REDUCTION MODULO $p$ OF CUSPIDAL REPRESENTATIONS AND WEIGHTS IN SERRE’S CONJECTURE

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Abstract. Let $O$ be the ring of integers of a $p$-adic field and $p$ its maximal ideal. We compute the Jordan-Hölder decomposition of the reduction modulo $p$ of the cuspidal representations of $GL_2(O/p^e)$ for $e \geq 1$. We also provide an alternative formulation of Serre’s conjecture for Hilbert modular forms.

1. Cuspidal representations and weights

1.1. Cuspidal representations. Let $K/\mathbb{Q}_p$ be a local field, where $p$ is a prime, and let $O$ be the ring of integers and $p$ its maximal ideal. Let $R_e = O/p^e$. In particular, $R_1 = O/p$ is the residue field; let $q = p^f$ be its cardinality. Let $\bar{K}$ be the unramified quadratic extension of $K$, and let $\bar{O}$ and $\bar{p}$ be its ring of integers and maximal ideal.

The cuspidal complex representations of $GL_2(R_e)$ are well known (see for instance [PS]) in the case $e = 1$ and have been constructed for general $e$, under various names, by several authors; see, for instance, [Shi], [Gér], [How], [Car], [BK], and [Hil]. Aubert, Onn, and Prasad proved ([AOP], Theorem B; note that the notions of cuspidal and strongly cuspidal representations coincide for $GL_2$ by Theorem A) that they are parametrized by $Gal(\bar{K}/K)$-orbits of strongly primitive characters $\xi : (\bar{O}/\bar{p}^e)^* \to \mathbb{C}^*$. A strongly primitive character of $(\bar{O}/\bar{p})^*$ is one that does not factor through the norm map $N : \bar{O}/\bar{p} \to O/p$. See [AOP], 5.2, for the definition of strongly primitive characters for general $e$. We denote by $\Theta_e(\xi)$ the cuspidal representation of $GL_2(R_e)$ corresponding to $\xi$. Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, and from now on we view $\xi$ and $\Theta_e(\xi)$ as $p$-adic representations.

In this note we compute the Jordan-Hölder constituents of $\Theta_e(\xi)$, the reduction mod $p$ of $\Theta_e(\xi)$, and use the notions introduced to reformulate the Serre-type conjecture for Hilbert modular forms of [Sch]. See the last section for some remarks about motivation.

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1.2. Brauer characters. Let $\Theta_e(\xi)$ be a cuspidal representation of $GL_2(R_e)$. The Jordan-Hölder constituents of $\Theta_e(\xi)$ are determined by its Brauer character, hence by the values of the character of $\Theta_e(\xi)$ at $p$-regular conjugacy classes. The $p$-regular conjugacy classes of $GL_2(R_e)$ are sent by the natural surjection $\pi : GL_2(R_e) \to GL_2(R_1)$ to $p$-regular conjugacy classes of $GL_2(R_1)$. Moreover,
since the kernel of $\pi$ is a $p$-group, irreducible mod $p$ representations of $\text{GL}_2(R_e)$ factor through $\pi$; see [Edi] for a proof of this. Thus the character of $\Theta_e(\xi)$ is constant on all $p$-regular conjugacy classes of $\text{GL}_2(R_e)$ lying above a given conjugacy class of $\text{GL}_2(R_1)$, and we will abusively write characters of $\text{GL}_2(R_e)$-representations as though they were functions on conjugacy classes of $\text{GL}_2(R_1)$.

Let $r \in R_1^*$ be a non-square element. Then we have an $R_1$-algebra embedding $i: \tilde{R}_1 \hookrightarrow M_2(R_1)$ given by

$$a + b\sqrt{r} \mapsto \begin{pmatrix} a & rb \\ b & a \end{pmatrix}, \quad a, b \in R_1.$$ 

