1. Growth of groups and amenability (cont.)

Recall from the last lecture:

**Definition 1.1.** Let $G$ be a finitely generated group, with a symmetric set of generators $S$. Consider the set $S^n$ of all words of length $n$ using the generators as letters. The set $B_S(n) = \bigcup_{i \leq n} S^i$ is the ball of radius $n$ centered at the identity, with respect to the word metric corresponding to $S$ (the distance between $g$ and $h$ in $G$ is defined as the shortest length of a word representing $h^{-1}g$). The group is said to have \textit{subexponential growth} if

$$\limsup_n |B_S(n)|^{1/n} = 1,$$

otherwise it has \textit{exponential growth}.

In fact, $\lim_{n \to \infty} |B_S(n)|^{1/n}$ exists, and it cannot be larger than $|S|$. It is also easy to verify that these properties do not depend on the set of generators.

**Theorem 1.2.** A group of subexponential growth is amenable.

The converse is false — there are amenable groups of exponential growth (see Example below).

By the way, I made a mistake in the 2nd lecture, saying that every \textit{elementary group} has polynomial growth. The class $EG$ of elementary groups is defined to be the collection of groups generated by finite and Abelian groups and closed under taking subgroups, quotients, extensions, and direct unions. Every elementary group is amenable. Results of Milnor and Wolf imply that every elementary group has either polynomial or exponential growth. Two important classes of elementary groups are \textit{solvable} and \textit{nilpotent} groups.

A group $G$ is \textit{solvable} if there is a series of normal subgroups $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_k = G$, such that every factor group $G_j/G_{j-1}$ is Abelian.

Equivalently, consider the \textit{derived series} of $G$, defined as

$$G = G_{(0)} \supseteq G_{(1)} \supseteq \cdots$$

where $G_{(n)} = [G_{(n-1)}, G_{(n-1)}]$ and

$$[A, B] = \{aba^{-1}b^{-1} : a \in A, b \in B\}$$

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denotes the commutator group of $A$ and $B$. A group is solvable iff there exists $n$ such that $G(n) = \{1\}$. On the other hand, consider the lower central series of $G$:

$$G = G_0 \supseteq G_1 \supseteq \ldots$$

where $G_n = [G, G_{n-1}]$. A group $G$ is nilpotent iff there exists $n$ such that $G_n = \{1\}$.

Clearly, every nilpotent group is solvable, and every solvable group is elementary. A celebrated theorem of Gromov is that a group has polynomial growth iff it is virtually nilpotent (the difficult direction is “only if”; virtually nilpotent means there is a nilpotent subgroup of finite index).

**Example.** A classical example of a solvable (hence amenable) group of exponential growth is $BS(1, 2)$, the Baumslag-Solitar group. It is defined by

$$BS(1, 2) = \langle a, b \mid bab^{-1} = a^2 \rangle.$$ More generally, $BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$. To show that $BS(1, 2)$ is solvable of exponential growth, notice that it can be represented as a group of affine transformations on the line: $f_a : x \mapsto x + 1$, $f_b : x \mapsto 2x$. (More precisely, we will work with the group $G = \langle f_a, f_b \rangle$, which can be shown isomorphic to $BS(2, 1)$.) Observe that any composition of $f_a$, $f_b$ and there inverses has the form

$$F : x \mapsto 2^k x + b, \quad k \in \mathbb{Z}, \quad b \in \mathbb{Z}[\frac{1}{2}],$$

where $\mathbb{Z}[\frac{1}{2}]$ is the group of 2-adic rationals (that is, rationals of the form $p/2^s$). Then it is easy to see that for any $F_1, F_2$ in the group, their commutator is

$$F_1 F_2 F_1^{-1} F_2^{-1}(x) = x + c, \quad \text{for some } c \in \mathbb{Z}[\frac{1}{2}].$$

It follows that $[G, G] \leq \mathbb{Z}[\frac{1}{2}]$ (in fact, it is not hard to see that they are equal). Since the commutator group is Abelian (but not finitely generated!), we obtain that $G$ is solvable.

On the other hand, notice that $G$ contains $h(x) = \frac{x}{2} = f_b^{-1}$ and $g(x) = \frac{x+1}{2} = f_b^{-1} f_a$, and these elements generate a free semigroup. This is because

$$h((0, 1)) = (0, 1/2), \quad g((0, 1)) = (1/2, 1),$$

and we can apply the following elementary “ping-pong” lemma. Then $BS(2n)$, the set of all words of length $\leq 2n$, using the generator set $S = \{f_a, f_b, f_a^{-1}, f_b^{-1}\}$, includes all words of length $n$ in the “alphabet” $\{g, h\}$, which are distinct by the lemma. Thus $|BS(2n)| \geq 2^n$, proving exponential growth.

