

FAKE REFLECTION

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ABSTRACT. We introduce a generalization of stationary set reflection which we call *filter reflection*, and show it is compatible with the axiom of constructibility as well as with strong forcing axioms. We prove the independence of filter reflection from ZFC, and present applications of filter reflection to the study of canonical equivalence relations of the generalized Cantor and Baire spaces.

1. INTRODUCTION

Throughout the paper, κ denotes an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$.

Motivated by questions arising from the study of the higher Cantor and Baire spaces, 2^κ and κ^κ , respectively, in this paper, we introduce and study a new notion of reflection which we call *filter reflection*.

Definition 1.1. Suppose X and S are stationary subsets of κ , and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, \mathcal{F}_α is a filter over α .

- (1) We say that $\vec{\mathcal{F}}$ *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary;
- (2) We say that X $\vec{\mathcal{F}}$ -*reflects to* S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary;
- (3) We say that X \mathfrak{f} -*reflects to* S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$ -reflects to S' .

It is not hard to verify (see Lemma 2.8 below) that if X \mathfrak{f} -reflects to S , then the equivalence relation $=_X$ on the space κ^κ (see Definition 2.5 below) admits a Lipschitz reduction to the equivalence relation $=_S$. A detailed study of the effect of filter reflection (and stronger forms of which) on the generalized Baire and Cantor spaces is given in Section 2 below, but as we feel that the notion of filter reflection is of independent interest, let us take a moment to inspect Definition 1.1.

- A sequence $\vec{\mathcal{F}}$ over a stationary $S \subseteq \text{cof}(>\omega)$ that captures clubs can be obtained in ZFC; for every $\alpha \in S$, simply take \mathcal{F}_α to be the club filter over α , $\text{CUB}(\alpha)$.

An alternative way to consistently get a sequence $\vec{\mathcal{F}}$ that captures clubs is by advising with a \diamond_S^* -sequence, in which case, each filter \mathcal{F}_α will have a small base.

- If we omit the requirement that $\vec{\mathcal{F}}$ captures clubs from Definition 1.1(2), then again such a sequence may be constructed in ZFC, with each \mathcal{F}_α being a principal ultrafilter. However, if S is an ineffable set, then capturing implies that $\mathcal{F}_\alpha \supseteq \text{CUB}(\alpha)$ for most $\alpha \in S$.

An alternative way to consistently get “reflection minus capturing” goes through forcing axioms; for instance, it follows from the main result of [Moo06] that MRP (a principle weaker than the Proper Forcing Axiom)

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entails that for $X := \omega_2 \cap \text{cof}(\omega)$ and $S := \omega_2 \cap \text{cof}(\omega_1)$, there exists a sequence \vec{F} with $\mathcal{F}_\alpha \subseteq \text{CUB}(\alpha)$ for all $\alpha \in S$, and satisfying that for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Thus, filter reflection is the conjunction of two requirements, each being a consequence of ZFC, and the challenge is in simultaneously satisfying both. The special case in which $\mathcal{F}_\alpha = \text{CUB}(\alpha)$ for all $\alpha \in S'$ is better known as the assertion that “every stationary subset of X reflects in S ” that goes back to some of the milestone papers in the history of forcing with large cardinals [Bau76, Mag82, HS85, MS89].

In this paper, we pay special attention to the case in which $\mathcal{F}_\alpha \not\subseteq \text{CUB}(\alpha)$ for all $\alpha \in S'$; we regard this special case by the name *fake reflection*. The obvious advantage of fake reflection over genuine reflection is that the former makes sense even for stationary sets S consisting of points of countable cofinality. The other — somewhat unexpected — advantage is that its consistency does not require large cardinals:

Theorem A. *For all stationary subsets X and S of κ , there exists a $<\kappa$ -closed κ^+ -cc forcing extension, in which X \mathfrak{f} -reflects to S .*

We shall also show that (a strong form of) fake reflection is a consequence of two axioms of contradictory nature, that is, the axiom of constructibility ($V = L$) and Martin’s Maximum (MM).

After realizing that fake reflection is so prevalent, we tried for a while to prove that it is a consequence of ZFC, or, at least, a consequence of \diamond_S^+ , but in vain. Eventually, we discovered that this is not the case:¹

Theorem B. *There exists a cofinality-preserving forcing extension in which for all two disjoint stationary subsets X, S of κ , X does not \mathfrak{f} -reflect to S .*

Theorems A and B give two extreme configurations of filter reflection. The next theorem allows considerably more subtle configurations, as it gives rise to models in which filter reflection is no more general than the classic notion of reflection:

Theorem C. *If κ is strongly inaccessible (e.g., a Laver-indestructible large cardinal), then, in the forcing extension by $\text{Add}(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, S of κ , the following are equivalent:*

- X \mathfrak{f} -reflects to S ;
- every stationary subset of X reflects in S .

As hinted earlier, filter reflection has a strong connection with the Borel-reducibility hierarchy of the higher Cantor and Baire spaces. To exemplify:²

Theorem D. *For all two disjoint stationary subsets X, S of κ . If X \mathfrak{f} -reflects to S and S \mathfrak{f} -reflects to X , then there is a nilpotent map $f : \kappa^\kappa \rightarrow \kappa^\kappa$ simultaneously witnessing*

$$=_X \hookrightarrow_1 =_S \ \& \ =_S \hookrightarrow_1 =_X.$$

Filter reflection also provides us with tools to answer a few questions from the literature. For instance, the following answers Question 2.12 of [AHKM19] for the case $\kappa = \omega_2$:

Theorem E. *The consistency of MM implies the consistency of*

$$=_{\omega_2 \cap \text{cof}(\omega)} \hookrightarrow_1 =_{\omega_2 \cap \text{cof}(\omega_1)}^2 \ \& \ =_{\omega_2 \cap \text{cof}(\omega_1)}^2 \not\hookrightarrow_{BM} =_{\omega_2 \cap \text{cof}(\omega)}.$$

¹See also Corollary 5.15 below.

²See Section 2 for missing definitions.

1.1. Organization of this paper. In Section 2, we introduce the notion of filter reflection and its variations and study their effect on canonical equivalence relations over the higher Baire and Cantor spaces. The proof of Theorem D will be found there.

In Section 3, we study strong and simultaneous forms of filter reflection, proving various sufficient and equivalent conditions for these principles to hold.

In Section 4, we present a poset that forces filter and fake reflection to hold, thus, proving Theorem A.

In Section 5, we present a poset that forces filter and fake reflection to fail, along the way, proving Theorems B and C.

