{ZFC} + (V = L)$ without the power set axiom. By $\text{ZF}^\circ$ we mean $\text{ZF}^-$ with the following axiom:

“For all regular cardinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in S^\alpha_\mu}$ is such that for all $\gamma < \alpha$, $F_0(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence.”

Whether or not $\text{ZF}^-$ proves $\text{ZF}^\circ$ is irrelevant for the present argument. We denote by $\text{Sk}(Y)^{L_\beta}$ the Skolem closure of $Y$ in $L_\beta$ under the definable Skolem functions.

Lemma 3.5. $(V = L)$ For any $\Sigma_1$-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, the following are equivalent for all $\eta \in 2^\mu$:

- $\varphi(\eta, x)$
- $S \setminus A$ is non-stationary, where $S = \{ \alpha \in S^\mu_\kappa \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha \}$ and $A = \{ \alpha \in C_\phi \cap \kappa \mid \exists \beta > \alpha (L_\beta \models \text{ZF}^\circ \land \varphi(\eta \upharpoonright \alpha, x \upharpoonright \alpha) \land r(\alpha)) \}$
- where $r(\alpha)$ is the formula “$\alpha$ is a regular cardinal”.

Remark 3.6. This Lemma is reminiscent of [9, Remark 1.10], but there is a big difference, because now $S$ depends on $\eta$ through the diamond-sequence. The proof in [9] is not applicable here.

Proof. Let $\mu < \kappa$ be a regular cardinal. Suppose that $\eta \in 2^\mu$ is such that $\varphi(\eta, x)$ holds. Let $\theta$ be a cardinal large enough such that $L_\theta \models \text{ZF}^\circ \land \varphi(\eta, x) \land r(\kappa)$. 

For each $\alpha < \kappa$, let 

$$H(\alpha) = \text{Sk}(\alpha \cup \{\kappa, \eta, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let 

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$ 

Then $D$ is a club set and $D \cap C_\beta$ is a club. Since $H(\alpha)$ is an elementary submodel of $L_\theta$ and the Mostowski collapse $\bar{H}(\alpha)$ is equal to $L_\beta$ for some $\beta > \alpha$, we have $D \cap C_\beta \subseteq A$.

Suppose $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Let $\theta$ be a large enough cardinal such that

$$L_\theta \models \text{ZF}^\circ \land \neg \varphi(\eta, x) \land r(\kappa).$$

Let $C$ be an unbounded set which is closed under $\mu$-limits (a $\mu$-club). Let 

$$H(\alpha) = \text{Sk}(\alpha \cup \{\kappa, C, \eta, x, (X_\gamma, C_\gamma)_{\gamma \in S_\mu}\})^{L_\theta}.$$ 

Let 

$$D = \{\alpha \in S_\mu^\kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$ 

Notice that since $H(\alpha)$ is an elementary substructure of $L_\theta$, then $H(\alpha)$ calculates all cofinalities correctly below $\alpha$. Then $D$ is an unbounded set, closed under $\mu$-limits. Let $S = \{\alpha \in S_\mu^\kappa \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and $\alpha_0$ be the least ordinal in $(\lim_\mu D) \cap S$ (where $\lim_\mu D$ is the set of ordinals of $D$ that are $\mu$-cofinal limits of elements of $D$). Since $\alpha_0 \in \lim_\mu D$, $\alpha_0 > \mu$. By the elementarity of each $H(\alpha)$ we conclude that $\alpha_0 \in C$.

Let $\beta$ be such that $L_\beta$ is equal to the Mostowski collapse of $H(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$, thus $\alpha_0 \in C_\beta \cap \kappa$. There exists $\beta > \alpha_0$ such that 

$$L_\beta \models \text{ZF}^\circ \land \varphi(\eta \mid \alpha_0, x \mid \alpha_0) \land r(\alpha_0).$$

Since $\varphi(\eta, x)$ is a $\Sigma_1$-formula, $\beta$ is a limit ordinal greater than $\bar{\beta}$.

Claim 3.6.1. $L_\beta$ satisfies the following:

1. For all $\gamma \in S \cap \alpha_0$, $\gamma$ has cofinality $\mu$.
2. $S \cap \alpha_0$ is a stationary subset of $\alpha_0$.
3. $D \cap \alpha_0$ is a $\mu$-club subset of $\alpha_0$.

Proof. 1. $H(\alpha_0)$ calculates all cofinalities correctly below $\alpha_0$. Thus $L_\beta$ calculates all cofinalities correctly below $\alpha_0$. Since $\beta$ is greater than $\bar{\beta}$, $L_\beta$ calculates all cofinalities correctly below $\alpha_0$. Since $S \cap \alpha_0 \subseteq S_\mu^\kappa$ in $L$, then $S \cap \alpha_0 \subseteq S_\mu^\kappa$ holds in $L_\beta$.

2. Since $\alpha_0 \in C_\beta \cap \kappa$ and $L_\beta$ satisfies $\text{ZF}^\circ$ and $r(\alpha_0)$, $L_\beta$ satisfies that $S \cap \alpha_0$ is a stationary subset of $\alpha_0$.

3. Let $\alpha < \alpha_0$ be such that $L_\beta \models \text{cf}(\alpha) = \mu \land \bigcup(D \cap \alpha) = \alpha$, we will show that $L_\beta \models \alpha \in D \cap \alpha_0$. Since $L_\beta$ calculates all cofinalities correctly below $\alpha_0$, $L \models \text{cf}(\alpha) = \mu \land \bigcup(D \cap \alpha) = \alpha$. $D$ is a $\mu$-club in $L$, thus $L \models \alpha \in D$. Since $\alpha < \alpha_0$, $L \models \alpha \in D \cap \alpha_0$. We will finish the proof by showing that $L \models \alpha \in D \cap \alpha_0$ implies $L_\beta \models \alpha \in D \cap \alpha_0$.

Notice that $H(\alpha_0)$ is a definable subset of $L_\theta$ and $D$ is a definable subset of $L_\theta$. By elementarity, $D \cap \alpha_0$ is a definable subset of $H(\alpha_0)$, we conclude
that $D \cap \alpha_0$ is a definable subset of $L_\beta$ and $D \cap \alpha_0 \in L_\beta$. Therefore $L_\beta \models \alpha \in D \cap \alpha_0$.  

Since $L_\beta \models (\lim_\mu D \cap \alpha_0)$, by the previous claim we concluded that $L_\beta$ satisfies “$\lim_\mu D \cap \alpha_0$ is a $\mu$-club”. Since $S \cap \alpha_0$ is a stationary subset of $\alpha_0$ in $L_\beta$, we conclude that

$L_\beta \models (\lim_\mu D \cap \alpha_0) \cap S \cap \alpha_0 \neq \emptyset$,

so $L \models (\lim_\mu D \cap \alpha_0) \cap S \cap \alpha_0 \neq \emptyset$.