**Lemma 1.3.** Let $X$ be any non-empty set and $g, h$ are 1-to-1 transformations from $X$ into itself, such that $g(X) \cap h(X) = \emptyset$. Then $g$ and $h$ generate a free semigroup.
2. Introduction to ergodic theory of group actions (see [3, 8.1, 8.4, 8.5])

**Definition 2.1.** Suppose that a $\sigma$-compact group $G$ acts on a measure space $(X, B, \mu)$ by measure-preserving transformations, which means that the $\sigma$-algebra $B$ is preserved under the action, and $g_*\mu = \mu$ for all $g \in G$. Recall that $g_*\mu(A) = \mu(g^{-1} \cdot A)$, $A \in B$.

(i) The $G$-action is called **ergodic** if

$$A \in B, \quad \mu(g^{-1} \cdot A \triangle A) = 0, \quad \text{for all } g \in G \implies \mu(A) \in \{0, 1\}.$$ 

(ii) There is an equivalent definition: the $G$-action is ergodic if

$$A \in B, \quad g^{-1} \cdot A = A, \quad \text{for all } g \in G \implies \mu(A) \in \{0, 1\}.$$ 

Obviously, (i) implies (i’). Proving the implication (i’) ⇒ (i) requires some work — it is easier when the group $G$ is countable.

(ii) The $G$-action is mixing if for any $A_0, A_1 \in B$ and sequence $g_n \in G$, $g_n \to \infty$ (in the sense that for any compact set $K \subseteq G$ there is $N$ such that $g_n \notin K$ for $n \geq N$), then

$$\mu(A_0 \cap g_n^{-1} \cdot A_1) \to \mu(A_0)\mu(A_1).$$

Note that if $G$ is a discrete group, then compact sets are finite, and the group is $\sigma$-compact iff it is countable.

**Lemma 2.2.** If the $G$-action is ergodic, then any measurable $G$-invariant function is equal to a constant a.e.

We next assume that a $\sigma$-compact group $G$ acts continuously on a $\sigma$-compact metric space $(X, \rho)$, and consider $B = B(X)$, the Borel $\sigma$-algebra. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on $X$ and by $\mathcal{M}^G(X)$ the set of $G$-invariant probability measures.

The following statements will be discussed without a complete proof:

**Theorem 2.3.** Under the standing assumptions, the space $\mathcal{M}^G(X)$ is a closed convex subset of $\mathcal{M}(X)$. A measure in $\mathcal{M}^G(X)$ is extremal iff it is $G$-ergodic.

**Corollary 2.4.** If $\mathcal{M}^G(X)$ is non-empty, then there exists at least one ergodic $G$-invariant measure.

2.1. **Amenable group actions.**

**Theorem 2.5.** Let $G$ be a countable discrete group. Then $G$ is amenable if and only if every continuous $G$-action on a compact metric space $X$ has an invariant probability measure.
Sketch of the proof of existence for $G$ amenable. Let $\{F_n\}$ be a Følner sequence in $G$, i.e.,

$$\frac{|F_n \triangle (h \cdot F_n)|}{|F_n|} \to 0, \quad n \to \infty, \quad \text{for all } h \in G.$$ 

Let $\nu \in \mathcal{M}(X)$ an arbitrary probability measure (e.g. $\nu = \delta_x$ for some $x \in X$). Consider

$$\mu_n := \frac{1}{|F_n|} \sum_{g \in F_n} g_* \nu \in \mathcal{M}(X).$$

By assumption, $X$ is compact, so by Banach-Alaoglu Theorem the unit ball in $C(X)^*$ (the dual of the Banach space of continuous functions on $X$) is compact and metrizable in the weak* topology. The set $\mathcal{M}(X)$ is a closed subset of the unit ball of $C(X)^*$ (recall that these are probability measures); therefore, there exists a subsequential limit $\mu = \lim_{k \to \infty} \mu_{n_k}$. We then verify that $\mu$ is $G$-invariant.

In fact, $\mu$ is $G$-invariant iff $\mu = h_* \mu$ for all $h \in G$. Equivalently, this is expressed as

$$\int_X \phi(x) \, d\mu(x) = \int_X \phi(h \cdot x) \, d\mu(x) \quad \text{for all } \phi \in C(X), \ h \in G.$$ 

By definition,

$$\int_X \phi(x) \, d\mu_n(x) = \frac{1}{|F_n|} \sum_{g \in F_n} \int_X \phi(g \cdot x) \, d\nu(x).$$

On the other hand,

$$\int_X \phi(h \cdot x) \, d\mu_n(x) = \frac{1}{|F_n|} \sum_{g \in F_n} \int_X \phi(hg \cdot x) \, d\nu(x) = \frac{1}{|F_n|} \sum_{g \in h \cdot F_n} \int_X \phi(g \cdot x) \, d\nu(x).$$

