1 Descriptive Set Theory

This is an intensive course, long proofs are discussed during the lectures but not included in the lecture notes.

Day 1

Definition 1.1 (The Baire space $B$). The Baire space is the set $\omega^n$ endowed with the following topology. For every $\eta \in \omega^n$ for some $n$, define the following basic open set

$$N_\eta = \{ f \in \omega^{\omega} \mid \eta \subseteq f \}$$

the open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets.

This topology is metrizable, let $d(f, g) = \frac{1}{n+1}$ where $n$ is the least natural number that satisfies $f(n) \neq g(n)$, in case it does not exist then $f = g$ and $d(f, g) = 0$.

Definition 1.2 (The Cantor space $C$). The cantor space is the set $2^{\omega}$ with the relative subspace topology.

Definition 1.3 (Borel class). Let $S \in \{B, C\}$. The class $\text{Borel}(S)$ of all Borel sets in $S$ is the least collection of subsets of $S$ which contains all open sets and is closed under complements, countable unions and countable intersections.

Definition 1.4 (Borel hierarchy). Let $S \in \{B, C\}$. Define the classes $\Sigma_{\alpha}(S)$ and $\Pi_{\alpha}(S)$, $\alpha < \omega_1$, as follows.

1. $\Sigma_1(S)$ is the class of open sets.
2. $\Pi_1(S)$ is the class of closed sets.
3. For all $\alpha > 1$, $\Sigma_{\alpha}(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Pi_{\beta}(S)$.
4. For all $\alpha > 1$, $\Pi_{\alpha}(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Sigma_{\beta}(S)$.

Exercise 1.1.

1. For all $n < \omega$ and all $\eta \in \omega^n$ the set $N_\eta$ is closed.
2. For all $\beta < \alpha < \omega_1$, $\Sigma_{\beta}(B) \subseteq \Sigma_{\alpha}B$.
3. $\text{Borel}(B) = \bigcup_{0<\alpha<\omega_1} \Sigma_{\alpha}(B)$.
4. $|\text{Borel}(B)| = 2^\omega$.
5. There are subsets of $B$ that are not Borel.

Definition 1.5. Let $S \in \{B, C\}$. We say that $A \subseteq S$ is co-meager, if it contains a countable intersection of open and dense subsets of $S$. A subset of $S$ is meager, if the complement of it is co-meager.

Definition 1.6. Let $S \in \{B, C\}$. We say that $X \subseteq S$ has the property of Baire (PB) if there is an open set $U \subseteq S$ such that $X \Delta U$ is meager.

Lemma 1.7. Every Borel subset of $B$ has the property of Baire.

Exercise 1.2. Prove Lemma 1.7. (Hint: prove that $X$ has the PB if and only if $B \setminus X$ has the PB.)

Definition 1.8 (Borel*-code). Let $X$ be a non-empty set.

1. A subset $T \subset X^{<\omega}$ is a tree if for all $f \in T$ with $n = \text{dom}(f) > 0$ and for all $m < n$, $f \upharpoonright m \in T$. 

5
2. A non-empty tree $T \subset X^{<\omega}$ is called an $\omega$-tree if the following holds:

(a) If $f : n \to X$ is in $T$ and $n > 0$, then for all $x \in X$, $f \upharpoonright (n-1) \cup \{(n-1, x)\} \in T$.

(b) There is no $f : \omega \to X$ such that for all $n < \omega$, $f \upharpoonright n \in T$.

3. We order $T$ by $\subseteq$. The maximal elements of $T$ are called leaves and the set of leaves is denoted by $L(T)$. The least element of $T$ is called root ($\emptyset$). For every $f \in T$ that is not the root, we denote by $f^-$ the immediate predecessor of $f$ in $T$. We call node every element that is not a leaf.

4. A Borel $^*$-code is a pair $(T, \pi)$, where $T \subseteq (\omega \times \omega)^{<\omega}$ is an $\omega$-tree and $\pi$ is a function from $L(T)$ to the basic open sets of $B$.

5. Given a Borel $^*$-code $(T, \pi)$ and $\eta \in B$, we define the game $GB^*(\eta, (T, \pi))$ as follows. The game $GB^*(\eta, (T, \pi))$ is played by two players, $I$ and $II$. In each move $0 \leq n < \omega$ the function $f_n : n + 1 \to (\omega \times \omega)$ from $T$ is chosen as follows: Suppose $f_{n-1} \in T$ is chosen, in case $n = 0$, $f_0 = \emptyset$. If $f_{n-1}$ is not a leaf, then $I$ chooses some $i < \omega$ and then $II$ chooses some $j < \omega$. This determines $f_n = f_{n-1} \cup \{(n, (i, j))\}$. If $f_{n-1}$ is a leaf, then the game ends and $II$ wins if $\eta \in \pi(f_{n-1})$.

6. A function $W : \omega^{<\omega} \to \omega$ is a winning strategy in $GB^*(\eta, (T, \pi))$ if $II$ wins by choosing $W(i_0, \ldots, i_n)$ on the move $n$, where $i_0, \ldots, i_n$ are the moves that $I$ made on the moves $0, \ldots, n$.

7. A Borel $^*$-code $(T, \pi)$ is a Borel $^*$-code for $X \subseteq B$ if for all $\eta \in B$, $\eta \in X$ if and only if $II$ has a winning strategy in $GB^*(\eta, (T, \pi))$. We say that $X \subseteq B$ is a Borel $^*$ set if it has a Borel $^*$-code. We denote by $Borel^*(B)$ the class of Borel $^*$ sets.

Theorem 1.5. $Borel(B) = Borel^*(B)$.

Proof. Let us start by showing that $Borel(B) \subseteq Borel^*(B)$. We will prove this by showing that every open set is a Borel $^*$ set. Let $\eta \in Borel^*(B)$ and if $(X_i)^{<\omega}$ is a countable collection of Borel $^*$ sets, then $\bigcup_{i<\omega} X_i$ and $\bigcap_{i<\omega} X_i$ are Borel $^*$ sets.

Suppose that $X$ is an open set. Let $\{\xi_i\}_{i<\omega}$ be a collection of elements of $\omega^{<\omega}$ such that $X = \bigcup_{i<\omega} N_{\xi_i}$. Let $T = (\omega \times \omega)^{<\omega}$ and $\pi$ the function given by $\pi((0, (i, j))) = N_{\xi_i}$. It is clear that for every $\eta \in X$, $II$ has a winning strategy in $GB^*(\eta, (T, \pi))$. Therefore $(T, \pi)$ is a Borel $^*$-code for $X$.