Let $\mathcal{X}$ be a set of representatives of equivalence classes in $(R_1^*)^2$ under the equivalence relation $(x, y) \sim (y, x)$. Then the following is a list of representatives of $p$-regular conjugacy classes of $\text{GL}_2(R_1)$:

$$m(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad (x, y) \in \mathcal{X}$$

$$i(z), \quad z \in (\tilde{R}_1^* - R_1^*)/\text{Gal}(\tilde{R}_1/R_1).$$

Consider $\xi: \tilde{R}_e^* \to \overline{\mathbb{Q}}_p$. Its reduction modulo $p$ is a character $\tilde{\xi}: \hat{R}_e^* \to \mathbb{F}_p^*$ that factors through the map $\hat{R}_e^* \to \tilde{R}_1^*$, whose kernel is a $p$-group. Let $\chi_\xi^r: \hat{R}_1^* \to \overline{\mathbb{Q}}_p$ be the canonical lift of the resulting character, and let $\chi_\xi = \chi_\xi^r|_{R_1^*}$, where we use the embedding of $R_1 \hookrightarrow \tilde{R}_1$ that is implicit in $i$. It follows from [AOP], Theorem C, that the Brauer character $\beta_\xi$ of $\Theta_e(\xi)$ is:

$$\beta_\xi(m(x, y)) = \begin{cases} q^{e-1}(q - 1)\chi_\xi(x) & : x = y \\ 0 & : x \neq y \end{cases}$$

$$\beta_\xi(i(z)) = (-1)^e(\chi_\xi(z) + \chi_\xi(z^q)).$$

We note that the conjugacy classes of $\text{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ are computed in [CDSG]. Its character table is given in [DD]; the characters of the cuspidal representations are the family denoted $\chi_{ij}^{(p-1)}$ there.

1.3. **Weights.** The notation in this section will mostly follow [Dia]. Recall ([BL], Prop. 1.1) that the distinct irreducible $\mathbb{F}_p$-representations of $\text{GL}_2(\mathbb{O}/p) = \text{GL}_2(R_1)$ are

$$\bigotimes_{\mu: R_1 \hookrightarrow \mathbb{F}_p} \left( (\text{det}^{m_\mu} \otimes \text{Sym}^{n_\mu-1} R_1^2) \otimes_{R_1, \mu} \mathbb{F}_p \right),$$

where $1 \leq n_\mu \leq p$ and $0 \leq m_\mu \leq p - 1$ for each $\mu$, and the $m_\mu$ are not all $p - 1$. These are called weights. Let $I$ be the set of field embeddings $R_1 \hookrightarrow \mathbb{F}_p$, and fix a labeling $\mu_0, \mu_1, \ldots, \mu_{f-1}$ of its elements such that $\mu_i = (\mu_{i-1})^p$ for all $i$. We write the representation in (2) as $V_{m, \vec{n}}$, where $m = \sum_{i=0}^{f-1} m_\mu, p^i$ and $\vec{n}$ is the vector $(n_{\mu_0}, n_{\mu_1}, \ldots, n_{\mu_{f-1}})$. Clearly $0 \leq m \leq q - 2$, and we can recover the $m_\mu$ by writing $m$ in base $p$.

The Brauer character $\beta_{m, \vec{n}}$ of $V_{m, \vec{n}}$ is not hard to compute. We copy it from [Dia] for easy reference. For each $0 \leq i \leq f - 1$, let $\mu_i': \tilde{R}_1 \hookrightarrow \mathbb{F}_p$ be either of the two field embeddings lifting $\mu_i$.
to $\tilde{R}_1$. Denote the canonical lift of $\mu_i$ (resp. $\mu'_i$) to a character $R'_1 \to \mathbb{Q}_p^*$ (resp. $\tilde{R}'_1 \to \mathbb{Q}_p^*$), by $\tau_i$ (resp. $\tau'_i$). Then,

$$
\beta_{m, \tilde{R}}(m(x, y)) = \prod_{i=0}^{f-1} \left( \tau_i(x)^{m_{\mu_i}} \sum_{\nu=0}^{n_{\mu_i} - 1} \tau_i(1)^{\nu} \tau_i(x)^{n_{\mu_i} - 1 - \nu} \right)
$$

$$
\beta_{m, \tilde{R}}(i(z)) = \prod_{i=0}^{f-1} \left( \tau'_i(z)^{(q+1)m_{\mu_i}} \sum_{\nu=0}^{n_{\mu_i} - 1} \tau'_i(z)^{n_{\mu_i} - 1 + (q-1)\nu} \right).
$$

The reader can easily check that this expression is independent of the choice of $\mu'_i$.

