Thus (MMS) suppose \( f \) is irreducible and

\[
\sigma_p = \bigotimes_{r \in \mathcal{L}_p} \text{det} \sigma_r \bigotimes \alpha_{r, c} \beta_{r, c} \gamma_{r, c} \delta_{r, c} \epsilon_{r, c}
\]

is the \( \sigma \)-component of a modular weight \( \sigma \). Suppose \( r + c < p \) for all \( r \in \mathcal{L} \). Then \( \sigma \in W_p(f) \).

Proof. Generalization of Fontaine's method. See below.

Under slightly stronger hypotheses on \( \sigma \) and technical hypotheses on \( f \), one can use modularity lifting to prove \( \sigma \) modular \( \Leftrightarrow \sigma_p \in W_p(f) \) in some cases.

\( p \) uniform. 
\( p \) tet. ramified Galois.

Finally we define modularity. Let \( D/F \) be a quaternion algebra split at all places above \( p \) and exactly one infinite place. \( G = \text{PGL}_2(D) \) algebraic group.

Consider compact open \( U \subseteq \mathcal{G}(\mathbb{A}^\infty) \) and \( U = \prod U_r \).

Get Shimura curve \( X_{u/F} \),

\[
X_u(C) = \mathcal{G}(\mathbb{A}^\infty) / U.
\]

We work with \( U \) of the form
If $U^0$ is sufficiently small, then $X_{1,u} \rightarrow X_{0,u}$ in a Cohen ring with jump $\nu$, and $\nu$. 

**Def** A Cohen ring $R = \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ is minimal if there exist $\nu$ and $U \in S(A_{\nu})$ such that

$$p \subseteq \mathfrak{F}(\nu, \mathbb{F}, X_{1,u}, \mathbb{F})$$

There is a Hecke algebra $T$ (away from local primes) acting on all these forms by correspondences.

If $\mathfrak{f}$ is a minimal ideal of $\mathbb{F}$, then there exists a maximal ideal $\mathfrak{m} \subseteq T$ with $\mathfrak{f} = \overline{\mathfrak{f}}$ and a "new $p$ Hilbert modular form" $\psi \in H^1(X_{0,u} \times \overline{\mathbb{F}}, \mathbb{F})_{\nu}$ such that $\mathfrak{f} = \overline{\mathfrak{f}}$. (Note: $\overline{\mathfrak{f}}$ as in Camell.)

Let $B(\mathfrak{f}) \leq \text{Cl}_1(\mathfrak{f})$ upper triangular matrices. Let $\Theta : B(\mathfrak{f}) \rightarrow \overline{\mathbb{F}}$ be a character such that $\mathfrak{f}$ is a Jordan–Hölder constituent of $\text{Cl}_1(\mathfrak{f})$. Then $\psi$ is modular if $\nu = \psi \Theta$. 

These $p$ modulars $\Rightarrow \nu = \psi \Theta$. 

Thus if $\psi$ is modular.
find a lift $\tilde{f} \in H^1_\text{et}(X_{0,\tilde{F}}, \hat{\mathcal{O}}, F_{\tilde{\pi}}) = H^1_\text{et}(X_{0,\tilde{F}}, \hat{\mathcal{O}}, \tilde{F}_{\pi})$

with some Hecke eigenvalue.

Moreover, $X_{0,\tilde{F}}$ has an integral model over $\mathcal{O}_k \cap \tilde{F}$.

$\mathcal{O}_k \cap \tilde{F} = \mathcal{W}(\mathfrak{p}_F)$

$D = \mathcal{W}(\mathfrak{p}_F)$

$D' = \mathcal{O}_k$

$K = \text{Frac} \, D = \tilde{F}^{-\infty}$

$K' = K(\tilde{\pi}^{1/2})$

Over $D'$, $X_{0,\tilde{F}}$ has pointless reduction, with special fiber consisting of two generic irreducible components intersecting transversally at finitely many points.

(Katz–Masuoka, Cornut,サーマル, Cici)

Let $\mathfrak{p} \in \mathcal{O}_k$.