Suppose that $\{X_i\}_{i<\omega}$ is a countable collection of Borel $^*$ sets. Let $(T_i, \pi_i)$ be a Borel $^*$-code of $X_i$. Let $T$ be the set of all functions $f : n \to (\omega \times \omega)$, for some $n < \omega$, such that if $f(0) = (i, j)$, then there is $g \in T_i$, $g : n - 1 \to (\omega \times \omega)$ with $dom(f) = dom(g) + 1$, and $f(m) = g(m - 1)$, for all $0 < m < dom(f)$. For every leaf $f$ of $T$ if $f(0) = (i, j)$, then there is $g \in L(T_i)$ such that $f(m) = g(m - 1)$, for all $0 < m < dom(f)$; define $\pi(f) = \pi_i(g)$.

Claim 1.10. $(T, \pi)$ is a Borel $^*$-code of $\bigcap_{i<\omega} X_i$, and $\bigcup_{i<\omega} X_i$ is a Borel $^*$ set.

Proof. Let $\eta \in \bigcap_{i<\omega} X_i$. Then for all $i < \omega$, there is a winning strategy $W_i$ of $II$ in $GB^*(\eta, (T_i, \pi_i))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = 0$ and $W(i_0, \ldots, i_n) = W_i(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that $W$ is a winning strategy of $II$ in $GB^*(\eta, (T, \pi))$.

Let $\eta \in B$ be such that $II$ has a winning strategy, $W$, in $GB^*(\eta, (T, \pi))$. Define $W_i : \omega^{<\omega} \to \omega$ by $W_i(i_0, \ldots, i_n) = W(i_0, \ldots, i_n)$. It is easy to see that $W_i$ is a winning strategy of $II$ in $GB^*(\eta, (T_i, \pi_i))$. Since this holds for all $i < \omega$, we conclude that $\eta \in X_i$, for all $i < \omega$.

Claim 1.11. $(T, \pi)$ is a Borel $^*$-code of $\bigcup_{i<\omega} X_i$, and $\bigcup_{i<\omega} X_i$ is a Borel $^*$ set.

Proof. Let $\eta \in \bigcup_{i<\omega} X_i$. Then there is $j < \omega$, such that there is a winning strategy $W_j$ of $II$ in $GB^*(\eta, (T_j, \pi_j))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = j$ and $W(i_0, \ldots, i_n) = W_j(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that $W$ is a winning strategy of $II$ in $GB^*(\eta, (T, \pi))$.

Let $\eta \in B$ be such that $II$ has a winning strategy, $W$, in $GB^*(\eta, (T, \pi))$. Define $W' : \omega^{<\omega} \to \omega$ by $W'(i_1, \ldots, i_n) = W(0, \ldots, i_n)$. It is easy to see that $W'$ is a winning strategy of $II$ in $GB^*(\eta, (T_{W(0)}, \pi_{W(0)}))$. Therefore $\eta \in X_{W(0)}$.

To show that $Borel^*(B) \subseteq Borel(B)$ we will define the rank of an $\omega$-tree and the rank of the elements of an $\omega$-tree.

Given an $\omega$-tree $T$, we define the rank function, $rk$, as follows:
The rank of a tree $T$ is defined by $rk(T) = rk(\emptyset)$.

**Exercise 1.3.**

1. Show that the rank of an $\omega$-tree is smaller than $\omega_1$.

2. Find an $\omega$-tree with infinite rank.

Let $X$ be a Borel* set, and $(T, \pi)$ a Borel*-code of $X$. We will prove by induction on $rk(T)$ that $X$ is a Borel set.

**Case** $rk(T) = 0$. It is clear that $T = \{\emptyset\}$ and $X = \pi(\emptyset)$, therefore $X$ is a Borel set.

Suppose $rk(T) = \alpha$ and if $Y$ is Borel* set with Borel*-code $(T', \pi')$ with $rk(T') < \alpha$, then $Y$ is a Borel set.

Let $T_{ij}$ be the set of all functions $f : n \rightarrow \omega$ such that there is a function $g \in T$ with $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Define $\pi_{ij}$ by $\pi_{ij}(f) = \pi(g)$, where $g \in T$ is such that $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Notice that for all $i, j < \omega$, $rk(T_{ij}) < \alpha$. By the induction hypothesis, for all $i, j < \omega$, $(T_{ij}, \pi_{ij})$ is a Borel*-code of a Borel set. Denote by $B_{ij}$ the Borel set with Borel*-code $(T_{ij}, \pi_{ij})$.

**Claim 1.12.** $X = \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$

**Proof.** Let $\eta \in X$, then $\Pi$ has a winning strategy, $W$, in $GB^*(\eta, (T, \pi))$. Define $W_{iW(\eta)} : \omega^{<\omega} \rightarrow \omega$ by $W_{iW(\eta)}(i_0, \ldots, i_n) = W(i, i_0, \ldots, i_n)$, it is clear that $W^{-iW(\eta)}$ is a winning strategy of $\Pi$ in $GB^*(\eta, (T_{iW(\eta)}, \pi_{iW(\eta)}))$, so $\eta \in B_{iW(\eta)}$. Therefore, for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, we conclude that $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$.

Let $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$. Then for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, denote by $h(i)$ this $j$. So there is $W_{iW(h(i))}$ a winning strategy of $\Pi$ in $GB^*(\eta, (T_{iW(h(i)), \pi_{iW(h(i))}}))$. Define $W : \omega^{<\omega} \rightarrow \omega$ by $W(i_0) = h(i_0)$ and $W(i_0, \ldots, i_n) = W_{iW(h(i))}(i_1, \ldots, i_n)$. It is clear that $W$ is a winning strategy of $\Pi$ in $GB^*(\eta, (T_{iW(h(i)), \pi_{iW(h(i))}}))$ and $\eta \in X$.

At the beginning the Borel*-codes look very artificial and complicated, but this codes will be very helpful in the future. In order to give a better understanding of the motivation behind the Borel*-codes we will define the Borel**-codes. This codes use intersections and unions as part of the coding of sets, this gives a better understanding on what is going on in the coding.

**Definition 1.13.**

1. A pair $(T, \pi)$ is a Borel**-code if $T \subseteq \omega^{<\omega}$ is an $\omega$-tree and $\pi$ is a function with domain $T$ such that if $f \in T$ is a leaf, then $\pi(f)$ is an open set, and in case $f$ is a node, $\pi(f) = \cap$ if $|\text{dom}(f)|$ is an even number and $\pi(f) = \cup$ if $|\text{dom}(f)|$ is an odd number.