Finally, suppose that the character $\xi : \tilde{R}'_1 \to \mathbb{Q}_p^*$ factors through the norm $N : \tilde{R}_1 \to R_1$. We will define a mod $p$ virtual representation $\Theta_1(\xi)$, which will simplify our arguments in the next section. The equality $\xi = (\tau'_0)^a$ holds for some $a$ divisible by $q + 1$. Denote by $\tilde{p}$ the vector $(p, p, \ldots, p)$, and define $\tilde{I}$ similarly. Let $a = m(q + 1)$, where $0 \leq m < q - 1$, and set

$$
\Theta_1(\xi) = V_{m, \tilde{R}} - V_{m, \tilde{I}}.
$$

It is easy to check that the Brauer character $\beta_\xi$ of this virtual representation satisfies (1).

2. JORDAN-HÖLDER CONSTITUENTS

Let $\Gamma_e$ be the Grothendieck group of virtual $\mathbb{F}_p$-representations of $GL_2(R_e)$. If $e' \geq e$, then inflation of representations from $GL_2(R_e)$ to $GL_2(R_{e'})$ induces a map $\Gamma_e \to \Gamma_{e'}$. In the sequel we will abusively consider elements of $\Gamma_e$ as lying in $\Gamma_{e'}$ for $e' \geq e$. As an element of the Grothendieck group, $\Theta_\epsilon(\xi)$ clearly depends only on $\chi'_\epsilon$. We write $\Theta_\epsilon(a)$ for $\Theta_\epsilon(\xi)$, where $a$ is such that $\chi'_\epsilon = (\tau'_0)^a$. For any integers $a$ and $w$, we define an element $P(a, w)$ of $\Gamma_1$, hence of any $\Gamma_e$, as follows:

$$
P(a, w) = \Theta_1(a - (q - 1)w).
$$

The element $P(a, w)$ may be described explicitly. If $a - (q - 1)w$ is divisible by $q + 1$, then $P(a, w)$ is described by (3) above. Otherwise, we can write $a - (q - 1)w = (q + 1)r + b$, where $1 \leq b \leq q$. Now express $b = 1 + \sum_{i=0}^{f-1} b_ip^i$, where $0 \leq b_i \leq p - 1$. Recall that $I$ is the set of embeddings $R_1 \subset \mathbb{F}_p$. For any $S \subset I$ and $\mu_i \in I$ we define $\delta_S(\mu_i)$ to be 1 if $\mu_{i-1} \in S$ and 0 otherwise. Then $P(a, w) = \sum_{S \subset I} V_{m_S, \tilde{n}_S}$, where for a subset $S \subset I$, we define $m_S$ and $\tilde{n}_S$ as follows (see [Dia], Prop.
1.3). Set \(m_{S,0} = \delta_S(\mu_0)\) if \(\mu_0 \in S\) and \(m_{S,0} = b_0 + 1\) if \(\mu_0 \notin S\). Then,

\[
m_S \equiv m_{S,0} + \sum_{i=1}^{f-1} (b_i + \delta_S(\mu_i))p^i + r \mod q - 1
\]

\[
n_{S,\mu_i} = \begin{cases} 
    b_i + 1 - \delta_S(\mu_i) & : \mu_i = \mu_0 \in S \\
    p - b_i - 1 + \delta_S(\mu_i) & : \mu_i = \mu_0 \notin S \\
    b_i + \delta_S(\mu_i) & : \mu_i \notin \mu_0, \mu_i \in S \\
    p - b_i - \delta_S(\mu_i) & : \mu_i \notin \mu_0, \mu_i \notin S.
\end{cases}
\]

Here we make the convention that if \(n_{S,\mu_i} = 0\) for any \(i\), then \(V_{m_S,\vec{\mu}_S} = 0\). Generically \(P(a, w)\) is a sum of \(2^f\) weights, but it may have fewer summands. For instance, if \(f = 1\) and \(a - (p - 1)w = (p + 1)r + b\), then the set of Jordan-Hölder constituents of \(\Theta_1(a - (p - 1)w)\) is \(\{V_{1+r,b-1} + V_{b,r,p-b}\}\).

Each constituent appears with multiplicity one. Note that if \(b \in \{1, p\}\), then \(P(a, w)\) is a single weight and not a sum of two weights.