In Section 6, we extend the analysis of Section 5 to get strong failures of Borel reductions. Along the way, we answer a few questions from the literature, and prove Theorem E.

1.2. Notation and conventions. For sets X, Y , their difference $\{x \in X \mid x \notin Y\}$ is denoted by $X \setminus Y$, whereas the notation $X - Y$ is reserved for the relative of Shelah's approachability ideal that we introduce in Section 5.

By a *filter* we always mean a proper filter, so that it does not contain the empty set. For a filter \mathcal{F} over a set X , we let $\mathcal{F}^+ := \{Y \in \mathcal{P}(X) \mid X \setminus Y \notin \mathcal{F}\}$. For an ideal \mathcal{I} over a set X and a subset $Y \subseteq X$, we write $\mathcal{I} \upharpoonright Y$ for $\mathcal{I} \cap \mathcal{P}(Y)$.

The class of ordinals is denoted by OR . The class of ordinals of cofinality μ is denoted by $\text{cof}(\mu)$, and the class of ordinals of cofinality greater than μ is denoted by $\text{cof}(>\mu)$. The class of infinite singular ordinals is denoted by Sing . The class of infinite regular cardinals is denoted by Reg . We write $\text{Sing}(\kappa)$ for $\text{Sing} \cap \kappa$, and $\text{Reg}(\kappa)$ for $\text{Reg} \cap \kappa$.

For a set of ordinals a , we write $\text{cl}(a) := \{\sup(a \cap \alpha) \mid \alpha \in \text{OR}, a \cap \alpha \neq \emptyset\}$, $\text{acc}(a) := \{\alpha \in a \mid \sup(a \cap \alpha) = \alpha > 0\}$, $\text{acc}^+ := \{\alpha < \sup(a) \mid \sup(a \cap \alpha) = \alpha > 0\}$ and $\text{nacc}(a) := a \setminus \text{acc}(a)$. For a stationary $Y \subseteq \kappa$, we write $\text{Tr}(Y) := \{\alpha \in \kappa \cap \text{cof}(>\omega) \mid Y \cap \alpha \text{ is stationary in } \alpha\}$.

We let $\text{Col}(\lambda, < \kappa)$ denote Lévy's notion of forcing for collapsing κ to λ^+ , and let $\text{Add}(\kappa, \theta)$ denote Cohen's notion of forcing for adding θ many functions from κ to 2. For all $\alpha < \kappa$, $p : \alpha \rightarrow 2$, and $i < 2$, we denote by $p \frown i$ the unique function p' extending p satisfying $\text{dom}(p') = \alpha + 1$ and $p'(\alpha) = i$.

2. FILTER REFLECTION AND LIPSCHITZ REDUCTIONS

Throughout this section, we let S, X denote stationary subsets of κ , and consider sequences $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$, where, for all $\alpha \in S$, \mathcal{F}_α is a filter over α . Recall that $\vec{\mathcal{F}}$ is said to *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary.

Definition 2.1 (Filter reflection).

- (1) We say that X $\vec{\mathcal{F}}$ -reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary;
- (2) We say that X *strongly* $\vec{\mathcal{F}}$ -reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary;
- (3) We say that X $\vec{\mathcal{F}}$ -reflects with \diamond to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \& \ Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Definition 2.2. We say that X \mathfrak{f} -reflects to S whenever there exists a stationary $S' \subseteq S$ and sequence of filters $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ such that X $\vec{\mathcal{F}}$ -reflects to S' .

The same convention applies to strong \mathfrak{f} -reflection and to \mathfrak{f} -reflection with \diamond .

Notice that a priori there is no reason to believe that “ X \mathfrak{f} -reflects to S ” implies the existence of a sequence $\vec{\mathcal{F}}$ for which “ X $\vec{\mathcal{F}}$ -reflects to S ”. To put our finger at the challenge, it is in the requirement of capturing clubs.

Proposition 2.3. *Suppose that X and S are stationary subset of κ .*

- (1) *If X strongly \mathfrak{f} -reflects to S , then X \mathfrak{f} -reflects to S ;*
- (2) *If X \mathfrak{f} -reflects with \diamond to S , then X strongly \mathfrak{f} -reflects to S .*

Proof. (2) Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ and $\langle Y_\alpha \mid \alpha \in S' \rangle$ witness together that X \mathfrak{f} -reflects with \diamond to S . Let $S'' := \{\alpha \in S' \mid Y_\alpha \in \mathcal{F}_\alpha^+\}$. For each $\alpha \in S''$, let $\bar{\mathcal{F}}_\alpha$ be the filter over α generated by $\mathcal{F}_\alpha \cup \{Y_\alpha\}$. Evidently, $\langle \bar{\mathcal{F}}_\alpha \mid \alpha \in S'' \rangle$ witnesses that X strongly \mathfrak{f} -reflects to S . \square

The following is obvious:

Lemma 2.4 (Monotonicity). *For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$:*

- (1) *If X \mathfrak{f} -reflects to S , then Y \mathfrak{f} -reflects to T ;*
- (2) *If X strongly \mathfrak{f} -reflects to S , then Y strongly \mathfrak{f} -reflects to T ;*
- (3) *If X \mathfrak{f} -reflects with \diamond to S , then Y \mathfrak{f} -reflects with \diamond to T .* \square

Definition 2.5. (1) For all $\eta, \xi \in \kappa^\kappa$, denote

$$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

- (2) For every $\theta \in [2, \kappa]$, the equivalence relation $=_{\mathcal{S}}^\theta$ over θ^κ is defined via

$$\eta =_{\mathcal{S}}^\theta \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is nonstationary.}$$

- (3) The special case $=_{\mathcal{S}}^\kappa$ is denoted by $=_S$.

The above equivalence relation is an important building block in the study of the connection between model theory and generalized descriptive set theory (cf. [FHK14, Chapter 6] and [HKM17, Theorem 7]).

Definition 2.6. For $i < 2$, let X_i be some space from $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$.

Let R_0 and R_1 be binary relations over X_0 and X_1 , respectively. A function $f : X_0 \rightarrow X_1$ is said to be:

- (a) a *reduction of R_0 to R_1* iff, for all $\eta, \xi \in X_0$,

$$\eta R_0 \xi \text{ iff } f(\eta) R_1 f(\xi).$$

- (b) *1-Lipschitz* iff for all $\eta, \xi \in X_0$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

The existence of a function f satisfying (a) and (b) is denoted by $R_0 \hookrightarrow_1 R_1$. We likewise define $R_0 \hookrightarrow_c R_1$, $R_0 \hookrightarrow_B R_1$ and $R_0 \hookrightarrow_{BM} R_1$ once we replace clause (b) by a continuous, Borel, or Baire measurable map, respectively.³ The following is obvious.