2. For an element $\eta \in B$ and a Borel**-code $(T, \pi)$, the game $B^*(\eta, (T, \pi))$ is played as follows. There are two players, $I$ and $\Pi$. The game starts from the root of $T$. At each move, if the game is at node $f \in T$ and $\pi(f) = \cap$, then $I$ chooses an immediate successor $g$ of $f$ and the game continues from this $g$. If $\pi(f) = \cup$, then $\Pi$ makes the choice. Finally, if $\pi(f)$ is an open set, then the game ends, and $\Pi$ wins if and only if $\eta \in \pi(x)$.

3. A set $X \subseteq \omega^w$ is a Borel**-set if there is a Borel**-code $(T, \pi)$ such that for all $\eta \in \omega^w$, $\eta \in X$ if and only if $\Pi$ has a winning strategy in the game $B^*(\eta, (T, \pi))$. We denote by Borel**(B) the set of Borel** sets.

**Exercise 1.4.** Borel*(B) = Borel**(B).

Notice that the rank was defined for $\omega$-trees in general. For every Borel** set, $X$, as the least ordinal $\alpha$ such that there is a Borel**-code of $X$.

**Exercise 1.5.** What is the relation between the rank of a Borel** set and the Borel hierarchy?

**Day 2**

**Definition 1.14.**

- $X \subseteq B$ is $\Sigma^1_1(B)$ if there is $Y \subseteq B \times B$ a Borel set such that $\text{pr}(Y) = X$.
- $X \subseteq B$ is $\Pi^1_1(B)$ if $B \setminus X$ is $\Sigma^1_1(B)$.
- $X \subseteq B$ is $\Delta^1_1(B)$ if it is $\Sigma^1_1(B)$ and $\Pi^1_1(B)$.

**Lemma 1.15.** The following are equivalent:

1. $X \subseteq B$ is $\Pi^1_1(B)$.
2. $X \subseteq B$ is $\Delta^1_1(B)$.
3. $X \subseteq B$ is $\Sigma^1_1(B)$.

\[ \text{If } \eta \in L(T), \text{ then } rk(\eta) = 0. \]

\[ \text{If } \eta \notin L(T), \text{ then } rk(\eta) = \bigcup \{ rk(f) + 1 \mid f^- = \eta \}. \]
• \( X \) is \( \Sigma^0_1(B) \).
• \( X = \text{pr}(Y) \) for some closed \( Y \subseteq B \times B \).

**Lemma 1.16.** If \( X \subseteq B \) is Borel, then \( X \) is \( \Delta^0_1(B) \).

**Proof.** Let \( X \subseteq B \) be a Borel set and \( (T, \pi) \) a Borel*-code for \( X \). Let \( h : \omega^\omega \to \omega \) be one-to-one and onto. For all \( f \in \omega^\omega \) define \( W_f : \omega^{<\omega} \to \omega \) by \( W_f(i_0, \ldots, i_n) = f(h(i_0, \ldots, i_n)) \). Let \( P \) be the set of all the tuples \( (\eta, f) \in \omega^\omega \times \omega^\omega \) such that \( W_f \) is a winning strategy for \( \Pi \) in the game \( GB^*(\eta, (T, \pi)) \). It is clear that \( \text{pr}(P) = X \).

**Claim 1.17.** \( P \) is closed.

**Proof.** Let \( (\eta, f) \notin P \) then there are \( n < \omega \) and \( \{j_0, \ldots, j_n\} \) such that if \( \Pi \) chooses \( j_m \) in the \( m \)-move and \( \Pi \) chooses \( W_{j_m}(j_0, \ldots, j_m) \) in the \( m \)-move, then after \( n \) moves the game stops in a leaf \( g \) and \( \eta \notin \pi(g) \). Therefore, there is \( r < \omega \), such that \( N_{\eta, r} \cap \pi(g) = \emptyset \), so \( (N_{\eta, r} \times N_{f(m)}) \cap P = \emptyset \).

We conclude that \( X \) is \( \Sigma^0_1(B) \) and since Borel(B) is closed under complements, we conclude that \( B \setminus X \) is Borel, therefore it is \( \Sigma^0_1(B) \). We conclude that \( X \) is \( \Delta^0_1(B) \).

**Exercise 1.6.** Prove the claims of the following proof.

**Theorem 1.18** (Separation). If \( X, Y \subseteq B \) are \( \Sigma^0_1(B) \) disjoint sets, then there is a Borel set \( Z \subseteq B \) that satisfies \( X \subseteq Z \subseteq B \setminus Y \).

**Proof.** Choose \( X^*, Y^* \subseteq B \times B \) such that \( \text{pr}(X^*) = X \) and \( \text{pr}(Y^*) = Y \). For all \( \eta \in B \), let \( X_\eta \) be the set of all \( \xi \in \omega^\omega \) that satisfy the following: If \( \text{dom}(\xi) = n \), then there are \( \eta, \xi' \in B \), \( \eta' \models n = \eta \upharpoonright n \) and \( \xi \subseteq \xi' \). Define \( Y_\eta \) in the same way. We denote by \( X_{\eta, n} \) the set of functions \( \xi \in \omega^n \) such that there is \( \eta' \in B \), and \( \xi \in X_{\eta', n} \) and \( \eta \upharpoonright n \subseteq \eta' \). It is clear that \( X_\eta = \bigcup_{n<\omega} X_{\eta, n} \).

Given two trees \( T, T' \subseteq \omega^{<\omega} \), we say that \( T \leq T' \) if there is a function \( f : T \to T' \) that satisfies the following: for all \( \eta, \xi \in T \), if \( \eta \subseteq \xi \), then \( f(\eta) \subseteq f(\xi) \). Let \( Z \) be the set of \( \eta \in B \) that satisfy \( Y_\eta \leq X_\eta \).

**Claim 1.19.**
- If \( \eta \in X \), then \( Y_\eta \leq X_\eta \).
- \( X \subseteq Z \subseteq B \setminus Y \).

for all \( T, T' \subseteq \omega^{<\omega} \) we define the game \( GC(T, T') \) as follows: in the \( n \)-th movement, \( \Pi \) chooses \( t_n \in T \) such that \( t_m \subseteq t_n \) holds for all \( m < n \), and \( \Pi \) chooses \( t'_n \in T' \) such that \( t'_m \subseteq t'_n \) holds for all \( m < n \). The game ends when a player cannot make a choice, the player that cannot make a choice loses.

