Characters, Descents and Matrices

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\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1 \\
\end{pmatrix}
\]
Abstract

A certain family of square matrices plays a major role in character formulas for the symmetric group and related algebras. These matrices are asymmetric variants of Walsh-Hadamard matrices, and have some fascinating properties which may be explained by use of Möbius inversion. They provide a tool for translation of statements about permutation statistics to results in representation theory, and vice versa.
Outline

1. Character Formulas

2. Matrices

3. Back to Characters
Character Formulas
A sequence \((a_1, \ldots, a_n)\) of distinct positive integers is \textbf{unimodal} if there exists \(1 \leq m \leq n\) such that

\[ a_1 > a_2 > \ldots > a_m < a_{m+1} < \ldots < a_n. \]
\textit{\(\mu\)-unimodal permutations}

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- Let \(\mu = (\mu_1, \ldots, \mu_t)\) be a \textbf{composition} of \(n\). A sequence of \(n\) positive integers is \textbf{\(\mu\)-unimodal} if the first \(\mu_1\) integers form a unimodal sequence, the next \(\mu_2\) integers form a unimodal sequence, and so on.
\(\mu\)-unimodal permutations

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- Let \(\mu = (\mu_1, \ldots, \mu_t)\) be a composition of \(n\). A sequence of \(n\) positive integers is \(\mu\)-unimodal if the first \(\mu_1\) integers form a unimodal sequence, the next \(\mu_2\) integers form a unimodal sequence, and so on.

- A permutation \(\pi \in S_n\) is \(\mu\)-unimodal if the sequence \((\pi(1), \ldots, \pi(n))\) is \(\mu\)-unimodal.
$\mu$-unimodal permutations, descent set

- Let $U_{\mu}$ be the set of all $\mu$-unimodal permutations in $S_n$. 
\(\mu\)-unimodal permutations, descent set

- Let \(U_\mu\) be the set of all \(\mu\)-unimodal permutations in \(S_n\).
- Example: \(n = 10, \mu = (3, 3, 4)\).

\[
\pi = (4, 2, 10, 9, 7, 6, 5, 3, 1, 8) \in U_\mu
\]

- \(\mu_1\)
- \(\mu_2\)
- \(\mu_3\)
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\begin{array}{|c|c|c|}
\mu_1 & \mu_2 & \mu_3 \\
\end{array}
\]

- The descent set of a permutation \(\pi \in S_n\) is

\[\text{Des}(\pi) := \{i : \pi(i) > \pi(i + 1)\}.\]
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-|-|-|-|

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- Example: \( \text{Des}(\pi) = \{1, 3, 4, 5, 6, 7, 8\} \)

- Denote \( I(\mu) := \{1, \ldots, n\} \setminus \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots\} \)
μ-unimodal permutations, descent set

• Let $U_\mu$ be the set of all $\mu$-unimodal permutations in $S_n$.

• Example: $n = 10$, $\mu = (3, 3, 4)$.

\[ \pi = (4, 2, 10, 9, 7, 6, 5, 3, 1, 8) \in U_\mu \]

\[ \mu_1 \quad \mu_2 \quad \mu_3 \]

• The descent set of a permutation $\pi \in S_n$ is

\[ \text{Des}(\pi) := \{ i : \pi(i) > \pi(i + 1) \} \]

• Example: $\text{Des}(\pi) = \{1, 3, 4, 5, 6, 7, 8\}$

• Denote $I(\mu) := \{1, \ldots, n\} \setminus \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots\}$

• Example: $I(\mu) = \{1, \ldots, 10\} \setminus \{3, 6, 10\} = \{1, 2, 4, 5, 7, 8, 9\}$

\[ \text{Des}(\pi) \cap I(\mu) = \{1, 4, 5, 7, 8\} \]
Formula 1: irreducible characters

Let $\lambda$ and $\mu$ be partitions of $n$, let $\chi^\lambda$ be the character of the irreducible $S_n$-representation corresponding to $\lambda$, and let $\chi^\lambda_\mu$ be its value on a conjugacy class of cycle type $\mu$. 
Formula 1: irreducible characters