This contradicts the minimality of $\alpha_0$.  

Now we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Suppose $Q$ is a $\Sigma^1_1$ quasi-order on $\kappa^\kappa$. Let $a : \kappa^\kappa \to 2^{\kappa \times \kappa}$ be the map defined by

$$a(\eta)(\alpha, \beta) = 1 \Leftrightarrow \eta(\alpha) = \beta.$$  

Let $b$ be a continuous bijection from $2^{\kappa \times \kappa}$ to $2^\kappa$, and $c = b \circ a$. Define $Q'$ by

$$(\eta, \xi) \in Q' \Leftrightarrow (\eta = \xi) \vee (\eta, \xi \in \text{ran}(c) \cap \{c^{-1}(\eta), c^{-1}(\xi)\} \in Q).$$

So $c$ is a continuous reduction of $Q$ to $Q'$, and $Q'$ is a $\Sigma^1_1$ quasi-order because it is a continuous image of $Q$. We can assume, without loss of generality, that $Q$ is a quasi-order on $2^\kappa$.

There is a $\Sigma^1_1$-formula of set theory $\psi(\eta, \xi) = \psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x) \supset \xi = \xi$ with $x \in 2^\kappa$, such that for all $\eta, \xi \in 2^\kappa$,

$$(\eta, \xi) \in Q \Leftrightarrow \psi(\eta, \xi),$$

we added $\eta = \xi$ to $\psi(\eta, \xi)$, to ensure that when we reflect $\psi(\eta \upharpoonright \alpha, \xi \upharpoonright \alpha)$ we get a reflexive relation. Let $r(\alpha)$ be the formula “$\alpha$ is a regular cardinal” and $\psi^Q(\kappa)$ be the sentence with parameter $\kappa$ that asserts that $\psi(\eta, \xi)$ defines a quasi-order on $2^\kappa$. For all $\eta \in 2^\kappa$ and $\alpha < \kappa$, let

$$T_{\eta, \alpha} = \{ p \in 2^\kappa \mid \exists \beta > \alpha (L_\beta \models ZF^c \land \psi(p, \eta \upharpoonright \alpha, x \upharpoonright \alpha) \land r(\alpha) \land \psi^Q(\alpha)) \}.$$  

Let $(X_\alpha)_{\alpha \in S^\kappa_\mu}$ be the diamond sequence of Definition 3.3, and for all $\alpha \in S^\kappa_\mu$, let $X_\alpha$ be the characteristic function of $X_\alpha$. Define $F : 2^\kappa \to 2^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} 
1 & \text{if } X_\alpha \in T_{\eta, \alpha} \text{ and } \alpha \in S^\kappa_\mu \\
0 & \text{otherwise}
\end{cases}.$$  

CLAIM 3.6.2. If $\eta \in Q \xi$, then $T_{\eta, \alpha} \subseteq T_{\xi, \alpha}$ for club-many $\alpha$'s.

PROOF. Suppose $\psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x)$ holds and let $k$ witnesses that. Let $\theta$ be a cardinal large enough such that $L_\theta \models ZF^c \land \varphi(k, \eta, \xi, x) \land r(\alpha)$. For all $\alpha < \kappa$ let $H(\alpha) = \text{Sk}(\alpha \cup \{ \kappa, k, \eta, \xi, x \})$. The set $D = \{ \alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha \land H(\alpha) \models \psi^Q(\alpha) \}$ is a club. Using the Mostowski collapse we have that

$$D' = \{ \alpha < \kappa \mid \exists \beta > \alpha (L_\beta \models ZF^c \land \varphi(k \upharpoonright \alpha, \eta \upharpoonright \alpha, \xi \upharpoonright \alpha, x \upharpoonright \alpha) \land r(\alpha) \land \psi^Q(\alpha)) \}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta, \alpha}$ we have that

$$\exists \beta_1 > \alpha (L_{\beta_1} \models ZF^c \land \psi(p \upharpoonright \alpha, x \upharpoonright \alpha) \land r(\alpha) \land \psi^Q(\alpha)).$$
Since for all $\eta \in \eta Q \xi$, therefore, for $\eta = \max\{\beta_1, \beta_2\}$ we have that $L_\beta \models \text{ZF}^\circ \wedge \psi(\eta \mid \alpha, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$.

Since $\psi^Q(\alpha)$ holds and so transitivity holds for $\psi(\eta, \xi)$ in $L_\beta$, we conclude that $L_\beta \models \text{ZF}^\circ \wedge \psi(p, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$.

so $p \in T_\xi, \alpha$ and $T_\eta, \alpha \subseteq T_\xi, \alpha$. This holds for all $\alpha \in D'$.

By the previous claim, we conclude that if $\eta \in \eta Q \xi$, then there is a $\mu$-club $C$ such that for every $\alpha \in C$ it holds that $X_\alpha \in T_\eta, \alpha \Rightarrow \chi_\alpha \in T_\xi, \alpha$. Therefore $(F(\eta)^{-1}\{1\} \setminus F(\xi)^{-1}\{1\}) \cap C = \emptyset$, and $F(\eta) \subseteq F(\xi)$.

For the other direction, suppose $\neg \psi(\eta, \xi, x)$ holds. Let $S = \{\alpha \in S_\mu^\circ \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$. Since $(X_\alpha)_{\alpha \in S_\mu^\circ}$ is a diamond sequence, $S$ is a stationary set. By Lemma 3.5 we know that $S \setminus A$ is stationary, where $A = \{\alpha \in C_\circ \cap \kappa \mid \exists \beta > \alpha \in \text{ZF}^\circ \wedge \psi(\eta \mid \alpha, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha)\}$.

Since for all $\alpha \in S \setminus A$ we have that $X_\alpha = \eta^{-1}\{1\} \cap \alpha$, so $X_\alpha \in T_\eta, \alpha$. We conclude that for all $\alpha \in S \setminus A, F(\eta)(\alpha) = 1$. On the other hand, for all $\alpha \in S \setminus A$ it holds that

\[ \forall \beta > \alpha \in \text{ZF}^\circ \wedge \psi(\eta \mid \alpha, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha) \]

so

\[ \forall \beta > \alpha \in \text{ZF}^\circ \wedge \psi(X_\alpha, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha). \]

Therefore

\[ \forall \beta > \alpha \in \text{ZF}^\circ \wedge \psi(X_\alpha, \xi \mid \alpha, x \mid \alpha) \wedge r(\alpha) \]

we conclude that $X_\alpha \notin T_\xi, \alpha$, and $F(\xi)(\alpha) = 0$. Hence, for all $\alpha \in S \setminus A, F(\eta)(\alpha) = 1$ and $F(\xi)(\alpha) = 0$. Since $S \setminus A$ is stationary, we conclude that $F(\eta)^{-1}\{1\} \setminus F(\xi)^{-1}\{1\}$ is stationary and $F(\eta) \subseteq F(\xi)$.