**Claim 1.20.** \( T \leq T' \) si y solo si \( \Pi \) has a winning strategy for the game \( GC(T, T') \).

Let \( T \) be the set of all functions with finite domain, \( f : n \to \bigcup_{m<\omega} (\omega^m)^3 \) such that for all \( i < n \) the following holds:
- \( f(i) \in (\omega^3)^3 \).
- If \( j + 1 < n \) and \( f(j) = (\xi_k)_{k<3} \), then \( \xi_1 \in X_{\xi_0} \) and \( \xi_2 \in X_{\xi_0} \).
- If \( j < l < n \) and \( f(j) = (\xi_k)_{k<3} \) and \( f(l) = (\xi'_k)_{k<3} \), then for all \( k < 3 \), \( \xi_k \subseteq \xi'_k \).

Define \( \pi \) with domain \( L(T) \) as \( \pi(f) = N_{\xi_0} \), if \( \text{dom}(f) = n + 1 \), \( f(n) = (\xi_k)_{k<3} \), and \( \xi_2 \notin X_{\xi_0} \). And \( \pi(f) = 0 \) in other case.

**Claim 1.21.** There is a Borel*-code \( (T', \pi') \) such that there is a tree isomorphism \( h : T' \to T \) that satisfies \( \pi'(f) = \pi(h(f)) \).

**Claim 1.22.** \( \Pi \) has a winning strategy in \( GB^*(\eta, (T', \pi')) \) if and only if \( GC(Y_\eta, X_\eta) \).

The following is a standard way to code structures with domain \( \omega \) with elements of \( 2^{<\omega} \). Fix a countable relational vocabulary \( L = \{ P_n \mid n < \omega \} \).

**Definition 1.23.** Fix a bijection \( \pi : \omega^{<\omega} \to \omega \). For every \( \eta \in 2^{<\omega} \) define the \( L \)-structure \( A_\eta \) with domain \( \omega \) as follows: For every relation \( P_m \) with arity \( n \), every tuple \( (a_1, a_2, \ldots, a_n) \) in \( \omega^n \) satisfies \( (a_1, a_2, \ldots, a_n) \in P_m^{A_\eta} \iff \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1 \).
Definition 1.24 (The isomorphism relation). Assume $T$ is a complete first order theory in a countable vocabulary. We define $\cong^T_\equiv$ as the relation

$\{(\eta, \xi) \in 2^\omega \times 2^\omega \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T)\}.$

A function $f : 2^\omega \to 2^\omega$ is Borel, if for every open set $A \subseteq 2^\omega$ the inverse image $f^{-1}[A]$ is a Borel subset of $2^\omega$. Let $E_1$ and $E_2$ be equivalence relations on $2^\omega$. We say that $E_1$ is Borel reducible to $E_2$, if there is a Borel function $f : 2^\omega \to 2^\omega$ that satisfies $(x, y) \in E_1 \iff (f(x), f(y)) \in E_2$, we denote it by $E_1 \leq_B E_2$.

Exercise 1.7. A function $f$ is Borel if and only if for all Borel set $X$, $f^{-1}[X]$ is Borel.

Example 1.1. Let $T_1$ be the theory of the order of the rational numbers, $\cong_{T_1}^\equiv$ has only two equivalent classes. Let $T_2$ be the theory of a vector space over the field of rational numbers. $\cong_{T_1}^\equiv \leq_B \cong_{T_2}^\equiv$.

This can be used to compare the complexity of two theories, from Example 1.1 we conclude that $T_1$ is less complex than $T_2$, in the Borel reducibility sense.

Question 1.25. Is there an equivalence relation $E$ on $2^\omega$ such that for every complete first order theory in a countable vocabulary $T$, either $E \not\leq_B \cong_{T_1}^\equiv$ or $\cong_{T_1}^\equiv \not\leq_B E$.

Let $T$ be a complete countable theory, we will denote by $I(\lambda, T)$ the amount of non-isomorphic models of $T$ of size $\lambda$. The following is the main theorem of [12].

Theorem 1.26 (The Main Gap Theorem, [12]). Let $T$ be a complete countable theory.

- If $T$ is not superstable, or deep, or with DOP or OTOP then for every uncountable cardinal $\lambda$, $I(\lambda, T) = 2^\lambda$.
- If $T$ is shallow superstable without DOP and without OTOP, then for every $\alpha > 0$, $I(\aleph_\alpha, T) \leq \beth_1(\aleph_\alpha)$.

Let $T$ be a complete countable theory, we say that $T$ is a classifiable theory if $T$ is superstable without DOP and without OTOP. $T_1$ in Example 1.1 is not classifiable and $T_2$ is classifiable. The Main Gap Theorem tells us that classifiable theories are less complex than non-classifiable ones, in the stability sense.

2 Generalized Descriptive Set Theory

Day 3

Definition 2.1 (The Generalized Baire space $B(\kappa)$). Let $\kappa$ be an uncountable cardinal. The generalized Baire space is the set $\kappa^\omega$ endowed with the following topology. For every $\eta \in \kappa^\omega$, define the following basic open set

$N_\eta = \{f \in \kappa^\omega \mid \eta \subseteq f\}
$ the open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets.

Definition 2.2 (The Generalized Cantor space $C(\kappa)$). Let $\kappa$ be an uncountable cardinal. The generalized Cantor space is the set $2^\kappa$ with the relative subspace topology.

From now on $\kappa$ is an uncountable cardinal that satisfies $\kappa^\omega$.

Definition 2.3 ($\kappa$-Borel class). Let $S \in \{B(\kappa), C(\kappa)\}$. The class $\kappa$-Borel($S$) of all $\kappa$-Borel sets in $S$ is the least collection of subsets of $S$ which contains all open sets and is closed under complements, unions and intersections both of length at most $\kappa$.

Definition 2.4 ($\kappa$-Borel$^*$-set in $C(\kappa)$).

1. A tree $T$ is a $\kappa^+$-$\kappa$-tree if does not contain chains of length $\kappa$ and its cardinality is less than $\kappa^+$. It is closed if every chain has a unique supremum.

2. A pair $(T, h)$ is a $\kappa$-Borel$^*$-code if $T$ is a closed $\kappa^+$-$\kappa$-tree and $h$ is a function with domain $T$ such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\emptyset, \kappa\}$.

3. For an element $\eta \in 2^\kappa$ and a $\kappa$-Borel$^*$-code $(T, h)$, the $\kappa$-Borel$^*$-game $B^*(T, h, \eta)$ is played as follows.

There are two players, I and II. The game starts from the root of $T$. At each move, if the game is at node $x \in T$ and $h(x) = \emptyset$, then I chooses an immediate successor $y$ of $x$ and the game continues from this $y$. If $h(x) = \kappa$, then II makes the choice. At limits the game continues from the (unique) supremum of the previous moves by Player I. Finally, if $h(x)$ is a basic open set, then the game ends, and II wins if and only if $\eta \in h(x)$.