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Theorem (Roichman ’97)

$$\chi^\lambda_\mu = \sum_{\pi \in \mathcal{C} \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap \text{I}(\mu)|},$$

where $\mathcal{C}$ is any Knuth class of shape $\lambda$. 
Let $\chi^{(k)}$ be the $S_n$-character corresponding to the symmetric group action on the $k$-th homogeneous component of its coinvariant algebra, and let $\chi_{\mu}^{(k)}$ be its value on a conjugacy class of cycle type $\mu$. 
Formula 2: coinvariant algebra, homogeneous component

Let $\chi^{(k)}$ be the $S_n$-character corresponding to the symmetric group action on the $k$-th homogeneous component of its coinvariant algebra, and let $\chi^{(k)}_{\mu}$ be its value on a conjugacy class of cycle type $\mu$.

**Theorem (A-Postnikov-Roichman '00)**

$$\chi^{(k)}_{\mu} = \sum_{\pi \in L(k) \cap U_{\mu}} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},$$

where $L(k)$ is the set of all permutations of length $k$ in $S_n$. 
A complex representation of a group or an algebra $A$ is called a **Gelfand model** for $A$ if it is equivalent to the multiplicity free direct sum of all irreducible $A$-representations. Let $\chi^G$ be the corresponding character, and let $\chi^G_\mu$ be its value on a conjugacy class of cycle type $\mu$. 

**Theorem (A-Postnikov-Roichman '08)**

The character of the Gelfand model of $S_n$ at a conjugacy class of cycle type $\mu$ is equal to 

$$\chi^G_\mu = \sum_{\pi \in \text{Inv}_n \cap \mathcal{U}(\mu)} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},$$

where $\text{Inv}_n := \{\sigma \in S_n : \sigma^2 = \text{id} \}$ is the set of all involutions in $S_n$. 


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*where $Inv_n := \{\sigma \in S_n : \sigma^2 = id\}$ is the set of all involutions in $S_n$.***
Iwahori-Hecke algebra

Let $\mathcal{H}_n(q)$ be the algebra over $\mathbb{Q}$ generated by $T_1, \ldots, T_{n-1}$ subject to the relations

\[(T_i + q)(T_i - 1) = 0 \quad (\forall i)\]

\[T_i T_j T_j T_i \quad (|j - i| > 1)\]

and

\[T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i < n - 1).\]
Iwahori-Hecke algebra characters

Theorem
In order to determine an Hecke algebra ordinary character it suffices to evaluate it on the elements $T_{\mu} := \prod_{i \in I} (\mu)_{T_i}$ over all partitions $\mu$ of $n$.

Remark.
All above formulas extend to $H_n(q)$ when replacing $(-1)$ by $(-q)$.

Example.
The character of the Gelfand model of $H_n(q)$ at the element $T_{\mu}$ is equal to $\sum_{\pi \in \text{Inv} n \cap U_{\mu}} (-q)^{|\text{Des} (\pi) \cap I(\mu)|}$,
Iwahori-Hecke algebra characters

**Theorem** In order to determine an Hecke algebra ordinary character it suffices to evaluate it on the elements $T_\mu := \prod_{i \in I(\mu)} T_i$ over all partitions $\mu$ of $n$. 
Iwahori-Hecke algebra characters

**Theorem** In order to determine an Hecke algebra ordinary character it suffices to evaluate it on the elements $T_\mu := \prod_{i \in \lambda(\mu)} T_i$ over all partitions $\mu$ of $n$.

**Remark.** All above formulas extend to $\mathcal{H}_n(q)$ when replacing $(-1)$ by $(-q)$. 
**Iwahori-Hecke algebra characters**

**Theorem** In order to determine an Hecke algebra ordinary character it suffices to evaluate it on the elements $T_\mu := \prod_{i \in I(\mu)} T_i$

over all partitions $\mu$ of $n$.