Lemma 2.7 (Monotonicity). *For*

- $\mathfrak{p} \in \{1, c, B, BM\}$,
- *stationary sets $X \subseteq X' \subseteq \kappa$ and $Y \subseteq Y' \subseteq \kappa$, and*
- *ordinals $2 \leq \theta \leq \theta' \leq \kappa$ and $2 \leq \lambda \leq \lambda' \leq \kappa$,*

$$=_{X'}^{\theta'} \hookrightarrow_{\mathfrak{p}} =_Y^\lambda \text{ entails } =_X^\theta \hookrightarrow_{\mathfrak{p}} =_{Y'}^{\lambda'}. \quad \square$$

Lemma 2.8. *If X \mathfrak{f} -reflects to S , then $=_X \hookrightarrow_1 =_S$.*

³See Section 6 for definitions and topological properties of the higher Cantor and Baire spaces.

Proof. Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ witnesses that X \mathfrak{f} -reflects to S . For every $\alpha \in S'$, define an equivalence relation \sim_α over ${}^\alpha\kappa$ by letting $\eta \sim_\alpha \xi$ iff there is $W \in \mathcal{F}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$. As there are at most $|\alpha|^\kappa$ many equivalence classes and as $\kappa^{<\kappa} = \kappa$, we may attach to each equivalence class $[\eta]_{\sim_\alpha}$ a unique ordinal (a *code*) in κ , which we shall denote by $\ulcorner [\eta]_{\sim_\alpha} \urcorner$. Next, define a map $f : \kappa^\kappa \rightarrow \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} \ulcorner [\eta \upharpoonright \alpha]_{\sim_\alpha} \urcorner, & \text{if } \alpha \in S'; \\ 0, & \text{otherwise.} \end{cases}$$

As, for all $\eta \in {}^\kappa\kappa$ and $\alpha < \kappa$, $f(\eta)(\alpha)$ depends only on $\eta \upharpoonright \alpha$, f is 1-Lipschitz. To see that it forms a reduction from $=_X$ to $=_S$, let η, ξ be arbitrary elements of ${}^\kappa\kappa$. There are two main cases to consider:

► If $\eta =_X \xi$, then let us fix a club C such that $C \cap X \subseteq \{\beta < \kappa \mid \eta(\beta) = \xi(\beta)\}$. Since $\vec{\mathcal{F}}$ captures clubs, let us fix a club $D \subseteq \kappa$ such that, for all $\alpha \in D \cap S'$, $C \cap \alpha \in \mathcal{F}_\alpha$. We claim that D is disjoint from $\{\alpha \in S \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$, so that indeed $f(\eta) =_S f(\xi)$. So see this, let $\alpha \in D$ be arbitrary.

►► If $\alpha \notin S'$, then $f(\eta)(\alpha) = 0 = f(\xi)(\alpha)$.

►► If $\alpha \in S'$, then for $W := C \cap \alpha$, we have that $W \in \mathcal{F}_\alpha$ and $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$, so that $[\eta \upharpoonright \alpha]_{\sim_\alpha} = [\xi \upharpoonright \alpha]_{\sim_\alpha}$ and $f(\eta)(\alpha) = f(\xi)(\alpha)$.

► If $\eta \neq_X \xi$, then $Y := \{\beta \in X \mid \eta(\beta) \neq \xi(\beta)\}$ is stationary. In effect, $T := \{\alpha \in S' \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary. Now, for every $\alpha \in T$, it must be the case that any $W \in \mathcal{F}_\alpha$ meets Y , so that $W \cap X \not\subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$, and $[\eta \upharpoonright \alpha]_{\sim_\alpha} \neq [\xi \upharpoonright \alpha]_{\sim_\alpha}$. It follows that $\{\alpha \in S' \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$ covers the stationary set T , so that $f(\eta) \neq_S f(\xi)$. \square

Lemma 2.9. *If X strongly \mathfrak{f} -reflects to S , then, for every $\theta \in [2, \kappa]$, $=_X^\theta \hookrightarrow_1 =_S^\theta$.*

Proof. By Proposition 2.3(1) and the preceding lemma, we may assume that $\theta \in [2, \kappa]$. Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ is a sequence witnessing that X strongly \mathfrak{f} -reflects to S . Define a map $f : \theta^\kappa \rightarrow \theta^\kappa$ as follows. For every $\alpha \in S'$ and $\eta \in \theta^\kappa$, if there exists $W \in \mathcal{F}_\alpha$ and $i < \theta$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = i\}$, then it is unique (since \mathcal{F}_α is a filter), and so we let $f(\eta)(\alpha) := i$. If there is no such i or if $\alpha \notin S'$, then we simply let $f(\eta)(\alpha) := 0$.

As, for all $\eta \in {}^\kappa\kappa$ and $\alpha < \kappa$, $f(\eta)(\alpha)$ depends only on $\eta \upharpoonright \alpha$, f is 1-Lipschitz. To see that it forms a reduction, let η, ξ be arbitrary elements of ${}^\kappa\kappa$. There are two cases to consider:

► If $\eta =_X^\theta \xi$, then since $\vec{\mathcal{F}}$ captures clubs, we infer that $f(\eta) =_S^\theta f(\xi)$, very much like the proof of this case in Lemma 2.8.

► If $\eta \neq_X^\theta \xi$, then, as $\theta < \kappa$, we may find $i \neq j$ for which $Y := \{\beta \in X \mid \eta(\beta) = i \ \& \ \xi(\beta) = j\}$ is stationary. In effect, $T := \{\alpha \in S' \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary. Now, for every $\alpha \in T$, we have that $f(\eta)(\alpha) = i$ while $f(\xi)(\alpha) = j$. It follows that $f(\eta) \neq_S f(\xi)$. \square

Given Lemmas 2.8 and 2.9, it is natural to ask whether $=_X^2 \hookrightarrow_1 =_S^2$ implies $=_X \hookrightarrow_1 =_S$. The following provides a sufficient condition.