4. A set $X \subseteq 2^\kappa$ is a $\kappa$-Borel$^*$-set if there is a $\kappa$-Borel$^*$-code $(T, h)$ such that for all $\eta \in 2^\kappa$, $\eta \in X$ if and only if II has a winning strategy in the game $B^*(T, h, \eta)$. 


We can define the $\kappa$-Borel* set in the generalized Baire space too, by using the same coding but with basic open sets of the generalized Baire space. Given two sets $X, Y \subseteq \kappa^\kappa$ we say that $X$ and $Y$ are duals if there is a $\kappa$-Borel* code $(T, h)$ such that for all $\eta, \eta \in \kappa^\kappa$, $\eta \in X$ if and only if $\eta$ has a winning strategy in the game $B^*(T, h, \eta)$, and $\eta \in Y$ if and only if $\eta$ has a winning strategy in the game $B^*(T, h, \eta)$. We will write $I \check{\rightarrow} B^*(T, h, \eta)$ when $I$ has a winning strategy in the game $B^*(T, h, \eta)$, and $I \check{\rightarrow} B^*(T, h, \eta)$ when $I$ has a winning strategy in the game $B^*(T, h, \eta)$.

Exercise 2.1. $X$ is a $\kappa$-Borel set if and only if there is a $\kappa$-Borel* code $(T, h)$ that codes $X$ and $T$ is a $\kappa^+, \omega$-tree.

Definition 2.5. $X \subseteq B(\kappa)$ is $\Sigma^1_1(\kappa)$ if there is $Y \subseteq B(\kappa) \times B(\kappa)$ a closed set such that $pr(Y) = X$.

Exercise 2.2. Complete the details in the proof of Theorem 2.6.

Theorem 2.7 ([2], Corollary 35). Theorem 2.6

Exercise 2.3. Prove the claims of the following proof.

Theorem 2.8 ([2], Theorem 17). $\Delta^1_1(\kappa)$ is $\kappa$-Borel*

Exercise 2.4. Prove the claims of the following proof.
Lemma 2.13 ([5], Lemma 5). Assume $V = L$. Suppose $\psi(x, \xi)$ is a $\Sigma_1$-formula in set theory with parameter $\xi \in 2^\kappa$ and that $r(\alpha)$ is a formula of set theory that says that “$\alpha$ is a regular cardinal”. Then for $x \in 2^\kappa$ we have $\psi(x, \xi)$ if and only if the set

$$A = \{\alpha < \kappa \mid \exists \beta > \alpha (L_\beta \models ZF^- \land \psi(x \upharpoonright \alpha, \xi \upharpoonright \alpha) \land r(\alpha))\}$$

contains a club.

Proof. Suppose that $x \in 2^\kappa$ is such that $\psi(x, \xi)$ holds. Let $\theta$ be a large enough cardinal such that

$$L_\theta \models ZF^- \land \psi(x, \xi) \land r(\alpha).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = Sk(\alpha \cup \{\kappa, \xi, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Claim 2.14. $D$ is a club set and $D \subseteq A$.

Suppose $x \in 2^\kappa$ is such that $\psi(x, \xi)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Take $\theta$ as above and let $C$ be an unbounded set, closed under $\mu$-limits (i.e. if $(\gamma_i)_i < \mu$ is an increasing succession of elements of $C$, then $\bigcup \{\gamma_i \mid i < \mu\} \in C$). Let

$$K(\alpha) = Sk(\alpha \cup \{\kappa, C, \xi, x\})^{L_\theta}$$

and

$$D = \{\alpha \in S_\mu^\kappa \mid K(\alpha) \cap \kappa = \alpha\}.$$

Claim 2.15. $D$ is an unbounded set, closed under $\mu$-limits.

Let $\alpha_0 \in D$ be the least ordinal that is a $\mu$-cofinal limit of elements of $D$.

Claim 2.16. $\alpha_0 \in C$ and $\alpha_0 > \mu$ (Hint: Use the elementarity of $K(\alpha)$ and the fact that $D \subseteq S_\mu^\kappa$).

Let $\beta$ be such that $L_\beta$ is equal to the Mostowski collapse of $K(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$. There exists $\beta > \alpha$ such that

$$L_\beta \models ZF^- \land \psi(x \upharpoonright \alpha, \xi \upharpoonright \alpha) \land r(\alpha).$$

Claim 2.17. $\beta$ is a limit ordinal greater than $\beta$ and $L_\beta$ satisfies “there exists a $\gamma \leq \alpha_0$ and an order-preserving bijection from $\gamma$ to $D \cap \alpha_0$” (Hint: Show that $K(\alpha_0)$ is a definable subset of $L_\theta$ and $D \cap \alpha_0$ is a definable subset of $K(\alpha_0)$, to conclude that $D \cap \alpha_0$ is a definable subset of $L_\beta$ and $D \cap \alpha_0 \in L_\beta$).

By the way $\alpha_0$ was chosen, $D \cap \alpha_0$ has order type $\mu$. Hence, by Claim 2.16 $\alpha_0$ is singular in $L_\beta$ but this contradicts that $L_\beta \models r(\alpha)$. □

Day 4

Let $\mu$ be a regular cardinal, we say that $X \subseteq \kappa$ is a $\mu$-club if $X$ is unbounded set and closed under $\mu$-limits.

Definition 2.18 ($E^\kappa_{\mu\text{-club}}$). Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in \kappa^\kappa$ we say that $\eta$ and $\xi$ are $E^\kappa_{\mu\text{-club}}$ equivalent ($\eta E^\kappa_{\mu\text{-club}} \xi$) if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a $\mu$-club.

Definition 2.19 ($E^2_{\mu\text{-club}}$). Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in \kappa^\kappa$ we say that $\eta$ and $\xi$ are $E^2_{\mu\text{-club}}$ equivalent ($\eta E^2_{\mu\text{-club}} \xi$) if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a $\mu$-club.

An equivalence relation $E$ on $X \subseteq \{\kappa^\kappa, 2^\kappa\}$ is $\Sigma^1_1(\kappa)$-complete if every $\Sigma^1_1(\kappa)$ equivalence relation is $\kappa$-Borel reducible to it.

Exercise 2.5. Prove the claims of the following proof.

Theorem 2.20 ([5], Theorem 7). Suppose that $V = L$. Then $E^\kappa_{\mu\text{-club}}$ is $\Sigma^1_1(\kappa)$-complete, for every regular $\mu$.