**Remark.** All above formulas extend to $H_n(q)$ when replacing $(-1)$ by $(-q)$.

**Example.** The character of the Gelfand model of $H_n(q)$ at the element $T_\mu$ is equal to

$$\sum_{\pi \in Inv_n \cap U_\mu} (-q)^{|\text{Des}(\pi) \cap I(\mu)|},$$
Inverse formulas?

Question
Are these formulas invertible?
In other words: to what extent do the character values $\chi^*_\mu (\forall \mu)$ determine the distribution of descent sets?
Matrices
Walsh-Hadamard matrices

Recursive definition

\[ H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \quad (n \geq 1) \]

with \( H_0 = (1) \).
Walsh-Hadamard matrices

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Example

\[ H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
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\[ H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} = H_1 \otimes 2 \]
Subsets as indices

Definition
Let $P_n$ be the power set (set of all subsets) of $\{1, \ldots, n\}$, with the anti-lexicographic linear order: for $I, J \in P_n$, $I \neq J$, let $m$ be the largest element in the symmetric difference $I \triangle J := (I \cup J) \setminus (I \cap J)$, and define: $I < J \iff m \in J$.

Example
The linear order on $P_3$ is

$$\emptyset < \{1\} < \{2\} < \{1, 2\} < \{3\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}. $$
Fact (explicit description of $H_n$)

The *Walsh-Hadamard matrix* $H_n$ of order $2^n$ has entries

$$h_{I,J} := (-1)^{|I \cap J|} \quad (\forall I, J \in P_n).$$

where rows and columns of $H_n$ are indexed by $P_n$ ordered as above.
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Note that

$$H_n^t = H_n$$

and

$$H_n H_n^t = 2^n I_{2^n}$$
The matrices $A$ and $B$

Recursive definition:

$$A_n = (A_{n-1}A_{n-1} - B_{n-1})$$

with $A_0 = (1)$.

and

$$B_n = (A_{n-1}0 - B_{n-1})$$

with $B_0 = (1)$.

For comparison:

$$H_n = (H_{n-1}H_{n-1} - H_{n-1})$$

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B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ 0 & -B_{n-1} \end{pmatrix} \quad (n \geq 1)
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Prefixes and runs

Definition
The prefix of length \( p \) of an interval \( \{m + 1, \ldots, m + \ell\} \) is the interval \( \{m + 1, \ldots, m + p\} \) \( (0 \leq p \leq \ell) \).
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For $l \in P_n$ let $l_1, \ldots, l_t$ be the sequence of runs (maximal consecutive intervals) in $l$. 
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Definition
For $I \in P_n$ let $I_1, \ldots, I_t$ be the sequence of runs (maximal consecutive intervals) in $I$.

Example
For $I = \{1, 2, 4, 5, 6, 8, 10\} \in P_{10}$:
$I_1 = \{1, 2\}$, $I_2 = \{4, 5, 6\}$, $I_3 = \{8\}$, $I_4 = \{10\}$. 
The matrices $A$ and $B$
Lemma (explicit description of $A_n$ and $B_n$)

For $I \in P_n$ let $I_1, \ldots, I_t$ be the runs in $I$. Define for any $J \in P_n$:

$$a_{I,J} := \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k; \\ 0, & \text{otherwise.} \end{cases}$$

Then $A_n = (a_{I,J})_{I,J \in P_n}$ and $B_n = (b_{I,J})_{I,J \in P_n}$ with $P_n$ ordered as above.
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    b_{I,J} := \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k, \\
    0, & \text{and } n \notin I \setminus J; \\
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$$A_n = (a_{I,J})_{I,J \in P_n} \quad \text{and} \quad B_n = (b_{I,J})_{I,J \in P_n}$$

with $P_n$ ordered as above.
A and B (examples)

\[ A_1 = (1) \quad B_1 = (1) \]
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$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$I = \{1, 2\}, \quad J = \{2\}$, $I \cap J = \{2\}$ is not a prefix of $I$.
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\[ A_n^t \neq A_n \quad (n \geq 2) \]
Determinant

Theorem

$A_n$ and $B_n$ are invertible for all $n \geq 0$. 
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$A_n$ and $B_n$ are invertible for all $n \geq 0$. In fact,

$$\det(A_n) = (n + 1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+4-k)} \quad (n \geq 2)$$

while $\det(A_0) = 1$ and $\det(A_1) = -2$. 
Theorem

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\]

while \( \det(A_0) = 1 \) and \( \det(A_1) = -2 \).