Lemma 2.10. *If κ is not a strong limit, then $=_X^2 \hookrightarrow_1 =_S^2$ implies $=_X \hookrightarrow_1 =_S$.*

Proof. Suppose that κ is not a strong limit. As we are working under the hypothesis that $\kappa^{<\kappa} = \kappa$, this altogether means that we may fix a bijection $h : \kappa \leftrightarrow {}^\lambda 2$ for some cardinal $\lambda < \kappa$. In effect, there is a bijection $\pi : \kappa^\kappa \leftrightarrow {}^\lambda(2^\kappa)$ satisfying that, for every $\eta \in \kappa^\kappa$, $\pi(\eta) = \langle \eta_i \mid i < \lambda \rangle$ iff, for all $i < \lambda$ and $\alpha < \kappa$:

$$\eta_i(\alpha) = h(\eta(\alpha))(i).$$

Suppose that $f : 2^\kappa \rightarrow 2^\kappa$ is a continuous reduction from $=_X^2$ to $=_S^2$. Define a map $g : \kappa^\kappa \rightarrow \kappa^\kappa$ via $g(\eta) := \zeta$ iff $\pi(\zeta) = \langle f(\eta_i) \mid i < \lambda \rangle$.

Claim 2.10.1. g is 1-Lipschitz.

Proof. Notice that π is 1-Lipschitz and g is equivalent to the function $\pi^{-1} \circ f \circ \pi$. The claim follows from the fact that f is 1-Lipschitz. \square

To see that g is a reduction from $=_X$ to $=_S$, let $\eta, \xi \in \kappa^\kappa$ be arbitrary. Denote $\langle \eta_i \mid i < \lambda \rangle := \pi(\eta)$ and $\langle \xi_i \mid i < \lambda \rangle := \pi(\xi)$.

► If $\eta =_X \xi$, then let us fix a club $C \subseteq \kappa$ such that $C \cap X \subseteq \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$. It is easy to see that for every $i < \lambda$, we have $C \cap X \subseteq \{\alpha < \kappa \mid \eta_i(\alpha) = \xi_i(\alpha)\}$, so that $\eta_i =_X^2 \xi_i$ and we may pick a club $D_i \subseteq \kappa$ such that $D_i \cap S \subseteq \{\alpha < \kappa \mid f(\eta_i)(\alpha) = f(\xi_i)(\alpha)\}$. Then $D := \bigcap_{i < \lambda} D_i$ is a club, and we have $D \cap S \subseteq \{\alpha < \kappa \mid g(\eta)(\alpha) = g(\xi)(\alpha)\}$, so that $g(\eta) =_S g(\xi)$.

► If $\neg(\eta =_X \xi)$, then $Y := \{\alpha \in X \mid \eta(\alpha) \neq \xi(\alpha)\}$ is stationary. Since π is a bijection, for every $\alpha \in Y$, there is some $i < \lambda$ such that $\eta_i(\alpha) \neq \xi_i(\alpha)$. It follows that there exists some $i < \lambda$ such that $Y_i := \{\alpha \in X \mid \eta_i(\alpha) \neq \xi_i(\alpha)\}$ is stationary. Consequently, the set $T := \{\alpha \in S \mid f(\eta_i)(\alpha) \neq f(\xi_i)(\alpha)\}$ is stationary. In particular, the set $\{\alpha \in S \mid g(\eta)(\alpha) \neq g(\xi)(\alpha)\}$ covers T , so that $\neg(g(\eta) =_S g(\xi))$. \square

Remark 2.11. The same holds true once replacing \hookrightarrow_1 in the preceding by $\hookrightarrow_{\mathfrak{p}}$ for any choice of $\mathfrak{p} \in \{c, B, BM\}$.

Definition 2.12. (1) The quasi-order \leq^S over κ^κ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

(2) The quasi-order \subseteq^S over 2^κ is defined via:

$$\eta \subseteq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

Remark 2.13. Note that \subseteq^S is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

In [FMR19], we addressed the problem of universality of \subseteq^S . The next theorem addresses a specific instance of this problem.

Theorem 2.14. *If X \mathfrak{f} -reflects with \diamond to S , then $\leq^X \hookrightarrow_1 \subseteq^S$.*

Proof. Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ and $\langle Y_\alpha \mid \alpha \in S' \rangle$ witness together that X \mathfrak{f} -reflects with \diamond to S . Let $S'' := \{\alpha \in S' \mid Y_\alpha \in \mathcal{F}_\alpha^+\}$. For each $\alpha \in S''$, let $\vec{\mathcal{F}}_\alpha$ be the filter over α generated by $\mathcal{F}_\alpha \cup \{Y_\alpha\}$.

Claim 2.14.1. *There exists a sequence $\langle \eta_\alpha \mid \alpha \in S'' \rangle$ such that, for every stationary $Y \subseteq X$ and every $\eta \in \kappa^\kappa$, the set $\{\alpha \in S'' \mid \eta_\alpha = \eta \upharpoonright \alpha \ \& \ Y \cap \alpha \in \vec{\mathcal{F}}_\alpha\}$ is stationary.*

Proof. Let $C := \text{acc}^+(X)$ and $B := X \setminus C$, so that C is a club in κ and B is a nonstationary subset of κ of cardinality κ . As $|B| = \kappa^{<\kappa}$,⁴ let us fix an injective enumeration $\{a_\beta \mid \beta \in B\}$ of the elements of ${}^{<\kappa}\kappa$. Then, for each $\alpha \in S''$, let $\eta_\alpha := (\bigcup\{a_\beta \mid \beta \in Y_\alpha \cap B\}) \cap (\alpha \times \alpha)$.

To see that $\langle \eta_\alpha \mid \alpha \in S'' \rangle$ is as sought, fix an arbitrary $\eta \in \kappa^\kappa$ and an arbitrary stationary $Y \subseteq X$. Let $f : \kappa \rightarrow B$ be the unique function to satisfy that, for all $\epsilon < \kappa$, $a_{f(\epsilon)} = \eta \upharpoonright \epsilon$. Evidently, $Y \cap C$ is a stationary subset of X disjoint from $\text{Im}(f)$. In particular, $Y' := (Y \cap C) \uplus \text{Im}(f)$ is a stationary subset of X , and hence $G := \{\alpha \in S' \mid Y_\alpha = Y' \cap \alpha \ \& \ Y' \cap \alpha \in \mathcal{F}_\alpha^+\}$ is a stationary subset of S'' . Now, as $\vec{\mathcal{F}}$ captures clubs, let us fix a club $D \subseteq \kappa$ such that, for all $\alpha \in D \cap S'$, $C \cap \alpha \in \mathcal{F}_\alpha$.

⁴By convention, we always assume that $\kappa^{<\kappa} = \kappa$, but this time this also follows from the hypothesis that X \mathfrak{f} -reflects with \diamond to S .

Consider $T := \{\alpha \in G \cap D \mid f[\alpha] \subseteq \alpha \ \& \ \eta[\alpha] \subseteq \alpha\}$ which is a stationary subset of S'' . Let $\alpha \in T$ be arbitrary.