Proof. Suppose $E$ is a $\Sigma^1_1(\kappa)$ equivalence relation on $\kappa^\kappa$. Let $a : \kappa^\kappa \rightarrow 2^{\kappa \times \kappa}$ the map defined by

$$a(\eta)(\alpha, \beta) = 1 \Leftrightarrow \eta(\alpha) = \beta.$$

Let $b$ be a continuous bijection from $2^{\kappa \times \kappa}$ to $2^\kappa$, and $c = b \circ a$. Define $E'$ by

$$(\eta, \xi) \in E' \Leftrightarrow (\eta = \xi) \lor (\eta, \xi \in ran(c) \land (c^{-1}(\eta), c^{-1}(\xi)) \in E)$$
Claim 2.21. c is a continuous reduction of $E$ to $E'$ and $E'$ is a $\Sigma^1_1(\kappa)$ equivalence relation.

We can assume without loss of generality, that $E$ is an equivalence relation on $2^{\kappa}$. It is enough to define $f : 2^\kappa \rightarrow (2^{<\kappa})^\kappa$ such that for all $\eta, \xi \in 2^{\kappa}$, $(\eta, \xi) \in E$ if and only if the set $\{ \alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha) \}$ contains a $\mu$-club and $f$ is continuous in the topology generated by the sets

$$\{ \eta \mid \eta \upharpoonright \alpha = p \}, p \in (2^{<\kappa})^\alpha, \alpha < \kappa.$$

Claim 2.22. $f$ can be coded by a $\kappa$-Borel function $F : 2^\kappa \rightarrow \kappa^\kappa$.

Claim 2.23. There is a $\Sigma_1$-formula of set theory $\psi(\eta, \xi) = \psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x)$ with $x \in 2^{\kappa}$, such that for all $\eta, \xi \in 2^\kappa$,

$$(\eta, \xi) \in E \iff \psi(\eta, \xi).$$

Let $r(\alpha)$ be the formula “$\alpha$ is a regular cardinal” and $\psi^E = \psi^E(\kappa)$ be the sentence with parameter $\kappa$ that asserts that $\psi(\eta, \xi)$ defines an equivalence relation on $2^{\kappa}$. For all $\eta \in 2^{\kappa}$ and $\alpha < \kappa$, let

$$T_{\eta, \alpha} = \{ p \in 2^\alpha \mid \exists \beta > \alpha(L_{\beta} \models ZF^- \land \psi(p, \eta \upharpoonright \alpha, x) \land r(\alpha) \land \psi^E) \}$$

and let

$$f(\eta)(\alpha) = \begin{cases} \min L T_{\eta, \alpha} & \text{if } T_{\eta, \xi} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We will show that $(\eta, \xi) \in E$ if and only if the set $\{ \alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha) \}$ contains a $\mu$-club.

Suppose $\psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x)$ holds and let $k$ witnesses that. Let $\theta$ be a cardinal large enough such that $L_\theta \models ZF^- \land \varphi(k, \eta, \xi, x) \land r(\alpha)$. For all $\alpha < \kappa$ let $H(\alpha) = Sk(\alpha \cup \{ k, \eta, \xi, x \})^{L_\theta}$. The set $D = \{ \alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha \land H(\alpha) \models \psi^E \}$ is a club. Using the Mostowski collapse we have that

$$D' = \{ \alpha < \kappa \mid \exists \beta > \alpha(L_{\beta} \models ZF^- \land \varphi(k \mid \alpha, \eta \mid \alpha, \xi \upharpoonright \alpha, x \upharpoonright \alpha \land r(\alpha) \land \psi^E) \}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta, \alpha}$ we have that

$$\exists \beta_1 > \alpha(L_{\beta_1} \models ZF^- \land \psi(p, \eta \upharpoonright \alpha) \land r(\alpha) \land \psi^E)$$

and

$$\exists \beta_2 > \alpha(L_{\beta_2} \models ZF^- \land \psi(\eta \upharpoonright \alpha, \xi \upharpoonright \alpha) \land r(\alpha) \land \psi^E).$$

Therefore, for $\beta = \max\{ \beta_1, \beta_2 \}$ we have that

$$L_{\beta} \models ZF^- \land \psi(p, \xi \upharpoonright \alpha) \land \psi(\eta \upharpoonright \alpha, \xi \upharpoonright \alpha) \land r(\alpha) \land \psi^E.$$

Since $\psi^E$ holds and so transitivity holds for $\psi(\eta, \xi)$, we conclude that

$$L_{\beta} \models ZF^- \land \psi(p, \xi \upharpoonright \alpha) \land r(\alpha) \land \psi^E$$

so $p \in T_{\xi, \alpha}$ and $T_{\eta, \alpha} \subseteq T_{\xi, \alpha}$. Using the same argument we can show that $T_{\xi, \alpha} \subseteq T_{\eta, \alpha}$ holds for all $\alpha \in D'$. We conclude that for all $\alpha \in D'$ it holds that $T_{\xi, \alpha} = T_{\eta, \alpha}$, and the set $\{ \alpha < \kappa \mid f(\eta)(\alpha) = f(\xi)(\alpha) \}$ contains a $\mu$-club.

Suppose that $\lnot \psi(\eta, \xi, x)$ holds. Then by Lemma 2.13 there is no $\mu$-club inside

$$\{ \alpha < \kappa \mid \exists \beta > \alpha(L_{\beta} \models ZF^- \land \psi(\eta \upharpoonright \alpha, \xi \upharpoonright \alpha) \land r(\alpha)) \}.$$
Exercise 2.7. Assume $f : 2^\kappa \to 2^\kappa$ is $\kappa$-Borel function and $B$ is a $\kappa$-Borel* set. Prove that $f^{-1}[B]$ is a $\kappa$-Borel* set.

Corollary 2.24 ([2], Theorem 18). Suppose that $V = L$. Then $\kappa$-Borel* = $\Sigma_1^1(\kappa)$.

Proof. It follows from Exercise 2.7, Exercise 2.6, and Theorem 2.20. \qed

Corollary 2.25 ([2], Theorem 18). Suppose that $V = L$. Then $\Delta_1^1(\kappa) \neq \kappa$-Borel*.

Proof. It follows from Theorem 2.10 and Corollary 2.24. \qed

Question 2.26. Is it consistent that $\Delta_1^1(\kappa) = \kappa$-Borel*?

Question 2.27. An equivalence relation $E$ on $X \in \{\kappa^+, 2^\kappa\}$ is $\kappa$-Borel* complete if every $\kappa$-Borel* equivalence relation is $\kappa$-Borel reducible to it. Does there exist a $\kappa$-Borel* complete relation that is not a $\Sigma_1^1$-complete relation?