For comparison,

\[
\det(H_n) = 2^{2^{n-1}}n \quad (n \geq 2)
\]

with \( \det(H_0) = 1 \) and \( \det(H_1) = -2 \).
From white light to rainbow colors
Möbius inversion

Let $Z_n$ be the zeta matrix of the poset $P_n$ with respect to set inclusion:

$$z_{I,J} := \begin{cases} 1, & \text{if } I \subseteq J; \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with $Z_0 = (1)$. Its inverse is the Möbius matrix $M_n = Z_n^{-1}$, with entries $m_{I,J}$ defined by

$$m_{I,J} := \begin{cases} (-1)^{|J \setminus I|}, & \text{if } I \subseteq J; \\ 0, & \text{otherwise}. \end{cases}$$

It satisfies

$$M_n = \begin{pmatrix} M_{n-1} & -M_{n-1} \\ 0 & M_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with $M_0 = (1)$. 

AM and BM

Denote now $AM_n := A_n M_n$, $BM_n := B_n M_n$ and $HM_n := H_n M_n$. It follows that

$$AM_n = \begin{pmatrix} AM_{n-1} & 0 \\ AM_{n-1} & -(AM_{n-1} + BM_{n-1}) \end{pmatrix} \quad (n \geq 1)$$

with $AM_0 = (1)$ and

$$BM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -BM_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with $BM_0 = (1)$, as well as

$$HM_n = \begin{pmatrix} HM_{n-1} & 0 \\ HM_{n-1} & -2HM_{n-1} \end{pmatrix} \quad (n \geq 1)$$

with $HM_0 = (1)$. **
**HM entries**

\[ HM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 & 0 \\
1 & -2 & -2 & 4 & -2 & 4 & 4 & -8 & \\
\end{pmatrix} \]
HM entries

\[
HM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 & 0 \\
1 & -2 & -2 & 4 & -2 & 4 & 4 & -8 & \\
\end{pmatrix}
\]

Lemma

- **Zero pattern:** \((HM_n)_I,J \neq 0 \iff J \subseteq I\)
The image contains a section on matrices, specifically the HM entries. The matrix is shown as follows:

\[
HM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 & 0 \\
1 & -2 & -2 & 4 & -2 & 4 & 4 & -8 & \\
\end{pmatrix}
\]

The text below the matrix is labeled as a lemma:

**Lemma**

- **Zero pattern:** \((HM_n)_{I,J} \neq 0 \iff J \subseteq I\)
- **Signs:** \((HM_n)_{I,J} \neq 0 \implies \text{sign}((HM_n)_{I,J}) = (-1)^{|J|}\)
1. Character Formulas

2. Matrices

3. Back to Characters

**HM entries**

\[
HM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\
1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 \\
1 & -2 & -2 & 4 & -2 & 4 & 4 & -8
\end{pmatrix}
\]

**Lemma**

- **Zero pattern:** \((HM_n)_{I,J} \neq 0 \iff J \subseteq I\)
- **Signs:** \((HM_n)_{I,J} \neq 0 \implies \text{sign}((HM_n)_{I,J}) = (-1)^{|J|}\)
- **Absolute values:** \((HM_n)_{I,J} \neq 0 \implies |(HM_n)_{I,J}| = 2^{|J|}\)
AM entries

\[ \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & -4
\end{pmatrix} \]
AM entries

\[
AM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 & 0 \\
\end{pmatrix}
\]