- As $\alpha \in D$, $C \cap \alpha \in \mathcal{F}_\alpha \subseteq \bar{\mathcal{F}}_\alpha$, and as $\alpha \in G$, $Y' \cap \alpha = Y_\alpha \in \bar{\mathcal{F}}_\alpha$. Therefore, the intersection $Y' \cap C \cap \alpha$ is in $\bar{\mathcal{F}}_\alpha$. But $Y' \cap C \cap \alpha = Y \cap C \cap \alpha$, and hence the superset $Y \cap \alpha$ is in $\bar{\mathcal{F}}_\alpha$, as well.

- As $\alpha \in G$, $Y_\alpha = Y' \cap \alpha$, and we infer that $Y_\alpha \cap B = \text{Im}(f) \cap \alpha$. As $f[\alpha] \subseteq \alpha$, we get that $f[\alpha] \subseteq Y_\alpha \cap B \subseteq \text{Im}(f)$. As $\eta[\alpha] \subseteq \alpha$, we get that $\eta \upharpoonright \alpha = \eta \cap (\alpha \times \alpha)$. Recalling the definition of f and the definition of η_α , it follows that $\eta \upharpoonright \alpha \subseteq \eta_\alpha \subseteq \eta$, so that $\eta_\alpha = \eta \upharpoonright \alpha$. \square

For every $\alpha \in S''$, define a quasi-order \preceq_α over ${}^\alpha\kappa$ by letting $\eta \preceq_\alpha \xi$ iff there is $W \in \bar{\mathcal{F}}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) \leq \xi(\beta)\}$. Let $\langle \eta_\alpha \mid \alpha \in S'' \rangle$ be given by the preceding claim. Define a map $f : \kappa^\kappa \rightarrow 2^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} 1, & \text{if } \alpha \in S'' \ \& \ \eta_\alpha \preceq_\alpha \eta \upharpoonright \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

As, for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$, $f(\eta)(\alpha)$ depends only on $\eta \upharpoonright \alpha$, f is 1-Lipschitz. To see that it forms a reduction from \leq^X to \subseteq^S , let η, ξ be arbitrary elements of ${}^\kappa\kappa$. There are two cases to consider:

- If $\eta \leq^X \xi$, then let us fix a club C such that $C \cap X \subseteq \{\beta < \kappa \mid \eta(\beta) \leq \xi(\beta)\}$. Since $\bar{\mathcal{F}}$ captures clubs, let us fix a club $D \subseteq \kappa$ such that, for all $\alpha \in D \cap S'$, $C \cap \alpha \in \mathcal{F}_\alpha$. We claim that D is disjoint from $\{\alpha \in S \mid f(\eta)(\alpha) > f(\xi)(\alpha)\}$, so that indeed $f(\eta) \subseteq^S f(\xi)$. Let $\alpha \in D$ with $f(\eta)(\alpha) = 1$. Then for $W := C \cap \alpha$, we have that $W \in \mathcal{F}_\alpha \subseteq \bar{\mathcal{F}}_\alpha$ and $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) \leq \xi(\beta)\}$, so that $\eta_\alpha \preceq_\alpha \eta \upharpoonright \alpha \preceq_\alpha \xi \upharpoonright \alpha$, and $f(\xi)(\alpha) = 1$.

- If $\neg(\eta \leq^X \xi)$, then $Y := \{\beta \in X \mid \eta(\beta) > \xi(\beta)\}$ is stationary. In effect, $T := \{\alpha \in S'' \mid \eta_\alpha = \eta \upharpoonright \alpha \ \& \ Y \cap \alpha \in \bar{\mathcal{F}}_\alpha\}$ is stationary. Now, for every $\alpha \in T$, the set $W := Y \cap \alpha$ is in $\bar{\mathcal{F}}_\alpha$ so that $f(\eta)(\alpha) = 1$ and $f(\xi)(\alpha) = 0$. Consequently, $\neg(f(\eta) \subseteq^S f(\xi))$. \square

Remark 2.15. The notion of “ $X \diamond$ -reflects to S ” from [FHK14, Definition 57] is the special case of “ $X \bar{\mathcal{F}}$ -reflects with \diamond to S ” in which, for every $\alpha \in S$, $\text{cf}(\alpha) > \omega$ and $\mathcal{F}_\alpha = \text{CUB}(\alpha)$. By [FHK14, Theorem 58], if $X \diamond$ -reflects to S , then $=_X^2 \leftrightarrow_c =_S^2$. So the preceding theorem is an improvement not only because CUB is replaced by abstract filters (allowing the case $S \subseteq \text{cof}(\omega)$), but also because $\leq^X \leftrightarrow_1 \subseteq^S$ entails $=_X \leftrightarrow_1 =_S^2$, so that $=_X^2 \leftrightarrow_1 =_X \leftrightarrow_1 =_S^2$ and hence also $=_X^2 \leftrightarrow_c =_S^2$.

Remark 2.16. Claim 2.14.1 establishes another interesting fact: If there exists a stationary $X \subseteq \kappa$ such that $X \mathfrak{f}$ -reflects with \diamond to S , then \diamond_S holds.

2.1. Nilpotent reductions. Now we turn to prove Theorem D. It is clear from Theorem 2.14 that if $X \mathfrak{f}$ -reflects with \diamond to X , then there exists a 1-Lipschitz map which is different from the identity, yet, it witnesses $=_X \leftrightarrow_1 =_X$. In certain cases, we can obtain a similar result with only \mathfrak{f} -reflection (without the need for \diamond).

Lemma 2.17. *Suppose X, Y, Z are stationary subsets of κ , with $X \cap Y$ nonstationary.*

- (1) *If $X \mathfrak{f}$ -reflects to Y and $Y \mathfrak{f}$ -reflects to X , then there is a function simultaneously witnessing*

$$=_X \leftrightarrow_1 =_Y \ \& \ =_Y \leftrightarrow_1 =_X.$$

- (2) *If $Z \mathfrak{f}$ -reflects to Y and $Z \mathfrak{f}$ -reflects to X , then there is a function simultaneously witnessing*

$$=_Z \leftrightarrow_1 =_Y \ \& \ =_Z \leftrightarrow_1 =_X.$$

Proof. We only prove Clause (1). The proof of Clause (2) is very similar.