It follows that

$$\left| \int_X \phi(x) \, d\mu_n(x) - \int_X \phi(h \cdot x) \, d\mu_n(x) \right| \leq \frac{1}{|F_n|} \sum_{g \in F_n \triangle (h \cdot F_n)} \int_X |\phi(g \cdot x)| \, d\nu(x)$$

$$\leq \frac{|F_n \triangle (h \cdot F_n)|}{|F_n|} \|\phi\|_{\infty} \to 0, \quad \text{as } n \to \infty,$$

and the desired property (1) follows by passing to the limit along the subsequence. \qed

2.2. Ergodic Theorems. We start with the statement of the classical pointwise Birkhoff Ergodic Theorem for measure-preserving $\mathbb{Z}$ and $\mathbb{R}$-actions, without proof.

**Theorem 2.6.** (i) Let $T$ be an invertible ergodic measure-preserving transformation of a probability measure space $(X, \mathcal{B}, \mu)$ (equivalently, an action of $\mathbb{Z}$ by $n(x) = T^n(x)$, $x \in X$, $n \in \mathbb{Z}$). Then for any integrable function $f \in L^1(X, \mu)$, the “time average” converges to the “space average” almost everywhere: for $\mu$-a.e. $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f \, d\mu, \quad \text{for } \mu\text{-a.e. } x \in X.$$
In case the action is not ergodic, the limit still converges almost everywhere, to a $T$-invariant function in $L^1(X, \mu)$.

(ii) Let $\{g_t\}_{t \in \mathbb{R}}$ be a measure-preserving $\mathbb{R}$-action of a probability measure space $(X, \mathcal{B}, \mu)$. Then for any integrable function $f \in L^1(X, \mu)$, the “time average” converges to the “space average” almost everywhere:

$$\lim_{A \to \infty} \frac{1}{A} \int_0^A f(g_t x) \, dx = \int_X f \, d\mu, \quad \text{for } \mu\text{-a.e. } x \in X.$$

In case the action is not ergodic, the limit still converges almost everywhere, to a function in $L^1(X, \mu)$, invariant under the action.

The pointwise Ergodic Theorem extends naturally to $\mathbb{Z}^d$ and $\mathbb{R}^d$-actions, but what about more general groups? Amenable groups are good candidates, because then we can average over a Følner sequence. The statement and proof(s) are, however, rather complicated. Instead, we will discuss Mean Ergodic Theorems, which are easier to handle.

Mean Ergodic Theorem for $\mathbb{Z}$ and $\mathbb{R}$-actions was proved by J. von Neumann, a little earlier than the Birkhoff Ergodic Theorem. Instead of the almost-everywhere convergence, it asserts converges “in the mean”; more precisely, for $f \in L^2(X, \mu)$, there is convergence in $L^2$ in (2), and similarly for $\mathbb{R}$-actions.

**Theorem 2.7.** Let $G$ be a countable discrete amenable group. Suppose that $G$ acts by measure-preserving transformations on a probability measure space $(X, \mathcal{B}, \mu)$. Let $P_G$ be the orthogonal projection on the closed subspace of $G$-invariant $L^2$-functions on $X$:

$$I = \{ \phi \in L^2(X, \mu) : \phi(x) = \phi(g \cdot x), \quad \text{for } \mu\text{-a.e. } x, \text{ for all } g \in G \}.$$

Then for any Følner sequence $\{F_n\}$ and $\phi \in L^2(X, \mu)$,

$$\frac{1}{|F_n|} \sum_{g \in F_n} \phi(g \cdot x) \to P_G \phi \quad \text{in } L^2(X, \mu), \quad \text{as } n \to \infty.$$

If the $G$-action is ergodic, then $P_G \phi = \int_X \phi \, d\mu$.

**Sketch of the proof.** For functions $\phi \in I$ the convergence in (3) is obvious. Consider

$$V = \text{Span}\{v(h \cdot x) - v(x) : v \in L^2(X, \mu), \ h \in G\}.$$

For functions $\phi \in V$ the convergence in (3) is proved, similarly to the Sketch of the proof of Theorem 2.5 above. By continuity this extends to the closure of $V$. It remains to observe that $I = V^\perp$, whence $\text{clos}(V) + I = L^2(X, \mu)$. □

**Remark.** In the above proof sketch it is important to note that the linear transformation $\phi(x) \mapsto \phi(g \cdot x)$ is a unitary operator (invertible isometry) on the Hilbert space $L^2(X, \mu)$,
by the measure-preserving condition. It follows that

$$\left\| \frac{1}{|F_n|} \sum_{g \in F_n} \phi(g \cdot x) - \frac{1}{|F_n|} \sum_{g \in F_n} \psi(g \cdot x) \right\|_2 \leq \|\phi - \psi\|_2$$

for $\phi, \psi \in L^2(X, \mu)$, which is used when extending (3) to the closure of $V$.

References