Theorem

- Zero pattern: \((AM_n)_{I,J} \neq 0 \iff J \subseteq I\)
$AM$ entries

$$AM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & 0 & -4 \\
\end{pmatrix}$$

**Theorem**

- **Zero pattern:** $(AM_n)_{i,j} \neq 0 \iff J \subseteq I$
- **Signs:** $(AM_n)_{i,j} \neq 0 \implies \text{sign}((AM_n)_{i,j}) = (-1)^{|J|}$
AM entries

\[
AM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 & 0
\end{pmatrix}
\]

Theorem

- **Zero pattern:** \((AM_n)_{I,J} \neq 0 \iff J \subseteq I\)
- **Signs:** \((AM_n)_{I,J} \neq 0 \implies \text{sign}((AM_n)_{I,J}) = (-1)^{|J|}\)
- **Absolute values:** ???
AM entries

Theorem

- **Zero pattern:** \((AM_n)_I,J \neq 0 \iff J \subseteq I\)
- **Signs:** \((AM_n)_I,J \neq 0 \implies \text{sign}((AM_n)_I,J) = (-1)^{|J|}\)
- **Absolute values:**

\[
(AM_n)_I,J \neq 0 \implies |(AM_n)_I,J| = \prod_{k=1}^{t} (|J_k| + 1)\delta_k(I)
\]

where \(J_1, \ldots, J_t\) are the runs in \(J\) and, for \(J_k = \{m_k + 1, \ldots, m_k + \ell_k\}\) (\(1 \leq k \leq t\)):

\[
\delta_k(I) := \begin{cases} 
0, & \text{if } m_k \in I; \\
1, & \text{otherwise.}
\end{cases}
\]
Diagonal and last row

**Corollary**

- **All entries in the diagonal and last row of** $AM_n$ **are non-zero.**
- **Diagonal:**
  
  $$|(AM_n)_{J,J}| = \prod_{k=1}^{t} (|J_k| + 1)$$

- **Last row:**
  
  $$|(AM_n)_{[n],J}| = \begin{cases} 
  |J_1| + 1, & \text{if } 1 \in J; \\
  1, & \text{otherwise.} 
  \end{cases}$$

- **Each nonzero entry** $(AM_n)_{I,J}$ **divides the corresponding diagonal entry** $(AM_n)_{J,J}$ **and is divisible by the corresponding last row entry** $(AM_n)_{[n],J}$. 
Diagonal and last row (example)

$AM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & -4
\end{pmatrix}$

$I = \{1, 2\}$

$I = \{1, 2, 3\}$

$J = \{1, 2\}$
**Diagonal and last row (example)**

$$AM_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\
1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\
1 & -2 & -1 & 3 & -1 & 2 & 1 & -4
\end{pmatrix}$$

$$I = \{2, 3\}$$

$$I = \{1, 2, 3\}$$

$$J = \{2, 3\}$$
Eigenvalues

\[ A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1
\end{pmatrix} \]

Question: What can be said about its eigenvalues?

Answer: char. poly. \( A_2 \) = \((x^2 - 4)(x^2 - 3)\)
Eigenvalues

\[ A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1
\end{pmatrix} \]

\[ A_2^t \neq A_2 \quad A_2 A_2^t \neq 4I_4 \]
Eigenvalues

\[ A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
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\end{pmatrix} \]

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Eigenvalues

\[ A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix} \]

\[ A_2^t \neq A_2 \quad A_2 A_2^t \neq 4I_4 \]

Question: What can be said about its eigenvalues?