Let $X' \subseteq X$ and $Y' \subseteq Y$ be stationary subsets such that there are $\vec{\mathcal{H}} = \langle \mathcal{H}_\alpha \mid \alpha \in Y' \rangle$ and $\vec{\mathcal{G}} = \langle \mathcal{G}_\alpha \mid \alpha \in X' \rangle$ witnessing that X \mathfrak{f} -reflects to Y , and Y \mathfrak{f} -reflects to X , respectively. Let us define $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in X' \triangle Y' \rangle$ via

$$\mathcal{F}_\alpha := \begin{cases} \mathcal{H}_\alpha, & \text{if } \alpha \in X'; \\ \mathcal{G}_\alpha, & \text{if } \alpha \in Y'. \end{cases}$$

For every $\alpha \in X' \triangle Y'$, define the equivalence relation \sim_α over ${}^\alpha\kappa$ and the codes $\ulcorner [\eta]_{\sim_\alpha} \urcorner$ as in Lemma 2.8. Next, define a map $f : \kappa^\kappa \rightarrow \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} \ulcorner [\eta \upharpoonright \alpha]_{\sim_\alpha} \urcorner, & \text{if } \alpha \in X' \triangle Y'; \\ 0, & \text{otherwise.} \end{cases}$$

At this point, the analysis is the same as in the proof of Lemma 2.8. \square

Corollary 2.18. *Suppose that X, Y are stationary subsets of κ , with $X \cap Y$ non-stationary. If X \mathfrak{f} -reflects to Y and Y \mathfrak{f} -reflects to X , then there is a function $f : \kappa^\kappa \rightarrow \kappa^\kappa$ different from the identity witnessing $=_X \hookrightarrow_1 =_X$.*

Proof. Take the square of a function witnessing Lemma 2.17(1). \square

Corollary 2.19. *Suppose $V = L$ and $\mu \in \text{Reg}(\kappa)$. Then there is a function simultaneously witnessing $=_{\kappa \cap \text{cof}(\mu)} \hookrightarrow_1 =_{\kappa \cap \text{cof}(\nu)}$ for all $\nu \in \text{Reg}(\kappa)$.*

Proof. By Lemma 2.17 and Corollary 4.6 below. \square

3. STRONG AND SIMULTANEOUS FORMS OF FILTER REFLECTION

Definition 3.1. For stationary sets $X, S \subseteq \kappa$ and a cardinal $\theta \leq \kappa$, the principle $\mathfrak{f}\text{-Refl}(\theta, X, S)$ asserts the existence of a sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ with $S' \subseteq S$, such that each \mathcal{F}_α is a filter over α , $\vec{\mathcal{F}}$ captures clubs, and, for every sequence $\langle Y_i \mid i < \theta \rangle$ of stationary subsets of X , the set $\{\alpha \in S' \mid \forall i < \max\{\theta, \alpha\} (Y_i \cap \alpha \in \mathcal{F}_\alpha^+)\}$ is stationary.

Remark 3.2. (1) In the special case $\theta = 1$, we shall omit θ , writing $\mathfrak{f}\text{-Refl}(X, S)$.

Note that the latter coincides with the notion of “ X \mathfrak{f} -reflects to S ”.

(2) In the special case in which $\mathcal{F}_\alpha = \text{CUB}(\alpha)$ for all $\alpha \in S$, we shall omit \mathfrak{f} , writing, e.g., “ $\text{Refl}(\theta, X, S)$ ” and “ X reflects with \diamond to S ”.

Another standard notion of reflection is *Friedman’s problem*, $\text{FP}(\kappa)$, asserting that every stationary subset of $\kappa \cap \text{cof}(\omega)$ contains a closed copy of ω_1 .

Lemma 3.3. *If $\text{FP}(\kappa)$ holds, then $\kappa \cap \text{cof}(\omega)$ strongly \mathfrak{f} -reflects to $\kappa \cap \text{cof}(\omega_1)$.*

Proof. For each $\alpha \in \kappa \cap \text{cof}(\omega_1)$, let $\mathcal{F}_\alpha := \text{CUB}(\alpha)$. \square

The following forms a partial converse to Proposition 2.3(2).

Proposition 3.4. *Suppose X strongly \mathfrak{f} -reflects to S .*

If \diamond_X holds, then so does \diamond_S .

Proof. Fix $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ with $S' \subseteq S$ such that X strongly $\vec{\mathcal{F}}$ -reflects to S' .

Claim 3.4.1. *The set $A := \{\alpha \in S' \mid \exists B \in \mathcal{F}_\alpha (\text{sup}(B) < \alpha)\}$ is nonstationary.*

Proof. Suppose not. Fix $\epsilon < \kappa$ for which $A_\epsilon := \{\alpha \in S' \mid \exists B \in \mathcal{F}_\alpha (\text{sup}(B) = \epsilon)\}$ is stationary. Now, consider the club $C := \kappa \setminus (\epsilon + 1)$. As $\vec{\mathcal{F}}$ captures clubs and A_ϵ is stationary, there must exist $\alpha \in A_\epsilon$ such that $C \cap \alpha \in \mathcal{F}_\alpha$. As $\alpha \in A_\epsilon$, let us pick $B \in \mathcal{F}_\alpha$ with $\text{sup}(B) = \epsilon$. Then $(C \cap \alpha) \cap B \in \mathcal{F}_\alpha$, contradicting the fact the former is empty. \square

Suppose that \diamond_X holds. Fix a \diamond_X -sequence $\langle Z_\beta \mid \beta \in X \rangle$. For every $\alpha \in S' \setminus A$, let $\mathcal{Z}_\alpha := \{Z \subseteq \alpha \mid \{\beta \in X \cap \alpha \mid Z \cap \beta = Z_\beta\} \in \mathcal{F}_\alpha\}$.

Claim 3.4.2. \mathcal{Z}_α contains at most a single set.

Proof. Towards a contradiction, suppose that $Z \neq Z'$ are elements of \mathcal{Z}_α . Let $B := \{\beta \in X \cap \alpha \mid Z \cap \beta = Z_\beta\}$ and $B' := \{\beta \in X \cap \alpha \mid Z' \cap \beta = Z_\beta\}$. Fix $\zeta \in Z \triangle Z'$. As $Z, Z' \in \mathcal{Z}_\alpha$, $B \cap B' \in \mathcal{F}_\alpha$. As $\alpha \notin A$, we may find some $\beta \in B \cap B'$ above ζ . But then $Z \cap \beta = Z' \cap \beta$, contradicting the fact that $\zeta \in (Z \cap \beta) \triangle (Z' \cap \beta)$. \square

For each $\alpha \in S$, fix a subset $Z_\alpha \subseteq \alpha$ such that, if $\alpha \in S' \setminus A$, then $\mathcal{Z}_\alpha \subseteq \{Z_\alpha\}$. To see that $\langle Z_\alpha \mid \alpha \in S \rangle$ is a \diamond_S -sequence, let Z be an arbitrary subset of κ . Consider the stationary set $Y := \{\beta \in X \mid Z \cap \beta = Z_\beta\}$. Pick $\alpha \in S' \cap \text{acc}^+(Y) \setminus A$ such that $Y \cap \alpha \in \mathcal{F}_\alpha$. Then $Z_\alpha = Z \cap \alpha$. \square

Remark 3.5. The preceding provides a way of separating \mathfrak{f} -reflection from strong \mathfrak{f} -reflection. Indeed, by the main result of [?], it is consistent that for some weakly compact cardinal κ , \diamond_κ holds, but $\diamond_{\text{Reg}(\kappa)}$ fails. It follows that in any such model, κ does not strongly \mathfrak{f} -reflects to $\text{Reg}(\kappa)$, while $\text{Refl}(\kappa, \text{Reg}(\kappa))$ does hold (see Lemma 3.7(1) below).

