Percolation in Interdependent and Interconnected Networks: Abrupt Change from Second to First Order Transition

Yanqing Hu1,2,3, Baruch Kasherim2, Reuven Cohen2, Shlomo Havlin2
1. School of Mathematics, Southwest Jiaotong University, Chengdu 610031, China.
2. Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
3. Department of Systems Science, School of Management and Center for Complexity Research, Beijing Normal University, Beijing 100875, China
4. Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

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Robustness of two coupled networks system has been studied only for dependency coupling (S. Buldyrev et. al., Nature, 2010) and only for connectivity coupling (E. A. Leicht and R. M. D’Souza, arxiv:09070894). Here we study, using a percolation approach, a more realistic coupled networks system where both interdependent and interconnected links exist. We find a rich and unusual phase transition phenomena including hybrid transition of mixed first and second order i.e., discontinuities like a first order transition of the giant component followed by a continuous decrease to zero like a second order transition. Moreover, we find unusual discontinuous changes from second order to first order transition as a function of the dependency coupling between the two networks.

During the last decade complex networks have been studied intensively, where most of the research was devoted to analyzing the structure and functionality isolated systems modeled as single non-interacting networks [1–11]. However, most real networks are not isolated, as they either complement other networks (“interconnected networks”), must consume resources supplied by other networks (“interdependent networks”) or both [12–16]. Thus, real networks continuously interact with each other, composing large complex systems, and with the enhanced development of technology, the coupling between many networks becomes more and more significant.

Two different types of coupled networks models have been studied. Buldyrev et. al. [17] investigated the robustness of coupled systems with only interdependence links. In these systems, when a node of one network fails, its dependent counterpart node in the other network also fails. They found that this interdependence makes the system significantly more vulnerable [17, 18]. In the same time, Leicht and D’Souza [19] studied the case where only connectivity links couple the networks, i.e., “interconnected networks”, and found that the interconnected links make the system significantly more robust. However, real coupled networks often contain both types of links, interdependent as well as interconnected links. For example, the airport and the railway networks in Europe are two coupled networks composing a transportation system. In order to arrive to an airport, one usually uses the railway. Also, people arriving to the country by airport usually use the railway. In this system, if the airport is disabled by some strike or accident, the passengers can still use the nearby railway station and travel to their destination or to another airport by train, so the two networks are coupled by connectivity links. On the other hand, if the railway network is disabled, the airport traffic is damaged, and if the airport is disabled, the railway traffic is damaged, so both networks are coupled by dependency links as well. The important characteristics of such systems, is that a failure of nodes in one network carries implications not only for this network, but also on the functionality of other dependent networks. In this way it is possible to have cascading failures between the coupled networks, that may lead to a catastrophic collapse of the whole system. Nevertheless, small clusters disconnected from the giant component in one network can still function through interconnected links connecting them to the giant component of other network. Thus, the inter-connectivity links increase the robustness of the system, while the inter-dependency links decrease its robustness. Here we study the competition of the two types of inter-links on robustness using a percolation approach, and find unusual types of phase transitions.

Let us consider a system of two networks, A and B, which are coupled by both dependency and connectivity links. The two networks are partially coupled by dependency links, so that a fraction \( q_A \) of A-nodes depends on nodes in network B, and a fraction \( q_B \) of B-nodes depends on the nodes in network A, with the following two exceptions: a node from one network depends on no more than one node from the other network, and assuming that node \( A_i \) depends on node \( B_j \), then if \( B_j \) depends on some \( A_h \), then \( h = i \) (see Fig. 1). In addition, the connectivity links within each network and between the networks (see Fig. 1) can be described by a set of degree distributions \( u_{x_k}^A \), \( u_{x_k}^B \), and \( p_{x_k}^A, q_{x_k}^B \), where \( p_{x_k}^A \) denotes the probability of an A-node (B-node) to have \( k_A \) (\( k_B \)) links to other A-nodes (B-nodes) and \( k_{AB} \) (\( k_{BA} \)) links towards B-nodes (A-nodes). In this manner we get a two dimensional generating function describing all the connectivity links [19], \( G_{x_A,x_B}^A = \sum_{k_A,k_B} p_{x_k}^A k_A \frac{x_A}{k_A} k_{BA} \frac{x_B}{k_B} \), and \( G_{x_A,x_B}^B = \sum_{k_A,k_B} q_{x_k}^B k_B \frac{x_B}{k_B} k_{BA} \frac{x_A}{k_A} \).