Answer: char. poly.\( (A_2) = (x^2 - 4)(x^2 - 3) \)
Eigenvalues

\[ A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1
\end{pmatrix} \]

\[ A_2^t \neq A_2 \quad A_2 A_2^t \neq 4I_4 \]

**Question:** What can be said about its eigenvalues?

**Answer:** \( \text{char. poly.}(A_2) = (x^2 - 4)(x^2 - 3) \)

\[ A_2^2 = \begin{pmatrix}
4 & 0 & 1 & 0 \\
0 & 4 & -1 & 0 \\
0 & 0 & 3 & 0 \\
1 & 1 & 0 & 3
\end{pmatrix} \]
Eigenvalues

\[ A_3^2 = \begin{pmatrix}
8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\
2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\
0 & 2 & 0 & 2 & 1 & 1 & 0 & 4 \\
\end{pmatrix} \]
Eigenvalues

\[ A_3^2 = \begin{pmatrix}
8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\
2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\
0 & 2 & 0 & 2 & 1 & 1 & 0 & 4
\end{pmatrix} \]

\[ \text{char. poly.}(A_3) = (x^2 - 8)(x^2 - 6)^2(x^2 - 4) \]
Conjecture
The eigenvalues of $A_n^2$ (counted by algebraic multiplicity) are in 1 : 1 correspondence with the diagonal entries of $A_n^2$ (which are explicitly known).
Theorem (G. Alon ’13)
The eigenvalues of $A^2_n$ (counted by algebraic multiplicity) are in 1 : 1 correspondence with the diagonal entries of $A^2_n$, and thus in 2:1 correspondence with the compositions $\mu = (\mu_1, \ldots, \mu_t)$ of $n$:

$$\pi_\mu = \prod_{i=1}^t (\mu_i + 1).$$
Eigenvalues

Theorem (G. Alon ’13)
The eigenvalues of $A_n^2$ (counted by algebraic multiplicity) are in 1 : 1 correspondence with the diagonal entries of $A_n^2$, and thus in 2:1 correspondence with the compositions $\mu = (\mu_1, \ldots, \mu_t)$ of $n$:

$$\pi_\mu = \prod_{i=1}^{t} (\mu_i + 1).$$

Similarly, The eigenvalues of $B_n^2$ are in 1 : 1 correspondence with the diagonal entries of $B_n^2$, and thus in 2:1 correspondence with the compositions of $n$:

$$\pi'_\mu = \prod_{i=1}^{t-1} (\mu_i + 1).$$
Eigenvalues

$$A_3 \sim \begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$
Back to Characters
Definition
Let $\mathcal{B}$ be a set of combinatorial objects.
Fine sets

Definition
Let $\mathcal{B}$ be a set of combinatorial objects. Let $\text{Des} : \mathcal{B} \rightarrow P_{n-1}$ be a map which associates a “descent set” $\text{Des}(b) \subseteq [n-1]$ to each element $b \in \mathcal{B}$. Then $\mathcal{B}$ is called a fine set for a complex $S_n$-representation $\rho$ if, for each composition $\mu$ of $n$, the character value of $\rho$ on a conjugacy class of cycle type $\mu$ satisfies
$$\chi_\rho(\mu) = \sum_{b \in \mathcal{B}} (-1)^{|\text{Des}(b)| - S(\mu)}$$.
Fine sets

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Let $\text{Des} : \mathcal{B} \rightarrow P_{n-1}$ be a map which associates a “descent set” $\text{Des}(b) \subseteq [n - 1]$ to each element $b \in \mathcal{B}$.
Denote by $\mathcal{B}^\mu$ the set of elements in $\mathcal{B}$ whose descent set $\text{Des}(b)$ is $\mu$-unimodal.
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Then $\mathcal{B}$ is called a fine set for a complex $S_n$-representation $\rho$ if,
Fine sets

Definition
Let $\mathcal{B}$ be a set of combinatorial objects. Let $\text{Des} : \mathcal{B} \to P_{n-1}$ be a map which associates a “descent set” $\text{Des}(b) \subseteq [n-1]$ to each element $b \in \mathcal{B}$. Denote by $\mathcal{B}^\mu$ the set of elements in $\mathcal{B}$ whose descent set $\text{Des}(b)$ is $\mu$-unimodal.