§4. Corollaries of Theorem 3.1.

4.1. $\Sigma_1^\circ$-completeness of $E_\mu^2$ in $L$.

**Theorem 4.1** ($V = L, \kappa > \omega$). $\subseteq_{\text{NS}}$ is a $\Sigma_1^\circ$-complete quasi-order.

**Proof.** Follows from Fact 2.4 and Theorem 3.1.

**Theorem 4.2** ($V = L$). Let $\mu$ be regular cardinal below $\kappa$, then $E_\mu^2$ is a $\Sigma_1^\circ$-complete equivalence relation.

**Proof.** This follows from Theorem 3.1, because $E_\mu^2$ is a symmetrisation of the quasi-order $\subseteq_{\mu}$.

The above result is not true in ZFC. It was shown in [4, Thm 56] that if $\kappa$ is not a successor of a singular cardinal, then in a cofinality preserving forcing extension $E_{\mu_1}$ and $E_{\mu_2}$ are $\leq_B$-incomparable for regular cardinals $\mu_1 < \mu_2 < \kappa$.

Theorem 4.1 gives consistently a positive answer to “Given a weakly compact cardinal $\kappa$, is $\subseteq_{\text{NS}}$ complete?” [17, Q. 11.4]. Theorem 4.2 answers the questions “Is it consistently true that $E_\mu^2 \leq_B E_\lambda^2$ for $\lambda < \mu$?” [4, 15, Q. 3.47] (take
\( \lambda = \omega, \mu = \omega_1 \) and \( \kappa = \omega_2 \), and gives consistently a positive answer to “Is \( E^\omega_2 \) Borel-reducible to \( E^2_\mu \) for a regular \( \mu \)?” [5, Q. 15], [15, Q. 3.46].

4.2. \( \Sigma_1^1 \)-completeness of \( \mathbb{DLO} \) in \( L \). [17, Q. 11.3] asks “Given a weakly compact cardinal \( \kappa \), is \( \mathbb{DLO} \) complete for \( \Sigma_1^1 \) quasi-orders? What about arbitrary regular cardinals \( \kappa \)?” In this section we apply Theorem 3.1 to show that the answer is positive if \( V = L \). To do that we first have to establish a general theorem about \( \mathbb{DLO} \):

**Theorem 4.3.** Suppose that for all \( \lambda < \kappa \) we have \( \lambda^\omega < \kappa \). Then there is a continuous reduction of \( \mathbb{DLO} \) to \( \mathbb{DLO} \).

**Proof.** We will first define a continuous function \( G: \mathcal{P}(\kappa) \to \mathcal{P}(\kappa) \) with the following properties for every \( A, B \subset \mathcal{P}(\kappa) \):

(G1) if \( A \subseteq_\omega B \), then there exists a continuous \( f: \kappa \to \kappa \) such that \( f[\mathcal{G}(A)] \subset \mathcal{G}(B) \)

(G2) if \( A \not\subseteq_\omega B \), then \( \mathcal{G}(A) \not\subseteq_\omega \mathcal{G}(B) \).

**Claim 4.3.1.** A function \( G \) as above exists.

**Proof.** Fix an \( \omega \)-club \( C \subset \kappa \) with the property that for all \( \alpha < \kappa \) and all \( \beta < \kappa \) there exists \( \gamma \) with \( \beta < \gamma < \kappa \) such that \( \gamma, \gamma + \alpha \cap C = \emptyset \), where \( \gamma, \gamma + \alpha = \{ \delta < \kappa \mid \gamma \leq \delta \leq \gamma + \alpha \} \), thus \( C \) is in a sense “sparse”. For \( A \subset \kappa \), let \( \mathcal{G}(A) = (A \cap C) \cup (\kappa \setminus C) \).

Let us show that then \( G \) is as needed. It is easy to see that it is continuous, because if \( A \cap C = A' \cap C \), then clearly \( \mathcal{G}(A) \cap C = \mathcal{G}(A') \cap C \) and vice versa for a club of \( \alpha \)'s. Suppose \( A \not\subset B \) is non-stationary. Let \( C' \) be a club such that \( A \cap C' \subset B \) and let \( D = C \cap C' \). Then define \( f: \kappa \to \kappa \) inductively as follows. Let \( \alpha_0 \) be the smallest ordinal in \( D \), find \( \gamma_0 > \alpha_0 \) such that \( \gamma_0, \gamma_0 + a_0 \cap C = \emptyset \) and let \( f \mid \alpha_0 \) be defined by \( f(\alpha) = \gamma_0 + \alpha \) for all \( \alpha < \alpha_0 \). Suppose that a sequence \( (\alpha_\pi)_{\pi \leq \pi'} \) has been defined as well as a sequence \( (\gamma_\pi)_{\pi \leq \pi'} \) such that for all \( \pi < \pi' \) we have

\[
(3) \gamma_\pi < \alpha_{\pi' + 1} < \gamma_{\pi' + 1}
\]

and \( f \mid (\alpha_{\pi' + 1}) \) is defined. Then let \( \alpha_{\pi' + 1} > \gamma_{\pi'} \) to be an element of \( D \), pick \( \gamma_{\pi' + 1} > \alpha_{\pi' + 1} \) to be such that \( \gamma_{\pi' + 1}, \gamma_{\pi' + 1} + \alpha_{\pi' + 1} \cap C = \emptyset \) and define \( f(\alpha) = \gamma_{\pi' + 1} + \alpha \) for all \( \alpha \in [\alpha_{\pi' + 1}, \alpha_{\pi' + 1}] \). Suppose that \( \pi' \) is a limit ordinal and \( (\alpha_{\pi' + 1})_{\pi < \pi'}, (\gamma_{\pi})_{\pi < \pi'} \) are defined and \( f(\alpha) \) is defined for all \( \alpha < sup_{\pi < \pi'} \alpha_\pi \). From (3) it follows that, \( \alpha_{\pi' + 1} = sup_{\pi < \pi'} \alpha_\pi \) and for all \( \alpha < \alpha_{\pi'} \) we have \( \alpha < f(\alpha) < \alpha_{\pi'} \). We also have that \( \alpha_{\pi' + 1} \in D \), because it is a limit of elements of \( D \), so we can now define \( f(\alpha_{\pi' + 1}) = \alpha_{\pi' + 1} \). In this way \( f \) is continuous and for every \( \alpha < \kappa \) we have either \( f(\alpha) \in \kappa \setminus C \) or \( \alpha \in D \) and \( f(\alpha) = \alpha \).

In both cases, if \( \alpha \in G(A), \text{ then } f(\alpha) \in G(B) \) for (G2), assume that \( A \not\subseteq \omega B \) and let \( S \subset A \setminus B \) be \( \omega \)-stationary. But then \( S \cap C \) is \( \omega \)-stationary and \( S \cap C \subset G(A) \setminus G(B) \).

