Diamond sharp and the Generalized Baire Spaces

Miguel Moreno
Bar-Ilan University

European Set Theory Conference 2019

July 2019
This is a joint work with Gabriel Fernandes and Assaf Rinot at BIU.

Our paper, entitled **Inclusion modulo nonstationary** is available at https://arxiv.org/abs/1906.10066

A month ago, the name of our preprint was *Analytic quasi-orders and two forms of diamond* since we had one diamond principle for non-ineffable sets and one for ineffable sets. In the meantime, we found a single principle that works uniformly.
Outline

1 Motivation

2 Another Kind of Universality

3 A Diamond reflecting second-order formulas

4 Universality of Inclusion Modulo Nonstationarity
A Quasi-order

A quasi-order is a binary relation which is reflexive and transitive.

Definition

The quasi-order $\leq^*$ over the Baire space $\mathbb{N}^\mathbb{N}$ is defined as follows:

$$\eta \leq^* \xi \iff \{ n \in \mathbb{N} | \eta(n) > \xi(n) \} \text{ is finite.}$$

Theorem (Hechler, 1974)

The structure $(\mathbb{N}^\mathbb{N}, \leq^*)$ is universal in the following sense:
For any $\sigma$-directed poset $\mathbb{P}$ with no maximal element, there is a ccc forcing extension in which $(\mathbb{N}^\mathbb{N}, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$. 
A Refinement of $\leq^*$

**Definition**

Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\leq^S$ over $\kappa^\kappa$ by letting, for any two elements $\eta : \kappa \to \kappa$ and $\xi : \kappa \to \kappa$,

$$\eta \leq^S \xi \iff \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \text{ is nonstationary.}$$

Galvin and Hajnal (1975) used this order to attach a rank $\|\eta\|$ to each $\eta$, in studying the behavior of the power function over the singular cardinals.
Comparing $\leq^S$

How $\leq^S$ compares with $\leq^{S'}$, for different $S$ and $S'$?

Theorem

Assume that $\kappa$ is a regular uncountable cardinal and GCH holds. Then there exists a cofinality-preserving GCH-preserving forcing extension in which for all stationary subsets $S, S'$ of $\kappa$, there exists a map $f : \kappa^{\leq \kappa} \rightarrow \kappa^{\leq \kappa}$ such that, for all $\eta, \xi \in \kappa^{\leq \kappa}$,

- $\text{dom}(f(\eta)) = \text{dom}(\eta)$;
- if $\eta \subseteq \xi$, then $f(\eta) \subseteq f(\xi)$;
- if $\text{dom}(\eta) = \text{dom}(\xi) = \kappa$, then $\eta \leq^S \xi$ iff $f(\eta) \leq^{S'} f(\xi)$. 

Miguel Moreno (ESTC 19)
Outline

1. Motivation
2. Another Kind of Universality
3. A Diamond reflecting second-order formulas
4. Universality of Inclusion Modulo Nonstationarity
The Generalized Baire Space

The generalized Baire space is the set $\kappa^\kappa$ endowed with the bounded topology, in which a basic open set takes the form $[\zeta] := \{ \eta \in \kappa^\kappa \mid \zeta \subseteq \eta \}$, with $\zeta$, an element of $\kappa^{<\kappa}$.

A subset $F \subseteq \kappa^\kappa$ is closed iff its complement is open iff there exists a tree $T \subseteq \kappa^{<\kappa}$ such that $[T] := \{ \eta \in \kappa^\kappa \mid \forall \alpha < \kappa (\eta \upharpoonright \alpha \in T) \}$ is equal to $F$.

A subset $A \subseteq \kappa^\kappa$ is analytic iff there is a closed subset $F$ of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $\text{pr}(F) := \{ \eta \in \kappa^\kappa \mid \exists \xi \in \kappa^\kappa (\eta, \xi) \in F \}$ is equal to $A$. 
The Generalized Cantor Space

*The generalized Cantor space* is the subspace $2^\kappa$ of $\kappa^\kappa$ endowed with the induced topology.