The following lemma shows that there is a model of set theory in which $\Delta_1^1(\kappa)$, $\kappa$-Borel*, and $\Sigma_1^1(\kappa)$ are different. The proof can be found in [4].

Lemma 2.28 ([4], Corollary 3.2). It is consistently that $\Delta_1^1(\kappa) \subseteq \kappa$-Borel* $\subseteq \Sigma_1^1(\kappa)$.

3 The Main Gap in $B(\kappa)$

Session in the logic seminar

Definition 3.1. For every $\eta \in \kappa^+$ define the structure $A_\eta$ with domain $\kappa$ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P^A_m \iff \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0.$$

Definition 3.2. For every $\eta \in 2^\kappa$ define the structure $A_\eta$ with domain $\kappa$ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P^A_m \iff \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Notice that the structure $|A_\eta| \upharpoonright \alpha$ is not necessary coded by the function $\eta \upharpoonright \alpha$.

Exercise 3.1. There is a club $C_\eta$ such that for all $\alpha \in C_\eta$, $A_\eta \upharpoonright \alpha = A_\eta|\alpha$.

With the structures coded by the elements of $2^\kappa$ and $\kappa^+$, it is easy to define the isomorphism relation of structures of size $\kappa$ in both spaces.

Definition 3.3 (The isomorphism relation). Assume $T$ is a complete first order theory in a countable vocabulary. We define $\equiv_T^1$ as the relation

$$\{(\eta, \xi) \in \kappa^+ \times \kappa^+ \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \models \neg T, A_\xi \models \neg T)\}.$$

Definition 3.4. Assume $T$ is a complete first order theory in a countable vocabulary. We define $\equiv_T^2$ as the relation

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \models \neg T, A_\xi \models \neg T)\}.$$

Notice that $\equiv_T^2 \leq \equiv_T^1 \leq \equiv_T^2$ holds for every theory $T$.

Definition 3.5. (Ehrenfeucht-Fraïssé game) Fix $(X_\gamma)_{\gamma < \kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $(f_\gamma)_{\gamma < \kappa}$ an enumeration of all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. For every $\alpha < \kappa$ we define the game $EF_\kappa^\alpha(\langle A \upharpoonright \alpha, B \upharpoonright \alpha \rangle)$ for structures $A$ and $B$ with domain $\kappa$, as follows. The game is played by two players, $I$ and $II$. In the $n$-th move, $I$ choose an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subseteq \alpha$, $X_{\beta_{n-1}} \subseteq X_{\beta_n}$, and then $II$ chooses an ordinal $\beta_n < \alpha$ such that $\text{dom}(f_{\beta_n}), \text{rang}(f_{\beta_n}) \subseteq \alpha$, $X_{\beta_n} \subseteq \text{dom}(f_{\beta_n}) \cap \text{rang}(f_{\beta_n})$ and $f_{\beta_{n-1}} \subseteq f_{\beta_n}$ (if $n = 0$ then $X_{\beta_{-1}} = 0$ and $f_{\beta_{-1}} = \emptyset$). The game finishes after $\omega$ moves. The player $II$ wins if $\cup_{i \leq \omega} f_{\beta_i} : A \upharpoonright \alpha \to B \upharpoonright \alpha$ is a partial isomorphism, otherwise the player $I$ wins.

We will write $I \uparrow EF_\kappa^\alpha(\langle A \upharpoonright \alpha, B \upharpoonright \alpha \rangle)$ when $I$ has a winning strategy in the game $EF_\kappa^\alpha(\langle A \upharpoonright \alpha, B \upharpoonright \alpha \rangle)$, similarly we write $II \uparrow EF_\kappa^\alpha(\langle A \upharpoonright \alpha, B \upharpoonright \alpha \rangle)$ when $II$ has a winning strategy.

Theorem 3.6. [12] If $T$ is a classifiable theory, then for every two models of $T$ with domain $\kappa$, $A, B$, it holds that $II \uparrow EF_\kappa^\alpha(A, B) \iff A \cong B$. 

9
Corollary 3.7 ([2], Theorem 70). If \( T \) is a classifiable theory, then \( \equiv^\mu_T \) is \( \Delta^1_1 \).

Lemma 3.8 ([7], Lemma 2.4). If \( A \) and \( B \) are structures with domain \( \kappa \), then the following hold:
- \( \mathbb{II} \uparrow EP^\kappa_\omega(A, B) \iff \mathbb{II} \uparrow EP^\kappa_\omega(A \upharpoonright \alpha, B \upharpoonright \alpha) \) for club-many \( \alpha \).
- \( \mathbb{I} \uparrow EP^\kappa_\omega(A, B) \iff \mathbb{I} \uparrow EP^\kappa_\omega(A \upharpoonright \alpha, B \upharpoonright \alpha) \) for club-many \( \alpha \).

Exercise 3.2. Prove Lemma 3.8 (Hint: look at the closed points of a winning strategy).

Definition 3.9. Assume \( T \) is a complete first order theory in a countable vocabulary. For every \( \alpha < \kappa \) and \( \eta, \xi \in \kappa^\omega \), we write \( R^\omega_{EF} \eta \) \( \xi \) if one of the following holds, \( A_\eta \upharpoonright \alpha \not\equiv T \) and \( A_\xi \upharpoonright \alpha \not\equiv T \), or \( A_\eta \upharpoonright \alpha \models T \), \( A_\xi \upharpoonright \alpha \models T \) and \( \mathbb{II} \uparrow EP^\kappa_\omega(A_\eta \upharpoonright \alpha, A_\xi \upharpoonright \alpha) \).

Exercise 3.3. Let \( T \) be a complete first order theory in a countable vocabulary. There are club many \( \alpha \) such that \( R^\omega_{EF} \) is an equivalence relation.

Theorem 3.10 ([7], Theorem 2.8). If \( T \) is a classifiable theory and \( \mu < \kappa \) a regular cardinal, then \( \equiv_T \) is continuously reducible to \( E^\kappa_{\mu \text{-club}} (\equiv^\mu_T \preceq c E^\kappa_{\mu \text{-club}}) \).

Proof. Define the reduction \( \mathcal{F} : \kappa^\kappa \rightarrow \kappa^\kappa \) by,
\[
\mathcal{F}(\eta)(\alpha) = \begin{cases} 
  f_\eta(\alpha) & \text{if } cf(\alpha) = \mu, A_\eta \upharpoonright \alpha \models T \text{ and } R^\omega_{EF} \text{ is an equivalence relation} \\
  0 & \text{in other case}
\end{cases}
\]
where \( f_\eta(\alpha) \) is a code in \( \kappa \setminus \{0\} \) for the \( R^\omega_{EF} \) equivalence class of \( A_\eta \upharpoonright \alpha \). The proof follows from Lemma 3.8 and Exercise 3.3.