For every \( p, q \in \kappa \leq \omega \) define \( p < q \) if either \( p \supset q \) or there is smallest such \( n < \omega \) that \( p(n) \neq q(n) \) and for this \( n \) we have \( p(n) < q(n) \). This defines a linear order on the set \( C(\kappa) \) of all strictly increasing functions \( p \in \kappa \leq \omega \). Let \( C^*(\kappa) \) be the set of all strictly increasing functions \( p \in \kappa \leq \omega \) whose range contains at least
one infinite ordinal. This ensures that none of our linear orders has an end-point.
Also let \( \lim^*(\kappa) \) be the set of all limit ordinals below \( \kappa \) except \( \omega \).

Now for \( A \in \mathcal{P}(\lim^*(\kappa)) \) define the linear order \( L(A) \) to be the set
\[
\{ p \in C^*(\kappa^{<\omega}) \mid \text{dom} \; p = \omega \text{ and } \sup \text{ran} \; p \in A \text{ and } p(0) = 0 \}
\]
equipped with the order \( \prec \). This is a modification of a construction given by
Baumgartner \([2]\). We will show that \( A \mapsto L(G(A)) \) is the desired reduction,
where \( G \) is as was proved to exist in Claim 4.3.1.

If \( f : \kappa \rightarrow \kappa \) is continuous and strictly increasing and \( A \subseteq \lim^*(\kappa) \) any set, it
is clear from the definition of \( L(A) \) that
\[
f[L(A)] = \{ f \circ p \mid p \in L(A) \} \subseteq L(f[A]).
\]
Thus, if \( A \subseteq_\omega B \) and \( f : \kappa \rightarrow \kappa \) is continuous such that \( f[G(A)] \subseteq G(B) \) (as
guaranteed by \((G1)\)), then \( p \mapsto f \circ p \) defines an embedding from \( L(G(A)) \) to
\( L(G(B)) \).

The other direction is essentially a simplification of the proof of Baumgartner
Theorem 5.3(ii) \([2]\). If \( A \subseteq_\omega B \), then by \((G2)\) also \( G(A) \subseteq_\omega G(B) \) and so \( G(A) \setminus
G(B) \) is \( \omega \)-stationary. So it is sufficient to show that that for any unbounded \( A, B \subseteq \lim^*(\kappa) \), if \( A \setminus B \) is \( \omega \)-stationary, then \( L(A) \) cannot be embedded into
\( L(B) \). Notice that \( L(A) = L(A \cap \lim^*(\kappa)) \).

So suppose that \( A \setminus B \) is stationary and assume towards a contradiction that
\( h : L(A) \rightarrow L(B) \) preserves the ordering \( \prec \). For any \( X \subseteq C^*(\kappa^{<\omega}) \), let \( T(X) =
\{ p \in C^*(\kappa^{<\omega}) \mid \exists q \in X (p \prec q) \} \). Note that for every strictly increasing \( p \in \kappa^{<\omega} \)
with \( p(0) = 0 \), we have \( p \in T(L(A)) \) and \( p \in T(L(B)) \). Let \( g \in T(L(B)) \), let
\[
\text{Right}(g) = \{ f \in L(A) \mid h(f) = g \text{ or } g \prec h(f) \},
\]
\[
\text{Left}(g) = \{ f \in L(A) \mid h(f) \prec g \}.
\]
Let
\[
\rho(g) = \{ f' \in T(\text{Right}(g)) \mid \text{for all } g' \in T(\text{Right}(g)), \text{ if } g' \prec f', \text{ then } f' \subset g' \},
\]
\[
\lambda(g) = \{ f' \in T(\text{Left}(g)) \mid \text{for all } g' \in T(\text{Left}(g)), \text{ if } f' \prec g', \text{ then } g' \subset f' \}.
\]
Note that \( \rho(g) \) and \( \lambda(g) \) are linearly ordered by \( \subset \). Now let \( C \subseteq \lim^*(\kappa) \) be the
set of all \( \alpha \in \lim^*(\kappa) \) satisfying
1. for all \( f \in L(A) \), \( \sup \text{ran} (f) < \alpha \iff \sup \text{ran} (h(f)) < \alpha \),
2. \( A \cap \alpha \) is unbounded in \( \alpha \),
3. if \( g \in T(L(B)) \) and \( \sup \text{ran}(g) < \alpha \), then \( \sup \{ \sup \text{ran}(f) \mid f \in \rho(g) \} < \alpha \)
and \( \sup \{ \sup \text{ran}(f) \mid f \in \lambda(g) \} < \alpha \),
4. if \( g \in T(L(B)) \), \( f \in T(L(\text{Left}(g))) \), \( \sup \text{ran}(g) \), \( \sup \text{ran}(f) < \alpha \), and there exists \( f' \in \text{Left}(g) \) such that \( f \prec f' \) and \( f' \not\subset f \), then there exists such an \( f' \) with
\( \sup \text{ran}(f') < \alpha \),
5. if \( g \in T(L(B)) \), \( f \in T(\text{Right}(g)) \), \( \sup \text{ran}(g), \sup \text{ran}(f) < \alpha \), and there exists \( f' \in \text{Right}(g) \) such that \( f \prec f' \) and \( f' \not\subset f \), then there exists such an \( f' \) with
\( \sup \text{ran}(f') < \alpha \),

Also assume w.l.o.g. that \( C \subseteq \lim^*(\kappa) \). Our cardinality assumption on \( \kappa \)
guarantees that \( C \) is a club. We will show that \( C \cap A \subseteq B \) which is a
contradiction. Let \( \alpha \in C \cap A \) and let \( f \in L(A) \) be such that \( \sup \text{ran}(f) = \alpha \). We will
show that \( \sup \text{ran}(h(f)) = \alpha \) and so \( h(f) \in L(B) \) and \( \alpha \in B \). Suppose not. If
sup ran(h(f)) < α, then by (i), sup ran(f) < α which is a contradiction. So we can assume that sup ran(h(f)) > α. Because we assumed that ρ(0) = 0 for all functions in question, there is n₀ < ω such that h(f)(n₀) < α ≤ h(f)(n₀ + 1).
Let
\[ g = h(f)↾(n₀ + 1). \]
In particular sup ran(g) < α (**). For every m < ω, pick α_m ∈ A such that f(m) < α_m < α. Such α_m exists by (ii). Now for each m fix f_m with sup ran(f_m) = α_m and f_m ↾ f ↾ (m + 1). We have two cases: either (A) sup{m < ω | f_m ∈ Left(g)} = ω or (B) sup{m < ω | f_m ∈ Right(g)} = ω. We will show that both (A) and (B) lead to a contradiction.

Let us start with (A) and suppose that there are infinitely many m < ω with f_m ∈ Left(g).