The notions of open, closed and analytic subsets of $2^\kappa$, $2^\kappa \times 2^\kappa$ and $\kappa^\kappa \times \kappa^\kappa$ are then defined in the obvious way.

**Definition**

*The restriction of the quasi-order $\leq^S$ to $2^\kappa$ is denoted by $\subseteq^S$.*
Lipschitz Reduction

Let $Q_1$ and $Q_2$ be quasi-orders on $X$, $Y \in \{2^{\kappa}, \kappa^{\kappa}\}$ respectively. We say that $Q_1$ is 1-\textit{Lipschitz reducible} to $Q_2$ iff there is a function $f : X \to Y$ that satisfies for all $a, b \in X$:

- $(a, b) \in Q_1 \iff (f(a), f(b)) \in Q_2$;
- $\forall \alpha \leq \kappa \ (a \upharpoonright \alpha = b \upharpoonright \alpha) \implies (f(a) \upharpoonright \alpha = f(b) \upharpoonright \alpha)$.

We write $Q_1 \rightarrow_1 Q_2$. 
The Universality of $\subseteq^S$

**Theorem**

Assume that $\kappa$ is a regular uncountable cardinal and GCH holds. Then there exists a cofinality-preserving GCH-preserving forcing extension in which, for every analytic quasi-order $Q$ over $\kappa^{\kappa}$ and every stationary $S \subseteq \kappa$, $Q \rightarrow^1_1 \subseteq^S$.

The universality statement under consideration is optimal, as $Q \rightarrow^1_1 \subseteq^S$ implies that $Q$ analytic.
Outline

1 Motivation

2 Another Kind of Universality

3 A Diamond reflecting second-order formulas

4 Universality of Inclusion Modulo Nonstationarity
The Universality Implications

Before we define the principle $\text{DI}_S^*(\Pi^1_2)$, let us see the universality implications of it and the history behind the abstract definition of $\text{DI}_S^*(\Pi^1_2)$.

**Theorem**

*Suppose $S$ is a stationary subset of a regular uncountable cardinal $\kappa$. If $\text{DI}_S^*(\Pi^1_2)$ holds, then, for every analytic quasi-order $Q$ over $\kappa^\kappa$, $Q \rightarrow^1_1 \subset^S$.***
A Diamond reflecting second-order formulas

Time Line

(1) Friedman, Hyttinen, and Kulikov (2014) identified a reflection principle while working on the question “Is every analytic set a Borel\(^*\) set?”. They found a positive answer under the assumption “\(V = L\)”. 

(2) Hyttinen and Kulikov (2015) used this principle to show the universality of the symmetric version \(=^S\) of \(\leq^S\) for \(S = \{\alpha < \kappa \mid \text{cf}(\alpha) = \omega\}\), assuming “\(V = L\)”. 

(3) Hyttinen, Kulikov, and Moreno (2019) merged the principle of (1) with a diamond sequence to answer, under the assumption “\(V = L\)”, the question “Is it consistently true that \(\subseteq^S\) is universal for \(S = \{\alpha < \kappa \mid \text{cf}(\alpha) = \omega\}\)”.
At the level of large cardinals, working on the consistency of “Every Borel* set is analytic and co-analytic”, Asperó, Hyttinen, Kulikov, and Moreno (2019) proved: If $\kappa$ is a $\Pi^1_2$-indescribable cardinal, and $S = \{\alpha < \kappa \mid \text{cf}(\alpha) = \alpha\}$, then the symmetric version $\equiv^S$ of $\leq^S$ is universal.

Questions

- Can the assumption $V = L$ exchange for $\Pi^1_2$-reflection in Hyttinen-Kulikov-Moreno (2019)?
- How can we merge the diamond principle in Hyttinen-Kulikov-Moreno (2019) with $\Pi^1_2$-reflection?
Diamond Sharp

For sets $N$ and $x$, we say that $N$ sees $x$ iff $N$ is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$

Definition (Devlin, 1982)

Let $\kappa$ be a regular and uncountable cardinal. $\Diamond^+_\kappa$ asserts the existence of a sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ such that:

1. for every infinite $\alpha < \kappa$, $N_\alpha$ is a set of cardinality $|\alpha|$ that sees $\alpha$;
2. for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C$, $X \cap \alpha, C \cap \alpha \in N_\alpha$;
3. for every $\Pi^1_2$-sentence $\phi$ valid in a structure $\langle \kappa, \in, (A_n)_{n<\omega} \rangle$, there exists $\alpha < \kappa$, such that

$$N_\alpha \models \text{"} \phi \text{ is valid in } \langle \alpha, \in, (A_n \upharpoonright \alpha)_{n<\omega} \rangle. \text{"}$$
Good and Bad News

The good news
Devlin proved that $\diamondsuit^\#_{\kappa}$ holds in $L$ for every regular uncountable cardinal $\kappa$ that is not ineffable.

The bad news
For every ineffable cardinal $\kappa$, $\diamondsuit^\#_{\kappa}$ fails.
Even a restricted version $\diamondsuit^\#_S$ for $S \subseteq \kappa$ will still fail for any ineffable $S$.

Conclusion
We need a finer principle.
A finer principle

Definition

Let $\kappa$ be a regular and uncountable cardinal and $S \subseteq \kappa$ a stationary set. $\text{Di}^*_S(\Pi^1_2)$ asserts the existence of a sequence $\langle N_\alpha \mid \alpha \in S \rangle$ such that:

1. for every $\alpha \in S$, $N_\alpha$ is a set of cardinality $< \kappa$ that sees $\alpha$;
2. for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;
3. for every $\Pi^1_2$-sentence $\phi$ valid in a structure $\langle \kappa, \in, (A_n)_{n<\omega} \rangle$, there exists $\alpha \in S$, such that $|N_\alpha| = |\alpha|$ and

$$N_\alpha \models \text{``} \phi \text{ is valid in } \langle \alpha, \in, (A_n \upharpoonright \alpha)_{n<\omega} \rangle. \text{''}$$

Always on the good side

In $L$, $\text{Di}^*_S(\Pi^1_2)$ holds for every $\kappa = \text{cf}(\kappa) > \aleph_0$ and every stationary $S \subseteq \kappa$. 
Local Club Condensation

The Local Club Condensation (LCC) principle was defined and used by Friedman and Holy (2011) to study comparability of large cardinals with inner-type models. LCC provides us with tools to study $\text{Di}^*_5(\Pi^1_2)$. Friedman and Holy proved that the LCC can be obtained everywhere by a class forcing. A set-forcing was then devised by Holy, Welch and Wu:

**Theorem (Holy-Welch-Wu, 2015)**

Assume GCH. For every regular cardinal $\kappa$, there is a (set-size) notion of forcing $\mathbb{P}$ which is $(<\kappa)$-directed-closed and has the $\kappa^+-cc$ such that, in $V^\mathbb{P}$, the two holds:

1. There is $\tilde{M}$ such that $\langle H_{\kappa^+}, \in, \tilde{M} \rangle \models \text{LCC}(\kappa, \kappa^+]$, and
2. There is a $\Delta_1$-formula $\Theta$ and a parameter $a \subseteq \kappa$ such that the order defined by $x <_\Theta y \iff H_{\kappa^+} \models \Theta(x, y, a)$ is a global well-order of $H_{\kappa^+}$. 
Forcing $\text{Dl}^*_S(\Pi^1_2)$

**Theorem**

Suppose that $\kappa$ is a regular uncountable cardinal, and $\vec{M}$ is such that $\langle H_{\kappa^+}, \in, \vec{M} \rangle \models \text{LCC}(\kappa, \kappa^+)$. Suppose further that there is a subset $a \subseteq \kappa$ and a formula $\Theta \in \Sigma_\omega$ which defines a well-order $<\Theta$ in $H_{\kappa^+}$ via $x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x, y, a)$. Then, for every stationary $S \subseteq \kappa$, $\text{Dl}^*_S(\Pi^1_2)$ holds.