Then $\mathcal{B}$ is called a **fine set** for a complex $S_n$-representation $\rho$ if, for each composition $\mu$ of $n$, the character value of $\rho$ on a conjugacy class of cycle type $\mu$ satisfies

$$
\chi^\rho_\mu = \sum_{b \in \mathcal{B}^\mu} (-1)^{|\text{Des}(b) \setminus S(\mu)|}.
$$
Character values and descent sets

Theorem (Fine Set Theorem)

If $B$ is a fine set for an $S_n$-representation $\rho$, then the character values of $\rho$ uniquely determine the overall distribution of descent sets over $B$. 

Idea of proof

For a subset $J = \{j_1, \ldots, j_k\} \subseteq [n-1]$ let 

$$s_J := s_{j_1}s_{j_2} \cdots s_{j_k} \in S_n.$$ 

Let $\chi_\rho$ be the vector with entries $\chi_\rho(s_J)$, for $J \in \mathcal{P}_{n-1}$, and let $v_B$ be the vector with entries $v_B(J) := |\{b \in B : \text{Des}(b) = J\}|$ ($\forall J \in \mathcal{P}_{n-1}$).

Then $B$ is a fine set for $\rho$ if and only if $\chi_\rho = A_{n-1}v_B$.

The result follows since $A_{n-1}$ is an invertible matrix.
Character values and descent sets

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$$\nu^B_J := |\{b \in B : \text{Des}(b) = J\}| \quad (\forall J \in P_{n-1}).$$
Character values and descent sets

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For a subset $J = \{j_1, \ldots, j_k\} \subseteq [n - 1]$ let $s_J := s_{j_1}s_{j_2}\cdots s_{j_k} \in S_n$. Let $\chi^\rho$ be the vector with entries $\chi^\rho(s_J)$, for $J \in P_{n-1}$, and let $v^\mathcal{B}$ be the vector with entries

$$v^\mathcal{B}_J := |\{b \in \mathcal{B} : \text{Des}(b) = J\}| \quad (\forall J \in P_{n-1}).$$

Then $\mathcal{B}$ is a fine set for $\rho$ if and only if

$$\chi^\rho = A_{n-1} v^\mathcal{B}.$$
Character values and descent sets

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If $B$ is a fine set for an $S_n$-representation $\rho$, then the character values of $\rho$ uniquely determine the overall distribution of descent sets over $B$.

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For a subset $J = \{j_1, \ldots, j_k\} \subseteq [n - 1]$ let $s_J := s_{j_1} s_{j_2} \cdots s_{j_k} \in S_n$. Let $\chi^\rho$ be the vector with entries $\chi^\rho(s_J)$, for $J \in \mathcal{P}_{n-1}$, and let $v^B$ be the vector with entries

$$v^B_J := |\{b \in B : \text{Des}(b) = J\}| \quad (\forall J \in \mathcal{P}_{n-1}).$$

Then $B$ is a fine set for $\rho$ if and only if

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The result follows since $A_{n-1}$ is an invertible matrix.
Example

Let $\rho$ be the regular representation of $S_2$. 
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The set of all permutations in \( S_2 \) is a fine set for \( \rho \).
Example

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$$v^{S_2} = \begin{pmatrix} 1 & \emptyset \\ 2 & \{1\} \\ 2 & \{2\} \\ 1 & \{1, 2\} \end{pmatrix}$$
Let $\rho$ be the regular representation of $S_2$.
The set of all permutations in $S_2$ is a fine set for $\rho$.
Then
\[
v^{S_2} = \begin{pmatrix}
1 \\
2 \\
2 \\
1
\end{pmatrix}
\begin{array}{c}
\emptyset \\
\{1\} \\
\{2\} \\
\{1, 2\}
\end{array}
\]

and
\[
A_2v^{S_2} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 \\
1 & -1 & 0 & 1 & 1
\end{pmatrix}
v^{S_2} = \begin{pmatrix}
6 \\
0 \\
0 \\
0
\end{pmatrix}
\]
Explicit inversion formula