A similar configuration can also be obtained at the level of accessible cardinals as small as \aleph_2 . Denote $S_m^2 := \aleph_2 \cap \text{cof}(\aleph_m)$. In [Zha20], Zhang constructed a model of GCH in which $\text{Refl}(S_0^2, S_1^2)$ holds, but $\diamond_{S_1^2}$ fails. By [Gre76, Lemma 2.1], GCH implies that \diamond_X holds for any stationary $X \subseteq S_0^2$. Thus, in Zhang's model, there exists no stationary subset of S_0^2 that strongly \mathfrak{f} -reflects to S_1^2 .

The rest of this section is motivated by Theorem 2.14.

Evidently, if X \mathfrak{f} -reflects with \diamond to S , then $\mathfrak{f}\text{-Refl}(X, S)$ and \diamond_S both hold. The next lemma deals with the converse implication.

Lemma 3.6. *Let $X \subseteq \kappa$ and $S \subseteq \kappa \cap \text{cof}(>\omega)$. For $\mathbb{P} := \text{Add}(\kappa, 1)$, the following are equivalent:*

- (1) $V^\mathbb{P} \models \text{Refl}(X, S)$;
- (2) $V^\mathbb{P} \models X$ reflects with \diamond to S .

Proof. It is clear from the definition that (2) \implies (1).

(1) \implies (2): For all $\alpha < \kappa$, we define a \mathbb{P} -name for a subset of α , as follows:

$$\dot{D}_\alpha := \{(\check{\beta}, p) \mid \beta < \alpha, p \in {}^{<\kappa}2 \setminus {}^{\leq \alpha + \beta}2, p(\alpha + \beta) = 1\}.$$

Now, suppose that \dot{Y} is a \mathbb{P} -name, $p \in \mathbb{P}$ and p forces that \dot{Y} is a stationary subset of X ; we shall find an extension r of p and an ordinal $\alpha \in S$ such that r forces that \dot{D}_α is a stationary subset of α which is also an initial segment of \dot{Y} .

Claim 3.6.1. *There are a condition q extending p and an ordinal $\alpha \in S$ such that:*

- $\text{dom}(q) = \alpha$;
- q decides \dot{Y} up to α , and decides it to be a stationary subset of α .

Proof. Let G be a \mathbb{P} -generic over V , with $p \in G$. Work in $V[G]$. Let $g := \bigcup G$. As \mathbb{P} does not add bounded subsets of κ , we may define a function $f : \kappa \rightarrow \kappa$ such that, for all $\epsilon < \kappa$, $g \upharpoonright (f(\epsilon))$ decides \dot{Y} up to ϵ . Consider the club $C := \{\alpha < \kappa \mid f[\alpha] \subseteq \alpha\}$, and note that, for all $\alpha \in C$, $g \upharpoonright \alpha$ decides \dot{Y} up to α . As $V^\mathbb{P} \models \text{Refl}(X, S)$, $p \in G$, and p forces that \dot{Y} is a stationary subset of X , $R := \{\alpha \in C \cap S \mid (\dot{Y}_G) \cap \alpha \text{ is stationary in } \alpha\}$ is stationary in $V[G]$. Fix $\alpha \in R$ with $\alpha > \text{dom}(p)$. Then α and $q := g \upharpoonright \alpha$ are as sought. \square

Let α and q be given by the claim. Fix a stationary $d \subseteq \alpha$ such that q forces that \dot{Y} up to α is equal to \check{d} . Define a function $r : \alpha + \alpha \rightarrow 2$ by letting, for all $\epsilon < \alpha + \alpha$,

$$r(\epsilon) := \begin{cases} q(\epsilon), & \text{if } \epsilon < \alpha; \\ 1, & \text{if } \epsilon = \alpha + \beta \text{ \& } \beta \in d; \\ 0, & \text{otherwise.} \end{cases}$$

Then r extends p and forces that \dot{D}_α is a stationary subset of α which is also an initial segment of \dot{Y} . \square

The following is well-known, the second item is pointed out in [Mag82]. As we will need the proof later on (in proving Theorem 4.5), we do include it.

Lemma 3.7. *Suppose that κ is weakly compact. Then:*

- (1) $\text{Refl}(\kappa, \kappa, \text{Reg}(\kappa))$ holds;
- (2) For every $\lambda \in \text{Reg}(\kappa)$, in the forcing extension by $\text{Col}(\lambda, < \kappa)$, $\kappa = \lambda^+$ and $\text{Refl}(\kappa, \kappa \cap \text{cof}(< \lambda), \kappa)$ holds.

Proof. (1) We shall consider a few first-order sentences in the language with unary predicate symbols \mathbb{O} and \mathbb{D} , and binary predicate symbols ϵ , \mathbb{A} and \mathbb{F} . Specifically, let φ_0 and φ_1 denote first-order sentences such that:

- $\langle V_\alpha, \epsilon, O, F \rangle \models \varphi_0$ iff F is a function from some $\beta \in O$ to O .
- $\langle V_\alpha, \epsilon, O, F \rangle \models \varphi_1$ iff F is a function with $\text{Im}(F) \subseteq \gamma$ for some $\gamma \in O$.

Evidently, $\Phi := \forall F(\varphi_0 \rightarrow \varphi_1)$ is a Π_1^1 -sentence such that $\langle V_\alpha, \epsilon, \text{OR} \cap \alpha \rangle \models \Phi$ iff α is a regular cardinal.