Question 3.11. Is it provable in ZFC that \( E^\kappa_{\mu \text{-club}} \preceq_B \equiv^\mu_T \) holds for every non-classifiable theory \( T \) and regular cardinal \( \mu \)?

Model theory session

Exercise 3.4. Prove the claim below (Hint: Use the proof of Theorem 3.10).

Lemma 3.12 ([6], Lemma 2). Assume \( T \) is a classifiable theory and \( \mu < \kappa \) is a regular cardinal. If \( \Diamond_\kappa(S^\mu_\kappa) \) holds then \( \equiv_T \) is continuously reducible to \( E^\kappa_{\mu \text{-club}} \).

Proof. Let \( \{S_\alpha \mid \alpha \in X\} \) be a sequence testifying \( \Diamond_\kappa(S^\mu_\kappa) \) and define the function \( \mathcal{F} : 2^\kappa \rightarrow 2^\kappa \) by
\[
\mathcal{F}(\eta)(\alpha) = \begin{cases} 
  1 & \text{if } \alpha \in S^\mu_\kappa \cap C_\pi \cap C_{EF}, \mathbb{II} \uparrow EP^\kappa_\omega(A_\eta \upharpoonright \alpha, A_{S_\alpha}) \text{ and } A_\eta \upharpoonright \alpha \models T \\
  0 & \text{otherwise}
\end{cases}
\]
Claim 3.13. \( \eta, \xi \) if and only \( \mathcal{F}(\eta) \equiv E^\kappa_{\mu \text{-club}} \mathcal{F}(\xi) \).

The proof of the following theorems can be found in [2].

Theorem 3.14 ([2], Theorem 79). Suppose that \( \kappa = \lambda^+ = 2^\lambda \) and \( \lambda < \lambda = \lambda \).

1. If \( \lambda \) is unstable or superstable with OTOP, then \( E^\kappa_{\lambda \text{-club}} \preceq c \equiv^\mu_T \).
2. If \( \lambda \geq 2^{<\omega} \) and \( T \) is superstable with DOP, then \( E^\kappa_{\lambda \text{-club}} \preceq c \equiv^\mu_T \).

Theorem 3.15 ([2], Theorem 86). Suppose that for all \( \gamma < \kappa, \gamma^{<\omega} < \kappa \) and \( T \) is a stable unsuperstable theory. Then \( E^\kappa_{\gamma^{<\omega} \text{-club}} \preceq c \equiv^\mu_T \).

Theorem 3.16 ([6], Theorem 4). Suppose that \( \kappa = \lambda^+ = 2^\lambda, \lambda < \lambda = \lambda \) and \( \Diamond_\kappa(S^\mu_\kappa) \) holds.

1. If \( T_1 \) is classifiable and \( T_2 \) is unstable or superstable with OTOP, then \( \equiv_T^\mu_{T_1} \preceq c \equiv^\mu_{T_2} \) and \( \equiv^\mu_T \preceq_B \equiv^\mu_{T_1} \).
2. If \( \lambda \geq 2^{<\omega}, T_1 \) is classifiable and \( T_2 \) is superstable with DOP, then \( \equiv^\mu_{T_1} \preceq c \equiv^\mu_{T_2} \) and \( \equiv^\mu_T \preceq_B \equiv^\mu_{T_1} \).

Notice that if \( V = L \), then \( \Diamond_\kappa(S^\mu_\kappa) \) holds for all \( \lambda < \kappa \). Therefore in \( L \) it holds that If \( T \) is classifiable and \( T' \) not, then \( \equiv^\mu_T \preceq c \equiv^\mu_{T'} \).

The last session was used to study Question 3.11. The following results answer Question 3.11 for two kind of non-classifiable theories, the proofs are omitted in this notes, due to the length of them. The proofs can be found in [7] and [11]. The main ideas of these proofs is the use of coloured trees, as it was discussed during the lecture. Coloured trees has been used to obtain Borel-reducibility results of isomorphism relations (see [2], [5], [7], and [11]).
Definition 3.17. Let $T$ be a stable theory. $T$ has the orthogonal chain property (OCP), if there exist $\lambda_r(T)$-saturated models of $T$ of power $\lambda_r(T)$, $\{A_i\}_{i < \omega}$, $a \notin \bigcup_{i < \omega} A_i$, such that $t(a, \bigcup_{i < \omega} A_i) \perp A_j$ and for every $i \leq j$, $A_i \subseteq A_j$.

Exercise 3.5. If $T$ has the OCP, then $T$ is unsuperstable.

Lemma 3.18 ([7], Corollary 5.10). Assume $T$ is stable and has the OCP, then $E_{\omega_{1\omega}} \leq_c T$.

Corollary 3.19 ([7], Corollary 5.11). Assume $T_1$ is a classifiable theory and $T_2$ is a stable theory with the OCP, then $\equiv_{T_1} \leq_c \equiv_{T_2}$.

Question 3.20. Does there exists a stable unsuperstable theory that doesn’t have OCP?

Definition 3.21. We say that a superstable theory $T$ has the strong dimensional order property (S-DOP) if the following holds:

There are $\mathcal{F}_{\omega_1}^\omega$-saturated models $(M_i)_{i < 3}$, $M_0 \subsetneq M_1 \cap M_2$, such that $M_1 \downarrow M_0$, $M_2$, and for every $M_3$ $\mathcal{F}_{\omega_1}^\omega$-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to $M_1$ and to $M_2$, such that it does not fork over $M_1 \cup M_2$.

Lemma 3.22 ([11], Corollary 4.15). Assume $T$ is a theory with S-DOP and let $\lambda$ be $(2^{<\omega})^+$, then $E_{\lambda_{1\omega}} \leq_c \equiv_T$.

Corollary 3.23 ([11], Corollary 4.16). Assume $T_1$ is a classifiable theory and $T_2$ is a superstable theory with S-DOP, then $\equiv_{T_1} \leq_c \equiv_{T_2}$.

Question 3.24. Does there exists a superstable theory with DOP that doesn’t have S-DOP?

Remark 3.25. By Theorem 2.20 we conclude from Lemma 3.18 and Lemma 3.22 that, if $V = L$, then $\equiv_T$ is $\Sigma_1^1$-complete for every $T$ stable with the OCP or superstable theory with S-DOP.

References