**CLAIM 4.3.2.** For all m < ω we have f ↾ (m + 1) ∈ λ(g).

**PROOF.** For every m, there is m’ > m such that f_m’ ∈ Left(g) and since f ↾ (m + 1) ⊂ f ↾ (m’ + 1) ⊂ f_m’, we have that f ↾ (m + 1) ∈ T(Left(g)). Suppose that f ↾ (m + 1) ∉ λ(g) for some m. Then by the definition of λ(g), there exists g’’ ∈ T(Left(g)) such that f ↾ (m + 1) ≺ g’’, but g’’ ∉ f ↾ (m + 1), so there exists n < m + 1 with g’’(n) > f(n) and n is the smallest such that g’’(n) ≠ f(n). This g’’ can be extended to g’ in Left(g) and by (iv) we can assume that sup ran(g’) < α. The number n witnesses that f ≺ g’ and so we must have h(f) ≺ h(g’). The latter implies that for the first n’ < ω with h(f)(n’) ≠ h(g’)(n’) we have h(g’)(n’) > h(f)(n’). If n’ > n₀ (n₀ is defined at (*) above) then sup ran(h(g’)) ≥ h(g’)(n’) > h(f)(n’) ≥ α, a contradiction. So n’ ≤ n₀ and h(g’)(n’) > h(f)(n’) = g(n’), so we have g ≺ h(g’). But this implies that g’ ∈ Right(g) which is a contradiction again. This proves the claim. □

Now \{ f ↾ (m + 1) | m < ω \} ⊂ λ(g) and since sup ran(f) = α we have sup{sup ran(k) | k ∈ λ(g)} = α contradicting (iii) above. This shows that (A) leads to a contradiction.

Assume (B) i.e. suppose that there are infinitely many m < ω with f_m ∈ Right(g).

**CLAIM 4.3.3.** For all m < ω we have f ↾ (m + 1) ∈ ρ(g).

**PROOF.** For every m, there is m’ > m such that f_m’ ∈ Right(g) and since f ↾ (m + 1) ⊂ f ↾ (m’ + 1) ⊂ f_m’, we have that f ↾ (m + 1) ∈ T(Right(g)). Suppose that f ↾ (m + 1) ∉ ρ(g) for some m. Then by the definition of ρ(g), there exists g’’ ∈ T(Right(g)) such that g’’ ≺ f ↾ (m + 1), but f ↾ (m + 1) ∉ g’’, so there exists n < m + 1 with g’’(n) < f(n) and n is the smallest such that g’’(n) ≠ f(n). This g’’ can be extended to g’ in Right(g) and by (v) we can assume that sup ran(g’) < α. The number n witnesses that g’ ≺ f and so we must have h(g’) < h(f). The latter implies that for the first n’ < ω with h(f)(n’) ≠ h(g’)(n’) we have h(g’)(n’) < h(f)(n’). If n’ > n₀ (n₀ is defined at (*) above), then g ≺ h(g’) and h(g’) < g which is a contradiction with g’ ∈ Right(g).

So n’ ≤ n₀ and so h(g’)(n’') < h(f)(n’') = g(n’’), so we have g ≺ h(g’) and again this implies that g’ ∈ Left(g), contradiction. This proves the claim. □
Now \( \{ f \mid (m+1) \mid m < \omega \} \subset \rho(g) \) and since \( \sup \text{ran}(f) = \alpha \) we have \( \sup \{ \sup \text{ran}(k) \mid k \in \rho(g) \} \geq \alpha \) contradicting (iii) above. This shows that (B) leads to a contradiction too.

**Theorem 4.4 (\( V = L \)).** If \( \kappa > \omega \) is a regular cardinal which is not the successor of an \( \omega \)-cofinal cardinal, then \( \sqsubseteq_{\text{DLO}} \) is \( \Sigma_1 \)-complete.

**Proof.** By Theorem 3.1 it is sufficient to reduce \( \sqsubseteq \) to \( \sqsubseteq_{\text{DLO}} \). But since \( V = L \) every cardinal \( \kappa > \omega \) which is not the successor of an \( \omega \)-cofinal cardinal satisfies the assumption of Theorem 4.3.

**4.3. Dichotomy for countable first-order theories in \( L \).** In [11] it was proved that if \( V = L \), \( \kappa \) is a successor of a uncountable regular cardinal \( \lambda \), then \( \equiv_T \leq_c \equiv_{T_1} \) and \( \equiv_{T_2} \not\leq_B \equiv_{T_1} \) holds for all \( T_1 \) classifiable and \( T_2 \) non-classifiable. This result can be improved using Theorem 4.2 together with some results from [4]:

**Theorem 4.5.** ([4, Thm 86]) Suppose that for all \( \gamma < \kappa \), \( \gamma^\omega < \kappa \) and \( T \) is a stable unsuperstable complete countable theory. Then \( E^2_\omega \leq_c \equiv_T \).

**Corollary 4.6 (\( V = L \)).** Suppose that \( \kappa \) is regular and not the successor of an \( \omega \)-cofinal cardinal \( \lambda \) and \( T \) is a stable unsuperstable complete countable theory. Then \( \equiv_T \) is a \( \Sigma_1 \)-complete relation.

**Proof.** Follows from Theorems 4.5 and 4.2 and GCH in \( L \).

**Theorem 4.7.** ([4, Thm 79]) Suppose that \( \kappa = \lambda^+ = 2^\lambda \) and \( \lambda^{<\lambda} = \lambda \).
1. If \( T \) is complete unstable or superstable with OTOP, then \( E^2_\lambda \leq_c \equiv_T \).
2. If \( \lambda \geq 2^\omega \) and \( T \) is complete superstable with DOP, then \( E^2_\lambda \leq_c \equiv_T \).

**Corollary 4.8 (\( V = L \)).** Suppose that \( \kappa \) is the successor of a regular uncountable cardinal \( \lambda \). If \( T \) is a non-classifiable complete countable theory, then \( \equiv_T \) is a \( \Sigma_1 \)-complete relation.

**Proof.** Follows from Theorems 4.2, 4.5, and 4.7.

By using yet another Theorem from [4] we obtain the following dichotomy in \( L \). The class of \( \Delta_1^1 \) sets consists of sets \( A \) such that both \( A \) and the complement of \( A \) are \( \Delta_1^1 \) [4].

**Theorem 4.9 (\( V = L \)).** Suppose that \( \kappa \) is the successor of a regular uncountable cardinal \( \lambda \). If \( T \) is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:
- \( \equiv_T \) is \( \Delta_1^1 \).
- \( \equiv_T \) is \( \Sigma_1^1 \)-complete.

**Proof.** For this proof it is useful to bare in mind how the isomorphism relation of a theory is defined, Definition 2.3. Sometimes in literature it is defined differently, but these are mutually Borel bi-reducible.