The cascading process is initiated by randomly removing a fraction \( 1 - p \) of the A-nodes and all their connectivity links. Because of the interdependence between the networks, the nodes in network B that depend on the removed A-nodes are also removed along with their connectivity links. As nodes and links are removed, each network breaks up into connected components (clusters). We assume that when the network is fragmented, the nodes belonging to the largest component (giant component) connecting a finite fraction of the network are still functional, while nodes that are parts of the remaining smaller clusters become dysfunctional, unless there exist a
path of connectivity-links connecting these small clusters to the largest component of the other network. Since the networks have different topologies, the removal of nodes and related dependency links, is not symmetric in both networks, so that, a cascading process occurs, until the system either becomes fragmented or stabilizes with a giant component.

Let \(g_A(\varphi, \phi)\) and \(g_B(\varphi, \phi)\) be the fraction of A-nodes and B-nodes in the giant components after the percolation process initiated by removing a fraction of \(1-\varphi\) and \(1-\phi\) of networks A and B respectively [11]. The functions \(g_A(\varphi, \phi)\) and \(g_B(\varphi, \phi)\) depend only on \(G_A(x_A, x_B)\) and \(G_B(x_A, x_B)\) (see Appendix) and the cascading process can be described by the following set of equations,

\[
\begin{align*}
\varphi_1 &= p, \quad \phi_1 = 1, \quad P^A_1 = \varphi_1 g_A(\varphi_1, \phi_1), \quad (1) \\
\varphi_2 &= 1 - q_B(1 - p g_A(\varphi_1, \phi_1)), \quad P^B_1 = \phi_1 g_B(\varphi_1, \phi_2), \\
\varphi_3 &= p(1 - q_A(1 - g_B(\varphi_2, \phi_2))), \quad P^A_2 = \varphi_2 g_A(\varphi_2, \phi_2), \\
\varphi_4 &= 1 - q_B(1 - p g_A(\varphi_3, \phi_2)), \quad P^B_2 = \phi_3 g_B(\varphi_2, \phi_3), \\
\varphi_n &= p(1 - q_A(1 - g_B(\varphi_{n-1}, \phi_n))), \quad (2) \\
\phi_n &= 1 - q_B(1 - p g_A(\varphi_{n-1}, \phi_{n-1})), \\
P^A_n &= \varphi_n g_A(\varphi_n, \phi_n), \quad P^B_n = \phi_n g_B(\varphi_{n-1}, \phi_n).
\end{align*}
\]

When the phase transition is of second order, i.e., the giant components at the percolation threshold is zero. Thus, according to the limit of system (6) at \(u_A = u_B = 0\) we obtain

\[
u_A = g_A(\varphi_\infty, \phi_\infty), \quad u_B = g_B(\varphi_\infty, \phi_\infty), \quad (3)
\]

we can write the equations at the end of the cascading process,

\[
\varphi_\infty = p(1 - q_A(1 - u_B)), \quad \phi_\infty = 1 - q_B(1 - p u_A), \quad (4)
\]

and the giant components are,

\[
P^A_\infty = u_A p (1 - q_A(1 - u_B)), \quad (5)
\]

\[
P^B_\infty = u_B p (1 - q_B(1 - p u_A)).
\]

In the case where all degree distributions of intra- and inter-links are Poisson distributed, the functions obtain a simple form. Assume \(\bar{\kappa}_A\) and \(\bar{\kappa}_B\) are the average intra-links degrees in networks A and B, and \(\bar{k}_{AB}, \bar{k}_{BA}\) are the average inter-links degrees between A and B (allowing the case \(\bar{k}_{AB} \neq \bar{k}_{BA}\), since the two networks may be of different sizes), we obtain,

\[
\begin{align*}
u_A &= 1 - e^{-\bar{\kappa}_A p u_A (1 - q_A(1 - \phi_\infty))}, \\
u_B &= 1 - e^{-\bar{\kappa}_B p u_A (1 - q_B(1 - \phi_\infty))}.
\end{align*}
\]

By introducing two new notations

\[
u_A = g_A(\varphi_\infty, \phi_\infty), \quad u_B = g_B(\varphi_\infty, \phi_\infty), \quad (3)
\]

we can write the equations at the end of the cascading process,

\[
\varphi_\infty = p(1 - q_A(1 - u_B)), \quad \phi_\infty = 1 - q_B(1 - p u_A), \quad (4)
\]

and the giant components are,

\[
P^A_\infty = u_A p (1 - q_A(1 - u_B)), \quad (5)
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P^B_\infty = u_B p (1 - q_B(1 - p u_A)).
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In the case where all degree distributions of intra- and inter-links are Poisson distributed, the functions obtain a simple form. Assume \(\bar{\kappa}_A\) and \(\bar{\kappa}_B\) are the average intra-links degrees in networks A and B, and \(\bar{k}_{AB}, \bar{k}_{BA}\) are the average inter-links degrees between A and B (allowing the case \(\bar{k}_{AB} \neq \bar{k}_{BA}\), since the two networks may be of different sizes), we obtain,

\[
\begin{align*}
u_A &= 1 - e^{-\bar{\kappa}_A p u_A (1 - q_A(1 - \phi_\infty))}, \\
u_B &= 1 - e^{-\bar{\kappa}_B p u_A (1 - q_B(1 - \phi_\infty))}.
\end{align*}
\]