**Corollary**

Assume $\text{GCH}$. For every regular cardinal $\kappa$, there is a (set-size) notion of forcing $\mathbb{P}$ which is $(<\kappa)$-directed-closed and has the $\kappa^+-\text{cc}$ such that, in $V^\mathbb{P}$, for every stationary $S \subseteq \kappa$, $\text{Dl}^*_S(\Pi^1_2)$ holds.
Outline

1 Motivation

2 Another Kind of Universality

3 A Diamond reflecting second-order formulas

4 Universality of Inclusion Modulo Nonstationarity
The Universality

Recall

Suppose $S$ is a stationary subset of a regular uncountable cardinal $\kappa$. If $\text{DI}^*_S(\Pi^1_2)$ holds, then, for every analytic quasi-order $Q$ over $\kappa^\kappa$, $Q \rightarrow_1 \subseteq S$.

Corollary

Assume that $\kappa$ is a regular uncountable cardinal and GCH holds. Then there is a $(<\kappa)$-directed-closed, $\kappa^+ - \text{cc}$ notion of forcing $\mathbb{P}$ such that, in $V^\mathbb{P}$, GCH holds and for every analytic quasi-order $Q$ over $\kappa^\kappa$ and every stationary $S \subseteq \kappa$, $Q \rightarrow_1 \subseteq S$. 
Universality of Inclusion Modulo Nonstationarity

$\Sigma^1_1$-completeness

Definition

A quasi-order $\preceq$ over a space $X \in \{2^\kappa, \kappa^\kappa\}$ is said to be $\Sigma^1_1$-complete iff it is analytic and, for every analytic quasi-order $Q$ over $X$, there exists a $\kappa$-Borel function $f : X \to X$ reducing $Q$ to $\preceq$.

Remark

As Lipschitz $\implies$ continuous $\implies$ $\kappa$-Borel, each $\subseteq S$ is a $\Sigma^1_1$-complete quasi-order. Such a consistency was previously only known for $S$’s of one of two specific forms, and the witnessing maps were not Lipschitz.
More on Universality

By the use of canonical functions coding (Friedman) or Kurepa tree coding (Lücke): For any given quasi-order $R$ over $\kappa^\kappa$, there is a forcing extension in which:

1. $R$ is an analytic quasi-order, and
2. for every analytic quasi-order $Q$ over $\kappa^\kappa$ and every stationary $S \subseteq \kappa$, $Q \leftrightarrow_1 S$.

So the main advantage of going outside of $L$ is that we can change the quasi-orders that belong to the class of analytic sets.
Conclusions

Suppose $\text{DI}^*_{\kappa \cap \text{cof} \lambda} (\Pi^1_2)$ holds and $T$ is a first-order countable relational theory (not necessarily complete).

- If $\lambda = \aleph_0$, then Borel$^* = \Sigma^1_1$.
- If $\lambda = \aleph_0$, $\kappa$ is $\aleph_0$-inaccessible, then the embedability of linear orders is $\Sigma^1_1$-complete.
- If $\lambda = \aleph_0$, $\kappa$ is $\aleph_0$-inaccessible, and $T$ is complete stable unsuperstable, then $\sim_T$ is $\Sigma^1_1$-complete.
- If $\lambda = 2^{\aleph_0}$, $\kappa$ is inaccessible, and $T$ is complete superstable with S-DOP, then $\sim_T$ is $\Sigma^1_1$-complete.
- If $\kappa = \lambda^+$, $\lambda^{<\lambda} = \lambda$, and $T$ is complete unstable or superstable with OTOP, then $\sim_T$ is $\Sigma^1_1$-complete.
- If $\kappa = \lambda^+ > \aleph_1$, $\lambda^{<\lambda} = \lambda$, and $T$ is complete superstable with DOP, then $\sim_T$ is $\Sigma^1_1$-complete.
- If $\kappa = \lambda^+$, $\lambda^{<\lambda} = \lambda > \aleph_0$, and $\text{DI}^*_{\kappa \cap \text{cof} \aleph_0} (\Pi^1_2)$, then $\sim_T$ is either $\Delta^1_1$ or $\Sigma^1_1$-complete.
We thus feel that we have identified the correct combinatorial principle behind a line of results that were previously obtained under the heavy hypothesis of “$V = L$”.

Thank you