Finiteness conditions on the injective hull of simple modules.

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jointly with

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**Definition (Injective hull)**

The injective hull $E(M)$ of a (left $R$)-module $M$ is an injective module such that $M$ embeds as an essential submodule in it, i.e. $M \cap U \neq 0$ for all $0 \neq U \subseteq E(M)$.

**Theorem (Matlis, 1960)**

The injective hull of a simple module over a commutative Noetherian ring is Artinian.

**Question**

What can be said if either “commutative” or “Noetherian” is dropped?
Non-commutativity

Definition (Jans, 1968)
A ring is co-Noetherian if the injective hull of any simple module is Artinian.

Proposition (Hirano, 2000)
The 1st Weyl algebra $A_1(\mathbb{Z})$ is co-Noetherian, but $A_1(\mathbb{Q})$ is not.

For any infinite set $\{x - a_1, x - a_2, \ldots\}$ in $\mathbb{Q}[x]$, the localisations $\mathbb{Q}[x]s_1 \supset \mathbb{Q}[x]s_2 \supset \cdots \supset \mathbb{Q}[x]s_n \supset \cdots \supset \mathbb{Q}[x]$ form a descending chain of $A_1(\mathbb{Q})$-modules, where $S_n$ is the multiplicatively closed set generated by $x - a_i$, for $i \geq n$.

$\Rightarrow$ $E(\mathbb{Q}[x])$ is not an Artinian $A_1(\mathbb{Q})$-module.
Locally Artinian

Definition

(\diamond) any injective hull of a simple $R$-module is locally Artinian.

Definition

A left ideal $I$ of $R$ is subdirectly irreducible (SDI) if $R/I$ has an essential simple socle.

$R$ satisfies (\diamond) if and only if $R/I$ is Artinian for all left SDI’s.

Proposition (Krull intersection)

Suppose finitely generated Artinian left $R$-modules are Noetherian. If $R$ satisfies (\diamond) then $\bigcap(I + \text{Jac}(R)^n) = I$ for any left ideal $I$. 

Any semiprime Noetherian ring of Krull dimension $\leq 1$ satisfies $\Diamond$.

Example (Goodearl-Schofield, 1986)

$\exists$ Noetherian ring with Krull dimension 1 not satisfying $\Diamond$.

Relies on a skew field extension $F \subseteq E$ with $E$ finite dimensional over $F$ on the right, but transcendental on the left. Then

$$
\begin{pmatrix}
E[t] & E[t] \\
0 & F[t]
\end{pmatrix}
$$

does not satisfy $\Diamond$, but is Noetherian and has Krull dimension 1.
Theorem (Jategaonkar, 1974)

Any fully bounded Noetherian ring satisfies \((\diamond)\). In particular any Noetherian semiprime PI-ring satisfies \((\diamond)\).

Theorem (Carvalho, Musson, 2011)

The q-plane \( R = K_q[x, y] = K\langle x, y \rangle/\langle xy - qyx \rangle \) satisfies \((\diamond)\) if and only if q is a root of unity.

If \( q \) is not a root of unity, then

\[
0 \to R/R(xy - 1) \to R/R(xy - 1)(x - 1) \to R/R(x - 1) \to 0
\]

is an essential embedding of a simple into a non-Artinian module.
A_1(K) satisfies $(\Diamond)$ since it is a Noetherian domain of Kdim 1

**Theorem (Stafford, 1985)**

Let $n > 1$ and $\lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\alpha = x_1 + \left( \sum_{i=2}^{n} \lambda_i y_i x_i \right) y_1 + \sum_{i=2}^{n} (x_i + y_i) \in A_n = A_n(\mathbb{C})$$

generates a maximal left ideal of $A_n$ and

$$0 \to A_n/A_n\alpha \to A_n/A_n\alpha x_1 \to A_n/A_n x_1 \to 0$$

is an essential embedding with $\text{Kdim}(A_n/A_n x_1) = n - 1$. 
Exploiting Stafford’s theorem

Example

Let $h_n = \text{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ with $[x_i, y_i] = z$. Then $U(h_n)$ satisfies (⋄) if and only if $n = 1$ as $U(h_n)/\langle z - 1 \rangle \simeq A_n$.