Generally, for fixed parameters \(\bar{\kappa}_A, \bar{\kappa}_B, \bar{k}_{AB}, \bar{k}_{BA}, q_A, q_B\) and \(p\), it is often impossible to achieve an explicit formula for the giant components \(P^A_n\) and \(P^B_n\). However, one can still solve Eqs. (6) graphically and substitute the numerical solution to Eqs. (5). For example, we study the case where \(\bar{\kappa}_A = \bar{\kappa}_B = \bar{\kappa}\) and \(\bar{k}_{AB} = \bar{k}_{BA} = \bar{k}\). Fig. 2a compares the numerical with the simulation results for \(P^A_n\) and \(P^B_n\) as a function of \(p\), showing that the analytical results of Eqs. (5) and (6) are in excellent agreement with the simulations.

Next we are interested in the properties of the phase transition under random attack, so first we determine the conditions when transition does not occur. This is the case when even all nodes of network A are removed (\(p = 0\)), for a given \(q_B < 1\), there still exists a giant component in network B (see circles in Fig. 2a) and no phase transition occurs. For Poisson degree distributions, if after the removal of all B-nodes that depend on the attacked A-nodes, the new average intra-link degree in network B is less than one, i.e.,

\[
\bar{k}_B(1 - q_B) < 1, \quad (7)
\]

a phase transition occurs. Therefore, the following analysis is based on condition (7). In addition, we always set both dependency strengths, \(q_A\) and \(q_B\), to be larger than zero.

When the phase transition is of second order, i.e., the giant components at the percolation threshold is zero. Thus, according to the limit of system (6) at \(u_A = u_B = 0\) we obtain

\[
\frac{1 - \bar{\kappa}_B(1 - q_B)}{(\bar{\kappa}_A + (\bar{k}_{BA} - \bar{k}_{AB})(1 - q_B))(1 - q_A)}.
\]

(8)
and the corresponding maximum $u_A$, $u_B$ of the minimal $p$ is the solution of the system at criticality.

When networks $A$ and $B$ are fully dependent, i.e., $q_A = q_B = 1$, system (6) yields a simple form

$$u_A = 1 - \exp[-pu_A u_B (\kappa_A + \kappa_B)],$$

$$u_B = 1 - \exp[-pu_A u_B (\kappa_B + \kappa_A)].$$

The size of the mutual giant component, $P_\infty$, is thus given by,

$$P_\infty = P_\infty^A = P_\infty^B = p(1 - e^{-\kappa_A u_A u_B}) (1 - e^{-\kappa_B u_A u_B}).$$

(11) which is similar to the solution of fully interdependent system [17], where the only difference is that the degrees of networks $A$ and $B$ are now replaced by $\kappa_A + \kappa_{AB}$ and $\kappa_B + \kappa_{BA}$, respectively. Thus, interestingly, in a fully interdependent coupled networks adding connectivity inter-links has the same effect as increasing the intra-degree of the corresponding networks and therefore, in this case, the phase transition must be of first order. From Eqs. (9) and (10), one can get the threshold,

$$p_t^I = \frac{1}{\kappa_A (1 - u_A) - 1 + (1 - u_A) - u_A \alpha (1 - u_A) - 1].$$

(12)

where, $\alpha \equiv (\kappa_B + \kappa_{BA}) / (\kappa_A + \kappa_{AB})$, and $u_A$ satisfies the equation,

$$u_A = 1 - \exp\left[\frac{u_A (1 - u_A)^{\alpha}}{(1 - u_A) - 1 + (1 - u_A)^{\alpha} - u_A \alpha (1 - u_A) - 1]\right].$$

(13)

For fully interdependent system, both networks are of the same size and therefore, $\kappa_{AB} = \kappa_{BA}$.