It has been shown [4, Thm 70] that if a complete theory \( T \) is classifiable, then \( \equiv_T \) is \( \Delta_1^1 \). So for a complete countable theory \( T \) the result follows from Corollary 4.8. Suppose \( T \) is not a complete theory. Let \( \mathcal{L} \) be the vocabulary of \( T \) and \( \{ T_\alpha \}_{\alpha < 2^\omega} \) be the set of all the complete theories in \( \mathcal{L} \) that extend \( T \).
Notice that $\cong_T = \bigcap_{\nu < \kappa} \cong_{T_\nu}$, therefore if $\cong_{T_\alpha}$ is a $\Delta^1_1$ equivalence relation for all $\alpha < \kappa$, then so is $\cong_T$ since $2^\omega < \kappa$.

Suppose $T'$ is a complete countable theory in $\mathcal{L}$ that extends $T$ such that $\cong_{T'}$ is not a $\Delta^1_1$ equivalence relation. Then $T'$ is a non-classifiable countable theory. By Corollary 4.8 $\cong_{T'}$ is a $\Sigma^1_1$-complete equivalence relation. We will show that $\cong_{T'} \leq_B \cong_T$ which finishes the proof. Define $\mathcal{F} : \kappa^\kappa \to \kappa^\kappa$ by

$$\mathcal{F}(\eta) = \begin{cases} \eta & \text{if } A_\eta \models T' \\ \xi & \text{otherwise.} \end{cases}$$

where $\xi$ is a fixed element of $\kappa^\kappa$ such that $A_\xi \not\models T'$. Since $T'$ extends $T$, $\eta \cong_T \zeta \Leftrightarrow \mathcal{F}(\eta) \cong_T \mathcal{F}(\zeta)$. To show that $\mathcal{F}$ is Borel, note that

$$\mathcal{F}^{-1}([\eta \upharpoonright \alpha]) = \begin{cases} \{\eta \upharpoonright \alpha\} \setminus \{\zeta \mid A_\zeta \not\models T'\} & \text{if } \xi \not\in [\eta \upharpoonright \alpha] \\ \{\zeta \mid A_\zeta \not\models T'\} \cup \{\eta \upharpoonright \alpha\} & \text{if } \xi \in [\eta \upharpoonright \alpha]. \end{cases}$$

Since $[\eta \upharpoonright \alpha]$ is a basic open set and $\{\zeta \mid A_\zeta \not\models T'\}$ is a Borel set, $[\eta \upharpoonright \alpha] \setminus \{\zeta \mid A_\zeta \not\models T'\}$ and $[\eta \upharpoonright \alpha] \cup \{\zeta \mid A_\zeta \not\models T'\}$ are Borel sets.

The dichotomy of Theorem 4.9 is not provable in ZFC. In [12, 13] it was shown, assuming $\kappa$ is a successor and $\kappa \in I[\kappa]$, that there is a stable unsuperstable countable theory $T$ in a countable vocabulary such that $\cong_T$ is Borel* (a generalisation of Borel sets to non-well-founded trees [4, 7]). Because of this, $\cong_T$ cannot be a $\Sigma^1_1$-complete equivalence relation, unless Borel* = $\Sigma^1_1$ and the fairly mild combinatorial assumptions mentioned above still hold. In $L$ it holds that Borel* = $\Sigma^1_1$ [9], but there is a model of ZFC in which $\Delta^1_1 \subsetneq \text{Borel}^* \subsetneq \Sigma^1_1$ [10]. In this model $E^2_2$ is not $\Delta^1_1$ and we still have $\kappa \in I[\kappa]$, so by Theorem 4.5 $\cong_T$ is neither $\Delta^1_1$ nor $\Sigma^1_1$-complete.

§5. The case $V \neq L$.

5.1. $\Sigma^1_1$-completeness of $\subseteq_{\text{NS}}$ for weakly ineffable $\kappa$. In Section 4 we answered in $L$ the questions [15, Q. 3.47], [17, Q.’s 11.3 and 11.4] and [5, Q. 15]. We used Theorem 4.2 as the starting point. But what if $V \neq L$? In this section we provide further partial answers to [17, Q.’s 11.3 and 11.4] outside of $L$. Recall that these questions ask “Given a weakly compact cardinal $\kappa$, are $\subseteq_{\text{NS}}$ and $\subseteq_{\text{DLO}}$ complete for $\Sigma^1_1$ quasi-orders?” We will use the following theorem:

THEOREM 5.1. ([17, Cor 10.24]) If $\kappa$ is weakly compact, then both the quasi-order of embeddability and the equivalence relation of bi-embeddability of graphs, $\subseteq_G$ and $\cong_T$ respectively, are $\Sigma^1_3$-complete.

DEFINITION 5.2 (Weakly compact diamond). Let $\kappa > \omega$ be a cardinal. The weakly compact ideal is generated by the sets of the form $\{\alpha < \kappa \mid \langle V_\alpha, \in, U \cap V_\alpha \rangle \models \neg \varphi\}$ where $U \subset V_\kappa$ and $\varphi$ is a $\Pi^1_1$-sentence such that $\langle V_\kappa, \in, U \rangle \models \varphi$. A set $A \subset \kappa$ is said to be weakly compact, if it does not belong to the weakly compact ideal. Note that $\kappa$ is weakly compact if and only if there exists $A \subset \kappa$ which is weakly compact, i.e. the weakly compact ideal is proper. For weakly compact $S \subset \kappa$, the $S$-weakly compact diamond, $\text{WC}_S(S)$, is the statement that there exists a sequence $(A_\alpha)_{\alpha < \kappa}$ such that for every $A \subset S$ the set

$$\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$$
is weakly compact. We denote $\text{WC}_\kappa = \text{WC}_\kappa(\kappa)$.

Weakly compact diamond was originally introduced in [18] and thoroughly analysed in [8]. In [1] it was used to study the reducibility properties of $E^\text{reg}_\kappa$. It has been sometimes called the dual diamond.

**FACT 5.3.** If $\kappa$ is weakly ineffable (same as almost ineffable), then $\text{WC}_\kappa$ holds. See [8] for proofs and references.

The proof of Lemma 5.4 can be found in [1] in complete detail.

**Lemma 5.4.** Let $\kappa$ be a weakly compact cardinal. The weakly compact diamond $\text{WC}_\kappa$ implies the following principle $\text{WC}^*_\kappa$. There exists a sequence $\langle f_\alpha \rangle_{\alpha \in \text{reg}(\kappa)}$ such that

- $f_\alpha : \alpha \rightarrow \kappa$,
- for all $g \in \kappa^\kappa$ and stationary $Z \subseteq \kappa$ the set
  $$\{ \alpha \in \text{reg}(\kappa) \mid g|\alpha = f_\alpha \wedge \alpha \cap Z \text{ is stationary} \}$$
  is stationary.