**Lemma 2.1.** The Brauer character of \(P(a, w)\) is the following:

\[
\beta_{P(a,w)} \left( \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{cases} 
    (q - 1)\tau_0(x)^a & : x = y \\
    0 & : x \neq y
\end{cases}
\]

\[
\beta_{P(a,w)}(i(z)) = -\tau'_0(z)^a(\tau'_0(z)^{q-1}(a+w) + \tau'_0(z)^{-(q-1)w}).
\]

**Proof.** This is immediate from (1) above. \(\square\)

**Lemma 2.2.** The following equality holds in the Grothendieck group of \(\text{GL}_2(R_2)\):

\[
\overline{\Theta_2(a)} = \sum_{w=1}^{q} P(a, w).
\]

**Proof.** We need to show that the Brauer characters of the summands on the right-hand side of the formula above add up to the Brauer character of \(\overline{\Theta_2(a)}\). This claim is obvious for the conjugacy classes \(m(x, y)\). Set \(\eta(z) = \tau'_0(z)^a\). By Lemma 2.1, we see that

\[
\sum_{w=1}^{q} \beta_{P(a,w)}(i(z)) = -\tau'_0(z)^a \sum_{w=1}^{q} (\eta(z)^{a+w} + \eta(z)^{-w}).
\]

Since \(\eta(z) \neq 1\), we have \(\sum_{y=0}^{q} \eta(z)^y = 0\), and therefore

\[
\sum_{w=1}^{q} \beta_{P(a,w)}(i(z)) = \tau'_0(z)^a(\eta(z)^a + 1) = \tau'_0(z)^a + \tau'_0(z^q)^a,
\]

which by (1) is the desired result. \(\square\)
We will now give a recursive description of the Jordan-Hölder constituents of $\Theta_e(a)$ for all $e \geq 1$ and all $a$. Given $e \geq 1$ and integers $a$ and $w$, we define the following element $P_e(a, w)$ of the Grothendieck group of $\text{GL}_2(R_e)$:

$$
P_e(a, 0) = \begin{cases} 
P(a, 0) & : e = 1 \\
\sum_{w=1}^{q} P_{e-1}(a, w) & : e > 1
\end{cases}
$$

$$
P_e(a, w) = P_e(a - (q - 1)w, 0)
$$

**Theorem 2.3.** In the Grothendieck group of $\text{GL}_2(R_e)$, for $e > 1$, we have the equality

$$
\Theta_e(a) = P_e(a, 0) = \sum_{w=1}^{q} P_{e-1}(a, w).
$$

**Proof.** By the argument of Lemma 2.2 and induction on $e$, the Brauer character of $P_e(a, w)$ is:

$$
\beta_{P_e(a,w)}(m(x, y)) = \begin{cases} 
q^{e-1}(q-1)\tau_0(x)^a & : x = y \\
0 & : x \neq y
\end{cases}
$$

$$
\beta_{P_e(a,w)}(i(z)) = (-1)^e \tau_0'(z)^a(\eta(z)^a + w + \eta(z)^{-w}),
$$

where $\eta(z) = \tau_0'(z)^{q-1}$ as before. The theorem follows by comparing Brauer characters. \(\square\)

**Remark 2.4.** Observe that since $\Theta_e(a)$ is an actual representation of $\text{GL}_2(R_e)$ and not just a virtual representation, every irreducible mod $p$ representation of $\text{GL}_2(R_e)$ appears with non-negative multiplicity in $P_e(a, 0)$. We thus obtain a recursive formula for the Jordan-Hölder constituents of $\Theta_e(a)$. Moreover, it follows from the definition of strongly primitive characters in [AOP] 5.2 that if $e \geq 2$, then $\Theta_e(a)$ is the reduction modulo $p$ of an irreducible cuspidal representation of $\text{GL}_2(R_e)$ for all $a$.

3. **Weights in Serre’s conjecture**

In this section we reformulate the Serre-type conjecture for Hilbert modular forms of [Sch], using the notions introduced earlier. First we recall the form of the conjecture.