By substituting $p_t^I$ from Eq. (8) into Eqs. (9) and (10) and evaluating both $u_A$ and $u_B$ we can draw and derive in the phase diagram, the boundary between the first and second order transitions (see dashed line in Fig. 2b). The most interesting phenomenon, which to the best of our knowledge, has not been observed before, is that when the phase transition changes from first to second, there are discontinuities (abrupt jumps) of $P_\infty^A(p_c), P_\infty^B(p_c)$ in the phase transition boundary (see Fig. 3a). On the boundary between first and second order phase transition, $p_t^I$ should always be the maximum non-negative solution in $[0,1]$. When Eq. (9) and (10) has more than one solution, we always choose the minimal non-negative value, $p_t^I$, and the corresponding maximum value solution $u_A^{max}, u_B^{max}$ as the physical solution at the threshold.

In some regime of the boundary, $u_A^{max} > 0$ and $u_B^{max} > 0$, and of course $p_t^I, u_A = 0, u_B = 0$ also is the system solution. It means that there exist two intersections and both of them satisfy the tangential condition (as shown in Fig. 4) on the boundary. This implies that when the order of the phase transition changes from first to second, $P_\infty^A(p_c), P_\infty^B(p_c)$ are discontinuous. This phenomenon contrasts most systems possessing both first and second order transitions. In physical systems usually, the first order jump in the order parameter, and

![FIG. 2: (Color online) a. Giant components $P_\infty^A$ and $P_\infty^B$ vs. fraction of remaining nodes, $p$, for $N = 10000, k = 2$ and $K = 1$. Networks $A$ (open symbols) and $B$ (full symbols) for different $(q_A, q_B)$ pairs: $(0.8, 0.1), (0.8, 0.8), (0.1, 0.1)$ (□). The symbols represent simulations and the lines the theory. We see three types of behaviors: no phase transition (○), second order phase transition (□) and first order phase transition (●). b. Phase diagram showing the first order, second order and hybrid phase transition regimes and the boundaries, for $q_B = 1, K = 3$. In the second order transition regime, between the two dashed curve (red and blue) is the hybrid phase transition regime (details in Fig. 3c and in the SI). Since the hybrid transition is continuous in the neighborhood of $p_c$, and jump occurs well above $p_c$ we classify a hybrid phase transition as a second order phase transition.](image)
In summary, we studied the cascade of failures in coupled networks, when both interdependent and interconnected links exist, using a percolation approach. Although our detailed analysis is for ER networks, the theory can be applied to any network systems topology. We find that the existence of inter-connectivity links between interdependent networks, introduces rich and intriguing phenomena through the process of cascading failures. Increasing the strength of interconnecting links can change the transition behavior significantly and often brings up some counterintuitive phenomenon, such as changing the transition from second order to first order (as seen in Fig. 2b). We also find an unusual abrupt jump in the boundary between first and second order phase transitions at the critical point, which, to the best of our knowledge, has not been observed earlier in physical systems. Moreover, when one of the networks strongly depends on the other network, unusual hybrid phase transitions are observed.

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**FIG. 3:** a. Size of giant components vs. dependency and connectivity links strength, for $q_A = 1$ and $\bar{K} = 3$. The giant components size at $p_c$ changes from zero to a finite value while changing $q_A$ and $\bar{K}$. When $q_A$ and $\bar{K}$ are at the boundary of different phase transitions, the jump occurs. b. The values of $P^A_{\infty}(p_c)$ ( ), $P^B_{\infty}(p_c)$ ( ) along the boundary for $q_A = 1$ and $\bar{K} = 3$. c. Hybrid phase transition, for $q_A = 1$, $q_B = 0.35$, $\bar{K} = 3$ and $\bar{K} = 0.1$. According to Eqs. (5), $P^A_{\infty}$ and $P^B_{\infty}$ have the same properties as $u_A$ and $u_B$ respectively. At $p \approx 0.66$ the values of $u_A$ and $u_B$ jump, and then for lower $p$ value continuously approach zero. In the inset, simulation and theoretical results are symbols and lines respectively.

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**FIG. 4:** Abrupt jump on the boundary, here $q_A = 0.394, q_B = 0.8, \bar{K} = 3, \bar{K} = 0.2$, $p'_1 = p''_1 = 0.5464$ which is the threshold of the system. Although, both intersections (one of which is at the origin) satisfy the tangential condition, the $u_A^{\text{max}}, u_B^{\text{max}}$ is the physical solution and the transition is of the first order.
FIG. 5: Hybrid transition analysis, for $q_B = 1$, $q_A = 0.35$, $K = 3$ and $\bar{R} = 0.1$, here $p_c \approx 0.556$, $p_h \approx 0.66$ The maximum intersection $S$ satisfies tangential condition. When continuously decreasing $p$, the solution of the system jumps from the maximum intersection $S$ to the minimum intersection $Q$ and then continuously decrease to zero.

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