Following this result, we will introduce the following principle $\text{WC}^*_G$. Let us denote by $G_{<\kappa}$ the set of all graphs with domain $\alpha < \kappa$. There exists a sequence $\langle f_\alpha \rangle_{\alpha < \kappa}$ such that

- $f_\alpha \in (G_{<\kappa})^\kappa$,
- if $(S,g)$ is a pair such that $S \subseteq \kappa$ is stationary and $g \in (G_{<\kappa})^\kappa$, the set
  $$\{ \alpha \in \text{reg}(\kappa) \mid g|\alpha = f_\alpha \wedge S \cap \alpha \text{ is stationary} \}$$
  is stationary.

**Fact 5.5.** If $\text{WC}^*_\kappa$ holds, then $\text{WC}^*_G$ holds.

**Proof.** Let $\langle f_\alpha \rangle_{\alpha < \kappa}$ be a sequence that witnesses $\text{WC}^*_\kappa$. Let $\{A_\beta\}_{\beta < \kappa}$ be an enumeration of the elements of $G_{<\kappa}$, and for every $\alpha < \kappa$, let $G_\alpha = \{A_\beta\}_{\beta < \alpha}$. Construct the sequence $\langle f_\alpha \rangle_{\alpha < \kappa}$ by $f_\alpha (\beta) = A_{f_\alpha (\beta)}$.

To show that $\langle f_\alpha \rangle_{\alpha < \kappa}$ witnesses $\text{WC}^*_G$, let $g \in (G_{<\kappa})^\kappa$ be any function and $S \subseteq \kappa$ be a stationary. There is a function $\tilde{g} : \kappa \rightarrow \kappa$ such that $g(\alpha) = A_{\tilde{g}(\alpha)}$. Because of $\text{WC}^*_\kappa$ we know that the set

$$\{ \alpha \in \text{reg}(\kappa) \mid \tilde{g}|\alpha = f_\alpha \wedge Z \cap \alpha \text{ is stationary} \}$$

is stationary. By the way $\langle f_\alpha \rangle_{\alpha < \kappa}$ and $\tilde{g}$ were defined, we conclude that the set

$$\{ \alpha \in \text{reg}(\kappa) \mid g|\alpha = f_\alpha \wedge Z \cap \alpha \text{ is stationary} \}$$

is stationary. $\dashv$

**Theorem 5.6.** If $\kappa$ is weakly compact and $\text{WC}^*_G$ holds, then $\subseteq_{\text{reg}}$ as well as $\subseteq_{\text{NS}}$ are $\Sigma^1_1$-complete.

**Proof.** The claim for $\subseteq_{\text{NS}}$ follows from Fact 2.4 once we prove the claim for $\subseteq_{\text{reg}}$. By Theorem 5.1 it is enough to show that $\subseteq_G \leq_B \subseteq_{\text{reg}}$. For all $K,H \in G_{<\kappa}$ we write $K \subseteq H$ if $K$ is embeddable to $H$. Let us denote by $Q$ the quasi-order $((G_{<\kappa})^\kappa, \leq_Q)$, where $f \leq_Q g$ holds if there is a club $C$ such that for all $\alpha \in C$, $f(\alpha) \subseteq g(\alpha)$ holds.
Let $H$ be the graph with domain 2 and no edges. Define $F : Mod^c_G \to (G_{<\kappa})^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} A_{\eta|\alpha} & \text{if } \alpha \in C_\pi \\ H & \text{otherwise.} \end{cases}$$

where $C_\pi$ is as in Definition 2.2.

**Claim 5.6.1.** $\eta \subseteq_G \xi$ if and only if $F(\eta) \leq_Q F(\xi)$.

**Proof.** Let us show that if $\eta \subseteq_G \xi$, then $F(\eta) \leq_Q F(\xi)$. Suppose $\eta \subseteq_G \xi$, then there is $f : \kappa \rightarrow \kappa$ an embedding of $A_{\eta}$ to $A_{\xi}$. Let $D$ be the set of closed points of $f$, $D$ is a club. Therefore $f \upharpoonright \alpha$ is an embedding of $A_{\eta|\alpha}$ to $A_{\xi|\alpha}$, for all $\alpha \in D \cap C_\pi$. We conclude that $F(\eta) \leq_Q F(\xi)$. Let us show that if $(\eta, \xi) \notin \subseteq_G$, then $F(\eta) \not\leq_Q F(\xi)$. Suppose $(\eta, \xi) \notin \subseteq_G$. The property

There is no embedding of $A_{\eta}$ to $A_{\xi}$, where $\kappa$ is regular, and $C_\pi$ is unbounded

is a $\Pi^1_1$-property of the structure $(\mathcal{V}_\kappa, \in, A)$, where $A = (\eta \times \{0\} \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\})$. Since $\kappa$ is weakly compact, there is stationary $\gamma$'s such that $C_\pi \cap \gamma$ is unbounded, $\gamma \in C_\pi$, $\gamma$ is regular, and there is no embedding of $A_{\eta|\gamma}$ to $A_{\xi|\gamma}$. We conclude that there are stationary many $\gamma$'s such that $F(\eta)(\gamma) \not\subseteq F(\xi)(\gamma)$, hence $F(\eta) \not\leq_Q F(\xi)$.

Let $(f_\alpha)_{\alpha < \kappa}$ be a sequence that witnesses $\text{WC}_G^\kappa$. For all $\alpha \in \text{reg}(\kappa)$ define the relation $\leq_Q^\alpha$ on $(G_{<\kappa})^\alpha$ by: $f \leq_Q^\alpha g$ if there is a club $C \subseteq \alpha$ such that for all $\beta \in C$, $f(\beta) \subseteq g(\beta)$ holds. Notice that since the intersection of two clubs is a club, then $\leq_Q$ is a quasi-order. Define the map $F : (G_{<\kappa})^\kappa \rightarrow 2^\kappa$ by

$$F(f)(\alpha) = \begin{cases} 0 & \text{if } f \upharpoonright \alpha \leq_Q f_\alpha \\ 1 & \text{otherwise.} \end{cases}$$

**Claim 5.6.2.** $f \leq_Q g$ if and only if $F(f) \subseteq_{\text{reg}} F(g)$.

**Proof.** Let us show that if $f \leq_Q g$, then $F(f) \subseteq_{\text{reg}} F(g)$. Suppose $f \leq_Q g$, then there is a club $C \subset \kappa$ such that for all $\alpha \in C$, $f(\alpha) \subseteq g(\alpha)$. Therefore, for all $\alpha \in C \cap \text{reg}(\kappa)$ it holds that $f \upharpoonright \alpha \leq_Q g \upharpoonright \alpha$. Now if $\alpha \in C \cap \text{reg}(\kappa)$ is such that $F(g)(\alpha) = 0$, then $g \upharpoonright \alpha \leq_Q f_\alpha$, so $f \upharpoonright \alpha \leq_Q f_\alpha$ and $F(f)(\alpha) = 0$. We conclude that $(F(f)^{-1}[1]) \cap \text{reg}(\kappa)$ is non-stationary. Hence $F(f) \not\subseteq_{\text{reg}} F(g)$.