Let $F$ be a totally real field, $p$ an odd rational prime, and $\rho : \text{Gal}(\overline{F}/F) \to \text{GL}_2(\overline{\mathbb{F}}_p)$ a continuous, irreducible, totally odd Galois representation. A weight is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\text{GL}_2(\mathcal{O}_F/p)$. Any weight factors through the quotient $\text{GL}_2(\mathcal{O}_F/p)$. Any weight factors through the quotient $\text{GL}_2(\mathcal{O}_F/p)$, since the kernel of this quotient is a $p$-group. One can define what it means for $\rho$ to be modular of a given weight; see, for instance, [Sch] §2. Serre’s conjecture has been generalized to this situation; conjectures due to Buzzard, Diamond, and Jarvis [BDJ] when $p$ is unramified in $F$ and to the author [Sch] for general $F$ specify a list $W(\rho)$ of modular weights for $\rho$. We note that the conjecture of [Sch] is formulated only when $\rho$ is tamely ramified at all places dividing $p$. Moreover, there exist sets
$W_v(\rho)$ of irreducible $\mathbb{F}_p$-representations of $\text{GL}_2(\mathcal{O}_F/v)$ for each prime $v$ of $F$ dividing $p$ such that

$$W(\rho) = \{ \sigma = \otimes_v \sigma_v : \forall v, \sigma_v \in W_v(\rho) \}.$$  

Let $p$ be a place of $F$ dividing $p$, suppose that $\rho$ is tame at $p$, let $G_p \subset \text{Gal}(\overline{F}/F)$ be a decomposition subgroup at $p$, and let $I_p \subset G_p$ be the inertia. Let $K = F_p$, let $\pi$ be a uniformizer, and write $K^{nr}$ for the maximal unramified extension of $K$. As before let $q = p^f$ be the cardinality of the residue field $k$ of $K$, and denote by $I = \{ \mu_0, \ldots, \mu_{f-1} \}$ be the set of embeddings $k \hookrightarrow \mathbb{F}_p$, where the labeling is chosen so that $\mu_i = \mu^p_{i-1}$. Let $k'$ be a quadratic extension of $k$ ($\tilde{R}_1$ in the notation of the previous sections), and let $\mu'_0, \mu'_1, \ldots, \mu'_{2f-1}$ be the collection of embeddings $k' \hookrightarrow \mathbb{F}_p$, labeled so that $\mu'_i = (\mu^p_{i-1})^p$ and so that, for $0 \leq i \leq f - 1$, we have $(\mu'_i)|_k = \mu_i$.

Suppose that the restriction of $\rho$ to $G_p$ is irreducible; it follows that the restriction of $\rho$ to $I_p \simeq \text{Gal}(\overline{K}/K^{nr})$ factors through $\text{Gal}(L/K^{nr}) \simeq (k')^* \simeq \mathbb{F}_p^*$, where $L = K^{nr}(\pi^{1/(q^2-1)})$ is the totally tamely ramified extension of $K^{nr}$ of degree $q^2 - 1$, and that

$$\rho|_{I_p} \sim \left( \begin{array}{cc} \phi & 0 \\ 0 & \phi^q \end{array} \right),$$

where $\phi : (k')^* \to \mathbb{F}_p^*$ is a character such that $\phi \neq \phi^q$. Let $\Theta(\phi)$ be the cuspidal representation of $\text{GL}_2(k)$ associated to the canonical lift of $\phi$. We say that a character $\xi : (k')^* \to \mathbb{F}_p^*$ is indecomposable if $\xi^q \neq \xi$.

If $V$ is any $\mathbb{F}_p$-representation, we write $JH(V)$ for the set of its Jordan-Hölder constituents. Let $e$ be the ramification index of $K/\mathbb{Q}_p$, and let $\Delta \subset \mathbb{Z}^f$ be the collection of $f$-tuples $(\delta_{\mu_0}, \delta_{\mu_1}, \ldots, \delta_{\mu_{f-1}})$ such that $0 \leq \delta_{\mu} \leq e - 1$ for each $\mu \in I$. Let $Y_p$ be the set of irreducible $\mathbb{F}_p$-representations of $\text{GL}_2(k)$. Then for each $\delta \in \Delta$ we defined in [Sch] a multi-valued map $\mathcal{R}_\phi : Y_p \to Y_p$ and conjectured that the set of (p-components of) modular weights of $\rho$ is

$$W_p(\rho) = \bigcup_{\delta \in \Delta} \mathcal{R}_\phi(\xi \in JH(V)).$$

Herzig observed in [Her], §11 that the conjecture of Buzzard, Diamond, and Jarvis [BDJ], which addresses the unramified case $e = 1$, could be reformulated in this way. In that case, $\Delta = \{ \tilde{0} \}$, and the definition of $W_p(\rho)$ involves a single map $\mathcal{R} = \mathcal{R}_1^0$. We will now reformulate Conjecture 1 of [Sch] so that it involves only the map $\mathcal{R}$, rather than a collection of maps that depends on the ramification of $K$.