Theorem (Hatipoglu-L. 2012)

Let $g$ be a finite dimensional nilpotent complex Lie (super)algebra. Then $U(g)$ satisfies (⋄) if and only if

1. $g$ has an Abelian ideal of codimension 1 or
2. $g \cong h \times a$ with $a$ Abelian and $h = \text{span}(e_1, \ldots, e_m)$ with either
   (i) $m = 5$ and $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$ or
   (ii) $m = 6$ and $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, $[e_1, e_2] = e_6$. 
Ore extensions

Theorem (Carvalho, Hatipoglu, L. 2015)

Let $\sigma$ be an automorphism of $K$ and $d$ a $\sigma$-derivation. Then $K[x][y;\sigma,d]$ satisfies $(\diamond)$ if and only if

(i) $\sigma = id$ and $d$ is locally nilpotent or
(ii) $\sigma \neq id$ has finite order.

Theorem (Vinciguerra, 2017)

Let $R = \mathbb{C}[x,y]$ and $d$ a non-zero derivation of it. Then $S = R[\theta, d]$ satisfies $(\diamond)$ if and only if

(i) every maximal ideal of $R$ contains an Darboux element
(ii) $d(R) \subseteq R_p$, for any Darboux element $p$ contained in a $d$-stable maximal ideal.

An element is Darboux if it generates a $d$-stable ideal.
Skew-polynomial rings

**Theorem (Brown, Carvalho, Matczuk 2017)**

Let $K$ be an uncountable field and $R$ a commutative affine $K$-algebra, and let $\alpha$ be a $K$-algebra automorphism of $R$. Then $S = R[\theta; \alpha]$ satisfies $(\diamond)$ if and only if all simple $S$-modules are finite dimensional over $K$.

Many more interesting results and open question can be found in the paper ”Simple modules and their essential extensions for skew polynomial rings” by Brown, Carvalho and Matczuk (arXiv:1705.06596).
Commutative, but not Noetherian?

From now on $R$ will be commutative.

Theorem (Vamos, 1968)

The following statements are equivalent for a commutative ring $R$.

(a) The injective hull of a simple module is Artinian.
(b) The localisation of $R$ by a maximal ideal is Noetherian.

Theorem

The following statements are equivalent for a commutative ring $R$.

(a) $R$ satisfies ($\diamond$)
(b) $R_m$ satisfies ($\diamond$) for all $m \in \text{MaxSpec}(R)$.

$E(R/m)$ is an injective hull of $R_m/mR_m$ as $R_m$-module.
Theorem

For a local ring \((R, \mathfrak{m})\) the following are equivalent:

(a) \(R\) is (co-)Noetherian.
(b) \(R\) satisfies \(\diamond\) and \(\mathfrak{m}/\mathfrak{m}^2\) is finitely generated.
(c) \(\bigcap(I + \mathfrak{m}^n) = I\) for any ideal \(I\) and \(\mathfrak{m}/\mathfrak{m}^2\) is finitely generated.
Local rings with nilpotent radical

**Proposition**

If $R$ has nilpotent radical, then the following are equivalent

(a) $R$ satisfies $(\diamondsuit)$

(b) For any module $M$: $\text{Soc}(M)$ f.g implies $\text{Soc}(M/\text{Soc}(M))$ f.g.

**Example**

Any local ring $(R, m)$ with $m^2 = 0$ satisfies $(\diamondsuit)$, because if $I$ is SDI, then $m/I \cap m$ has dimension $\leq 1$ as vector space over $R/m$, i.e. $R/I$ has length at most 2.

For example the trivial extension

$$R = \left\{ \begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \mid a \in K, v \in V \right\}.$$
**Theorem (Local rings with radical cube zero)**

Let \((R, \mathfrak{m})\) be local with \(\mathfrak{m}^3 = 0\). Then there exists a bijective correspondence between SDI's \(I\) not containing \(\text{Soc}(R)\) and non-zero \(f \in \text{Hom}(\text{Soc}(R), F)\). Corresponding pairs \((I, f)\) satisfy:

\[
\text{Soc}(R) + I = V_f := \{a \in \mathfrak{m} | f(\mathfrak{m}a) = 0\}.
\]

Then \(R\) satisfies \((\Diamond)\) iff \(\dim(\mathfrak{m}/V_f) < \infty\) for all \(f \in \text{Soc}(R)^*\).