Let us show that if $f \not\leq_Q g$, then $F(f) \not\subseteq_{\text{reg}} F(g)$. Suppose that $f \not\leq_Q g$, then there is a stationary set $S \subseteq \kappa$ such that for all $\alpha \in S$, $f(\alpha) \not\subseteq g(\alpha)$. Because of $\text{WC}_G^\kappa$ we know that the set

$$A = \{ \alpha \in \text{reg}(\kappa) | g \upharpoonright \alpha = f_\alpha \wedge S \cap \alpha \text{ is stationary} \}$$

is a stationary set. Therefore, for all $\alpha \in A$, $F(g)(\alpha) = 0$, and for all $\beta \in S \cap \alpha$, $f(\beta) \not\subseteq g(\beta)$. Since for all $\alpha \in A$, $g \upharpoonright \alpha = f_\alpha$, and $S \cap \alpha$ is stationary, we conclude that $f \upharpoonright \alpha \not\subseteq_Q f_\alpha$ holds for all $\alpha \in A$. Hence, for all $\alpha \in A$, $F(g)(\alpha) = 0$ and $F(f)(\alpha) = 1$. We conclude that $A \subseteq (F(f)^{-1}[1]) \cap \text{reg}(\kappa)$, and since $A$ is stationary, $F(f) \not\subseteq_{\text{reg}} F(g)$.

Clearly $F \circ F : Mod^c_G \rightarrow 2^\kappa$ is a Borel reduction of $\subseteq_G$ to $\subseteq_{\text{reg}}$.

**Theorem 5.7.** If $\kappa$ is weakly ineffable, then $\subseteq_{\text{NS}}$ is $\Sigma^1_1$-complete.

**Proof.** Follows from Fact 5.3, Lemma 5.4, Fact 5.5, and Theorem 5.6.
Thus, the only case concerning \cite[Q. 11.4]{17} that is still open is the case where \( V \neq L \) and \( \kappa \) is a weakly compact, but not weakly ineffable cardinal. For example the first weakly compact is such \cite[Lemma 1.12]{3}. For successor cardinals, we know from \cite{6} that it can be forced the relation \( E^2_{\wedge} \) to be a \( \Delta^1_1 \) equivalence relation. So it is consistently true that \( \leq_{\text{NS}} \) is not \( \Sigma^1_2 \)-complete.

5.2. \( \Sigma^1_1 \)-completeness of \( \equiv_{\text{DLO}} \) for weakly compact \( \kappa \). In this section we prove:

**Theorem 5.8.** Suppose that \( \kappa \) is weakly compact. Then the isomorphism relation on dense linear orders is \( \Sigma^1_1 \)-complete.

Note that the isomorphism of linear orders is reducible to graph isomorphism, so \( \equiv_G \) is also \( \Sigma^1_1 \)-complete for weakly compact \( \kappa \). Before proving this, we first prove the following:

**Lemma 5.9.** If \( \kappa \) is weakly compact, then the bi-embeddability of graphs \( \approx_G \) is reducible to \( E^\kappa_{\text{reg}} \) (Definition 2.3).

**Proof.** Let \( C_\pi \) be the club as in Definition 2.2 and for all \( \alpha \in C_\pi \) define the relation \( \approx^{\alpha}_G \) as follows. For all \( \eta, \xi \in \text{Mod}_G^\alpha \), let \( \eta \approx^{\alpha}_G \xi \), if \( A_{\eta|\alpha} \) is embeddable in \( A_\xi \), and \( A_{\eta|\alpha} \) is embeddable in \( A_\xi \).

There are at most \( \kappa \) many equivalence classes of \( \approx^{\alpha}_G \), so let \( g_\alpha : \text{Mod}_G^\alpha \to \kappa \) be a function with the property that for all \( \eta, \xi \in \text{Mod}_G^\alpha \) we have \( g_\alpha(\eta) = g_\alpha(\xi) \) if and only if \( \eta \approx^{\alpha}_G \xi \)

Define the reduction \( F : \text{Mod}_G^\alpha \to \kappa^\kappa \) by

\[
F(\eta)(\alpha) = \begin{cases} g_\alpha(\eta) & \text{if } \alpha \in C_\pi \\ 0 & \text{otherwise} \end{cases}
\]

Let us show that if \( \eta \approx_G \xi \), then \( (\eta, \xi) \in E^\kappa_{\text{reg}} \). Suppose that \( \eta \approx_G \xi \). Then there are embeddings \( F_1 : \kappa \to \kappa \) and \( F_2 : \kappa \to \kappa \) from \( A_\eta \) to \( A_\xi \), and from \( A_\xi \) to \( A_\eta \) respectively. Let \( D_1 \) and \( D_2 \) be the sets of closed points of \( F_1 \) and \( F_2 \) respectively. These are closed unbounded sets in \( \kappa \). Then for all \( \alpha \in D_1 \cap D_2 \cap C_\pi \), \( A_\eta|\alpha \) and \( A_\xi|\alpha \) are bi-embeddable. Hence for all \( \alpha \in D_1 \cap D_2 \cap C_\pi \), \( F(\eta|\alpha) = F(\xi|\alpha) \).

We conclude that \( (\eta, \xi) \in E^\kappa_{\text{reg}} \).

Let us show that if \( (\eta, \xi) \not\approx_G \), then \( \eta \) and \( \xi \) are not \( E^\kappa_{\text{reg}} \)-equivalent. Suppose that \( (\eta, \xi) \not\approx_G \), without loss of generality, suppose that there is no embedding of \( A_\eta \) into \( A_\xi \). The property

\[ \text{There is no embedding of } A_\eta \text{ to } A_\xi \land \kappa \text{ is regular } \land C_\pi \text{ is unbounded} \]

is a \( \Pi^1_1 \)-property of the structure \( (V_\kappa, \in, A) \), where \( A = (\eta \times \{0\}) \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\}) \). Since \( \kappa \) is weakly compact, there are stationary many ordinals \( \gamma < \kappa \) such that \( C_\pi \cap \gamma \) is unbounded, \( \gamma \in C_\pi \), \( \gamma \) is regular, and there is no embedding of \( A_\eta|\gamma \) to \( A_\xi|\gamma \). We conclude that there are stationary many points \( \gamma \) with \( F(\eta|\gamma) \neq F(\xi|\gamma) \), hence \( \eta \) and \( \xi \) are not \( E^\kappa_{\text{reg}} \)-equivalent.

**Corollary 5.10.** If \( \kappa \) is weakly compact, then \( E^\kappa_{\text{reg}} \) is \( \Sigma^1_1 \)-complete.

**Proof.** Follows from Theorem 5.1 and Lemma 5.9.

Now we can prove Theorem 5.8:

**Proof of Theorem 5.8.** By \cite[Thm 3.9]{1} we have \( E^\kappa_{\text{reg}} \leq c \equiv_{\text{DLO}} \), so the result follows from Corollary 5.10.
REFERENCES