Given an $f$-tuple $\delta \in \Delta$, we define an integer $w(\delta) = \sum_{i=0}^{f-1} \delta_i p^i$. Now set $d = \sum_{i=0}^{f-1} p^i = (q - 1)/(p - 1)$, and for $\delta \in \Delta$ let $\xi_\delta$ be the character $\phi \cdot \mu_0^{w(\delta)} \cdot (\mu_0^p)^{2w(\delta) - (e - 1)d} : (k')^* \to \mathbb{F}_p^*$. If $\mu_i \in I$, where $0 \leq i \leq f - 1$, we write $\delta_{\mu_i}$ for $\delta_i$. Let $\Delta'$ be the set of $\delta \in \Delta$ such that $\xi_\delta$ does not factor through the norm map $N_{k'/k}$. 


Proposition 3.1. If the notation is as above and $\rho|_{G_p}$ is irreducible, then

$$W_p(\rho) = \bigcup_{\delta \in \Delta'} \mathcal{R}(JH(\Theta(\xi_{\delta'}))).$$

Proof. Let $\sigma = \bigotimes_{\mu \in I} ((\det^m \otimes \text{Sym}^{k_\mu} - 2k^2) \otimes k_{\mu} \mathbb{F}_p)$ be an irreducible $\mathbb{F}_p$-representation of $\text{GL}_2(k)$, where $2 \leq k_\mu \leq p+1$ for every $\mu \in I$. Suppose that $\sigma \in W_p(\rho)$. Then by [Sch], Theorem 2.4, for each $\mu$ there exists a $\delta \in \Delta$ and a labeling $\{\alpha_\mu, \beta_\mu\}$ of the two embeddings $k' \hookrightarrow \mathbb{F}_p$ lifting $\mu$ such that

$$\phi = \prod_{\mu \in I} m_{\mu} \prod_{\mu} \alpha_\mu^{k_\mu-1+\delta_\mu} \beta_\mu^{e-1-\delta_\mu}. \quad (4)$$

Let $T \subset I$ be the set of $\mu \in I$ such that $\alpha_\mu = \mu_i^e$ with $0 \leq i \leq f-1$. Define $\delta' \in \Delta$ by $\delta'_\mu = e-1-\delta_\mu$ for $\mu \in T$ and $\delta'_\mu = \delta_\mu$ for $\mu \not\in T$. Then,

$$\phi \cdot (\mu'_0)^{2w(\delta')-(e-1)d} = \prod_{\mu \in I} \mu^{m_\mu} \prod_{\mu \in T} \alpha_\mu^{k_\mu-2+e-\delta_\mu} \beta_\mu^{e-1-\delta_\mu} \prod_{\mu \not\in T} \alpha_\mu^{k_\mu-1+\delta_\mu} \beta_\mu^{\delta_\mu} = \prod_{\mu \in I} \mu^{m_\mu} \prod_{\mu \in T} \mu^{e-1-\delta_\mu} \alpha_\mu^{k_\mu-1} \prod_{\mu \not\in T} \mu^{\delta_\mu} \alpha_\mu^{k_\mu-1}.$$  

If $\xi_{\delta'}$ is indecomposable, then it follows from [Her], Theorem 11.3, that $\sigma \in \mathcal{R}(JH(\Theta(\xi_{\delta'})))$.

From the expression above it is easy to see that $\xi_{\delta'}$ is decomposable if and only if there exist numbers $-1 = r_0 < r_1 < r_2 < \cdots < r_s = f-1$ such that, possibly after a cyclic relabeling of the embeddings $\mu_i$, the set $I$ can be split into intervals $I = \{\mu_0, \mu_1, \ldots, \mu_r\}$ $\cup \{\mu_{r+1}, \mu_{r+2}, \ldots, \mu_{r+2}\}$ $\cup$ $\cdots \cup \{\mu_{r_s-1}, \mu_{r_s-2}, \ldots, \mu_{r_s}\}$ with the following properties. Each interval contains at least two elements, and for every such interval $I_i = \{\mu_{r_i-1}, \ldots, \mu_{r_i}\}$ we have $k_{r_{i-1}+1} = p+1$ and $k_{r_{i+1}+2} = \cdots = k_{r_i-1} = p$ and $k_{r_i} = 2$. Moreover, $T \subset I$ must be such that for each $I_i$ we have either $T \cap I_i = \{\mu_{r_i}\}$ or $T \cap I_i = I_i \setminus \{\mu_{r_i}\}$.