LCFT: $\text{Gal}(K'/K) = \mathcal{O}_k^{\times}/1 + \mathfrak{p} \mathcal{O}_k = \mathcal{O}_k^{\times}$.

$\sigma \mapsto \tilde{f}(\sigma)$

$\text{Gal}(K'/K)$ acts on the special fiber $X_{0,\tilde{F}}$ by $D' \to \tilde{D}'$.

Cassels' congruence relation $\Rightarrow \sigma \in \text{Gal}(K'/K)$ acts on $E$ by

$\begin{pmatrix} 1 & \tilde{f}(\sigma) \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} (1, 0) \\ (0, 1) \end{pmatrix}$

By way of this, Cassel's congruence relation and

Bartenwerfer--Kunita--Ribet $\Rightarrow \mathcal{O}_k^{\times} = \mathcal{O}(\mathfrak{p}_F^{1/2})$. 
Let $F$ be a finite field such that $i, j \in F$ and $F_q = F$.

**Def.** A vector space $V$ over a commutative group scheme $G$ with action of a finite field $F$. Let $d \in G$ be the augmentation ideal. Any $V$ satisfies (**) if $d_x$ is invertible for all $X: F \rightarrow D$.

$$d_x = \{ f \in I : \alpha^d f = X(\alpha)f \quad \forall \alpha \in F^2 \}.$$

$G[\ell]$ has a point $h$ of rank $q^2$ such that $\text{Gal}(\overline{F}/F) = I_q$ acts on $H(G)$ by $g \cdot \phi = (\phi |_{G_p} \cdot (\overline{g}, \overline{0}))$.

**Remark:** have two Galois actions

1) $\text{Gal}(\overline{F}/F)$ acts on $H(G) = W(G)$ via a character $\chi$.

$$\chi = \omega_2 \omega_3 \omega_4 \cdots \omega_{q^2 - 1}.$$

2) $\text{Gal}(\overline{F}/F)$ acts on the co-$\text{cot}(H_p, x_0, \overline{F}_p)$ for a given extension $H_p$ of $H$ to $D$ (this isn't unique if $q > p$).

Define parameters $b_0, \ldots, b_{q^2 - 1}$ such that $\text{Gal}(\overline{F}/F)$ acts on this co-$\text{cot}$ via $f(x) \cdot b_i$ (an appropriate generator of the cot when it is non-zero).

The parameters are related by

$$a'_i = b_{q^i} - pb_i + (q - 1)a_i \text{ for } 0 \leq a'_i \leq q(q - 1).$$

Define $\Theta : (a, b) \rightarrow d$. Thus, $\text{Gal}(\overline{F}/F)$ acts
\[ a 
\text{ with } b \text{ by } b_i, b_{i+q} = 0, c_i + p \sum_{j=0}^{q-1} c_{i+j} = 3. \]

Case 1: \( b_{i+1} = 0, b_i = 0 \)
\[ a_i' = (q-1)a_i, \quad a_i = 0, 1, \ldots, \circ \]

Case 2: \( b_{i+1} = 0, b_i = c_i \)
\[ a_i' = -pc_i + (q-1)a_i, \quad a_i = c_i + 1, \ldots, c_i + t \]

Case 3: \( b_{i+1} = c_{i+1} \), \( b_i = 0 \)
\[ a_i' = c_{i+1} + (q-1)a_i, \quad a_i = 0, \ldots, \circ \]

Case 4: \( b_{i+1} = c_{i+1}, b_i = c_i \)
\[ a_i' = (q-1)a_i - (q-1)c_{i+1}, \quad a_i = c_i + 1, \ldots, c_i + t. \]

We get some extraneous solutions. To get rid of these, consider all \( O: B^{(0)} \rightarrow \mathbb{F}^p \) and that \( S \in \text{SH}(\text{add} c_{i+1}, \text{add} c_i, \Theta) \), without possible \( \circ \).

Reason for the hypothesis \( c_i + t \), in that otherwise, the interaction is too big. At the level of \( O \), the above result is optimal.