**Theorem**

Let $B$ be a fine set for an $S_n$-representation $\rho$. For every $I \subseteq [n-1]$, the number of elements in $B$ with descent set $D$ satisfies

$$|\{b \in B : \text{Des}(b) = D\}| = \sum_J \chi^\rho(c_J) \sum_{I : D \cup J \subseteq I} (-1)^{|\backslash D|} (AM_{n-1}^{-1})_{I,J}$$
Theorem

Let $B$ be a fine set for an $S_n$-representation $\rho$. For every $I \subseteq [n-1]$, the number of elements in $B$ with descent set $D$ satisfies

$$|\{b \in B : \text{Des}(b) = D\}| = \sum_J \chi^\rho(c_J) \sum_{I: D \cup J \subseteq I} (-1)^{|I\setminus D|} (AM_{n-1}^{-1})_{I,J}$$

where

$$(AM_{n-1}^{-1})_{I,J} = \frac{(-1)^{|J|}}{|\langle I \rangle|} \prod_{k=1}^{t} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1),$$

$I_1, \ldots, I_t$ are the runs in $I$ and $c_J := \prod_{j \in J} s_j$ is a Coxeter element in the parabolic subgroup $\langle J \rangle$. 

Equivalence of classical theorems

Corollary

*Given two symmetric group modules with fine sets, the isomorphism of these modules is equivalent to equi-distribution of the descent set on their fine sets.*
The major index of a permutation $\pi$ is $\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i$, and its length $\ell(\pi)$ is the number of inversions in $\pi$. For a subset $I \subseteq [n-1]$ denote $x^I := \prod_{i \in I} x_i$.

**Theorem (Foata-Schützenberger; Garsia-Gessel)**

$$\sum_{\pi \in S_n} x^{\text{Des}(\pi)} q^{\ell(\pi)} = \sum_{\pi \in S_n} x^{\text{Des}(\pi)} q^{\text{maj}(\pi^{-1})}.$$
For $0 \leq k \leq \binom{n}{2}$ let $R_k$ be the $k$-th homogeneous component of the coinvariant algebra of the symmetric group $S_n$.

For a partition $\lambda$, let $m_{k,\lambda}$ be the number of standard Young tableaux of shape $\lambda$ with major index $k$.

**Theorem (Lusztig-Stanley)**

$$R_k \cong \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda,$$

where the sum runs over all partitions of $n$ and $S^\lambda$ denotes the irreducible $S_n$-module indexed by $\lambda$. 

Equivalence of classical theorems

The Fine Set Theorem implies

**Corollary**

*The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.*
Equivalence of classical theorems

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**Idea of proof**
Equivalence of classical theorems

The Fine Set Theorem implies

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*The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.*

**Idea of proof**

\[ B_k = \{ \pi \in S_n : \text{maj}(\pi^{-1}) = k \} \]

is a fine set for the representation

\[ \rho_k := \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda. \]
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The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.

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\[ L_k = \{ \pi \in S_n : \ell(\pi) = k \} \]

is a fine set for \( R_k \).
Equivalence of classical theorems

The Fine Set Theorem implies

Corollary

The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.

Idea of proof

$B_k = \{ \pi \in S_n : \text{maj}(\pi^{-1}) = k \}$ is a fine set for the representation $\rho_k := \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda$. $L_k = \{ \pi \in S_n : \ell(\pi) = k \}$ is a fine set for $R_k$.

Thus $\rho_k \cong R_k$ if and only if the distributions of the descent set over $B_k$ and $L_k$ are equal.
Summary

• Asymmetric variants of Walsh-Hadamard matrices serve as a bridge between characters and combinatorial permutation statistics.
• They have fascinating properties, with a strong combinatorial flavor.
• And offer many more riddles, awaiting (your) solution!
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Summary

- Asymmetric variants of Walsh-Hadamard matrices
- ... serve as a bridge between characters and combinatorial permutation statistics
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- ... and offer many more riddles, awaiting (your) solution!
$A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1
\end{pmatrix}$
$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$

THANK YOU!