Consider the interval $I_1$, and suppose that $T \cap I_1 = \{\mu_{r_1}\}$; the other case is analogous. We must have $e > 1$, since otherwise it is easy to see that $\phi^\delta = \phi$. Then at least one of $\delta_{r_1}$ and $e-1-\delta_{r_1}$ must be non-zero. Suppose that $\delta_{r_1} \neq 0$. We write $\alpha_i$ for $\alpha_{\mu_i}$, and similarly with $\beta_i$. Then we have $\alpha_i^p = \alpha_{i+1}$ for $0 \leq i \leq r_1 - 1$ and $\alpha_{r_1-1}^p = \beta_{r_1}$. Hence the piece of $\phi$ corresponding to the elements of $I_1$ is:

$$\left( \prod_{i=0}^{r_1} \mu_{r_i}^{m_{\mu_i}} \right) \alpha_{r_1+\delta_{r_1}}^{p+\delta_{r_1} e-1-\delta_{r_1}} \left( \prod_{i=1}^{r_1-1} \alpha_i^{p-1+\delta_i} \beta_i^{e-1-\delta_i} \right) \left( \alpha_{r_1}^{e-1-\delta_{r_1}} \beta_{r_1}^{e-1-\delta_{r_1}} \right) = \left( \prod_{i=0}^{r_1} \mu_{r_i}^{m_{\mu_i}} \right) \delta_{r_1}^{p+\delta_{r_1} e-1-\delta_{r_1}} \left( \prod_{i=1}^{r_1-1} \alpha_i^{p-1+\delta_i} \beta_i^{e-1-\delta_i} \right) \left( \alpha_{r_1}^{e-1-\delta_{r_1}} \beta_{r_1}^{e-1-\delta_{r_1}} \right).$$
Therefore, if \( \tilde{\delta} \) instead of \( \delta \), then we obtain a character \( \chi \) for some characters \( \phi, \phi' \). If \( \rho|_{G_p} \) is reducible, then \( \rho|_{I_p} \) factors through \( k^* \), and, since \( \rho \) is assumed to be tame at \( p \),

\[
\rho|_{I_p} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}
\]

for some characters \( \phi, \phi' : k^* \to \mathbb{F}_p^* \). In this case, a similar argument to the above, using [Sch], Theorem 2.5, proves the following:

**Proposition 3.2.** If \( \rho|_{G_p} \) is reducible and tamely ramified, then

\[
W_p(\rho) = \bigcup_{\delta \in \Delta} \mathcal{R}(JH(\mathcal{B}_{\rho}(\overline{\Theta(\xi_\delta^\prime})))).
\]

In the unramified case \( e = 1 \), Herzig’s restatement in [Her] §11 of the conjecture of [BDJ] discovered a remarkable correspondence between irreducible characteristic zero representations of \( GL_2(k) = GL_2(R_1) \) and restrictions to inertia \( I_p \) of mod \( p \) Galois representations that are tame at \( p \). A Galois representation \( \rho \) corresponds to a representation \( V(\rho) \) of \( GL_2(k) \) such that \( W_p(\rho) = \mathcal{R}(JH(V(\rho))) \). Locally irreducible (resp. reducible) Galois representations correspond to cuspidal representations (resp. principal series). Our motivation for computing the Jordan-Hölder constituents of the reductions modulo \( p \) of representations of \( GL_2(R_e) \) was a hope that this correspondence could be generalized to all \( e \) and still be characterized in a similar way using the conjectural sets of modular weights. This hope failed, as for \( e \geq 2 \) Theorem 2.3 shows that all weights with the appropriate central character appear as constituents of \( \overline{\Theta(\xi)} \). However, Propositions 3.1 and 3.2 establish a correspondence between restrictions to inertia of tamely ramified \( \mathbb{F}_p \)-representations of \( \text{Gal}(\mathbb{F}_p/F_p) \) and collections, generically of cardinality \( e^f \), of characteristic zero representations of \( GL_2(k) \).
References


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