**Theorem**

Let \((R, \mathfrak{m})\) be a local ring with residue field \(F\) and \(\mathfrak{m}^3 = 0\). Then \(R\) satisfies \((\Diamond)\) if and only if \(\text{gr}(R) = F \oplus (\mathfrak{m}/\mathfrak{m}^2) \oplus \mathfrak{m}^2\) does.
Definition

For a field $F$, vector spaces $V$ and $W$ and a symmetric bilinear form $\beta : V \times V \to W$ we can consider the generalised matrix ring

$$
\begin{align*}
\left\{ \begin{pmatrix} a & v & w \\ 0 & a & v \\ 0 & 0 & a \end{pmatrix} \right| a \in F, v \in V, w \in W \right\}
\end{align*}
$$

which we identify by $S = F \times V \times W$ with multiplication

$$(a_1, v_1, w_1)(a_2, v_2, w_2) = (a_1 a_2, a_1 v_2 + v_1 a_2, a_1 w_2 + \beta(v_1, v_2) + w_1 a_2).$$

Then $\text{Soc}(S) = 0 \times V^\perp_\beta \times W$ where

$$V^\perp_\beta = \{ v \in V \mid \beta(V, v) = 0 \}.$$ 

Clearly $m = 0 \times V \times W$ and $m^2 = 0 \times 0 \times \text{Im}(\beta)$. 

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Examples

Let $F = \mathbb{R}$ and $V = C([0, 1])$, space of continuous real valued functions on $[0, 1]$. Define $\beta : V \times V \to \mathbb{R}$ by

$$\beta(f, g) = \int_{0}^{1} f(x)g(x)\,dx,$$

then $S = \mathbb{R} \times V \times \mathbb{R}$ has an 1-dimensional essential socle, but $S$ is not Artinian, i.e. $S$ does not satisfy $(\diamond)$. 
Let $F$ be any field and $V$ be any vector space with basis $\{v_i : i \geq 0\}$. Define

$$\beta(v_i, v_j) = \begin{cases} 1 & (i, j) = (0, 0) \\ 0 & \text{else} \end{cases}$$

Then $S = F \times V \times F$ satisfies $(\diamond)$, because

$$\mathfrak{m}/\text{Soc}(S) = (0 \times V \times F)/(0 \times V_\beta \times F) \simeq V/V_\beta \simeq F$$

Note that $S = \text{gr}(F[x_0, x_1, x_2 \ldots]/\langle x_0^3, x_i x_j : (i, j) \neq (0, 0)\rangle)$.

Here: $\beta$ not non-degenerated $\Rightarrow$ pass to $F \times V/V_\beta \times F$. 
Theorem

Let \((R, \mathfrak{m})\) be a local ring with residue field \(F\) and \(\mathfrak{m}^3 = 0\). Then the following are equivalent:

(a) \(R\) does not satisfy \((\diamond)\)

(b) \(R\) has a factor \(R/I\) such that \(\text{gr}(R/I)\) has the form \(F \times V \times F\) for a non-degenerated form \(\beta : V \times V \to F\) and \(\dim(V) = \infty\).
Let $A$ be an $F$-algebra. Then $S = F \times A \times A$ becomes a ring using the multiplication $\mu$ as bilinear form. Since $\mu$ is non-degenerated, $\text{Soc}(S) = 0 \times 0 \times A$. Hence $\text{Soc}(S)^* = A^*$. For any $f \in A^*$:

$$V_f = \{ a \in A : f(Aa) = 0 \}$$

is the largest ideal contained in $\ker(f)$.

Hence $A/V_f$ is finite dimensional if and only if $f \in A^0$.

**Proposition**

$S = F \times A \times A$ satisfies $(\Diamond)$ if and only if for any $A^* = A^0$.

**Example:** $A = F \times V$ the trivial extension satisfies $A^* = A^0$. 
Example

Let \( \text{char}(F) = 0 \) and \( A = F[x] \). Set \( f(x^n) = \frac{1}{n+1} \) for any \( n \geq 0 \). Then the only ideal contained in \( \ker(f) \) is the zero ideal, i.e.

\[
V_f = \{0\}.
\]

Therefore, \( \beta = f \circ \mu : F[x] \times F[x] \to F \) is a non-degenerated symmetric bilinear form and \( S = F \times F[x] \times F \) does not satisfy \((\Diamond)\).
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Thank you for your attention!