Also, let ψ be a first-order sentence such that $\langle V_\kappa, \epsilon, \text{OR} \cap \kappa, D \rangle \models \psi$ iff D is a club in κ . Then, let Ψ be the following Π_1^1 -sentence:

$$\forall D \forall \iota ((\psi(D) \wedge \mathbb{O}(\iota)) \rightarrow \exists \beta (D(\beta) \wedge (\mathbb{A}(\iota, \beta)))).$$

Now, to verify $\text{Refl}(\kappa, \kappa, \text{Reg}(\kappa))$, fix an arbitrary sequence $\langle Y_\iota \mid \iota < \kappa \rangle$ of stationary subsets of κ , and we shall find an $\alpha \in \text{Reg}(\kappa)$ such that $Y_\iota \cap \alpha$ is stationary for all $\iota < \alpha$.

Set $A := \{(\iota, \beta) \mid \iota < \kappa \text{ \& } \beta \in Y_\iota\}$. A moment reflection makes it clear that

$$\langle V_\kappa, \epsilon, \text{OR} \cap \kappa, A \rangle \models \Phi \wedge \Psi.$$

As weak compactness is equivalent to Π_1^1 -indescibability (cf. [Kan08, Theorem 6.4]), there exists an uncountable $\alpha < \kappa$ such that:

$$\langle V_\alpha, \epsilon, \text{OR} \cap \alpha, A \cap (\alpha \times \alpha) \rangle \models \Phi \wedge \Psi.$$

Clearly, α is as sought.

(2) Let $\lambda \in \text{Reg}(\kappa)$. As in [AHKM19, Claim 2.11.1], we work with a partial order \mathbb{P} which is isomorphic to $\text{Col}(\lambda, < \kappa)$, but, in addition, $\mathbb{P} \subseteq V_\kappa$. Namely, $\mathbb{P} = (P, \leq)$, where

$$P := \{r \upharpoonright (\text{sup}(\text{supp}(r)) + 1) \mid r \in \text{Col}(\lambda, < \kappa)\},$$

and $q \leq p$ iff $q \supseteq p$. Note that, for every $\alpha \in \text{Reg}(\kappa)$ above λ , \mathbb{P} is also isomorphic to $\mathbb{P}_\alpha \times \mathbb{P}^{\geq \alpha}$, where

- $\mathbb{P}_\alpha := (P_\alpha, \leq)$, with $P_\alpha := \{r \upharpoonright (\text{sup}(\text{supp}(r)) + 1) \mid r \in \text{Col}(\lambda, < \alpha)\}$, and
- $\mathbb{P}^{\geq \alpha} := (P^{\geq \alpha}, \leq)$, with $P^{\geq \alpha} := \{r \upharpoonright (\text{sup}(\text{supp}(r)) + 1) \mid r \in \text{Col}(\lambda, [\alpha, \kappa])\}$.

Now, suppose $p_0 \in P$ and \dot{f} is a \mathbb{P} -name such that p_0 forces \dot{f} is a function with domain κ , and, for each $\iota < \kappa$, $\dot{f}(\iota)$ is a stationary subset of $\kappa \cap \text{cof}(< \lambda)$.

Define a set H to consist of all quadruples (ι, β, p, q) such that:

- $\iota < \kappa$,
- $\beta \in \kappa \cap \text{cof}(< \lambda)$,
- $p, q \in V_\kappa$;

- if $p \in P$ and $p \leq p_0$, then $q \in P$, $q \leq p$ and $q \Vdash_{\mathbb{P}} \check{\beta} \in \dot{f}(\check{\iota})$.

Let $\text{op}(\dot{x}, \dot{y})$ denote the canonical name for the ordered pair whose left element is \dot{x} and right element is \dot{y} , and set $\dot{A} := \{(\text{op}(\check{\iota}, \check{\beta}), q) \mid \exists p \leq p_0 (\alpha, \beta, p, q) \in H\}$, so that the interpretation of \dot{A} plays the role of the set A from the proof of the previous clause. Furthermore, $p_0 \Vdash \dot{A} = \dot{f}$.

Let ψ be a first-order sentence such that $\langle V_\kappa, \in, \text{OR} \cap \kappa, D \rangle \models \psi$ iff D is a club in κ . Then, let Ψ be the following Π_1^1 -sentence:

$$\forall p \forall D \forall \iota ((\psi(D) \wedge \mathbb{Q}(\iota)) \rightarrow \exists q \exists \beta (D(\beta) \wedge (\mathbb{H}(\iota, \beta, p, q))).$$

Recalling that for any condition $p \leq p_0$, any club D in κ , and any $\iota < \kappa$, there is a condition $q \leq p$ deciding the existence of an ordinal in $\dot{f}(\iota)$ and in D , we have:

$$\langle V_\kappa, \in, \text{OR} \cap \kappa, H \rangle \models \Psi.$$

Since κ is Π_1^1 -indescribable, we may fix a strongly inaccessible cardinal $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, \text{OR} \cap \alpha, H \cap V_\alpha \rangle \models \Psi.$$

As \mathbb{P}_α has the α -cc, every club in α in $V^{\mathbb{P}_\alpha}$ covers a club in α from V . Therefore:

$p_0 \Vdash_{\mathbb{P}_\alpha} \langle \dot{f}(\iota) \cap \alpha \mid \iota < \alpha \rangle$ is an α -sequence of stationary subsets of $\alpha \cap \text{cof}(<\lambda)$.

By [She79, Theorem 20] and [She91, Lemma 4.4(1)] (see also [Eis10, §2]), for λ regular, every stationary subset of $\lambda^+ \cap \text{cof}(<\lambda)$ is preserved by a $<\lambda$ -closed notion of forcing. In $V^{\mathbb{P}_\alpha}$, $\alpha = \lambda^+$ for the regular cardinal λ , and $\mathbb{P}^{\geq \alpha}$ is $<\lambda$ -closed. Therefore:

$p_0 \Vdash_{\mathbb{P}_\alpha \times \mathbb{P}^{\geq \alpha}} \langle \dot{f}(\iota) \cap \alpha \mid \iota < \alpha \rangle$ is an α -sequence of stationary subsets of α . \square

The upcoming corollary was announced first by Shelah and Väänänen in [SV05] without a proof. It amounts to the special case of κ reflects with \diamond to $\text{Reg}(\kappa)$, when κ is weakly compact (see [SV05, Definition 9]).

Corollary 3.8 (Hellsten, [Hel03, Lemma 5.2.3]). *If κ is weakly compact, then, in some cofinality-preserving forcing extension, κ reflects with \diamond to $\text{Reg}(\kappa)$.*

Proof. Recall that Silver proved that for every weakly compact cardinal κ , there is a forcing extension $V[G]$ such that $V[G][H] \models \kappa$ is weakly compact, whenever H is an $\text{Add}(\kappa, 1)$ -generic over $V[G]$. (cf. [Cum10, Example 16.2]). Now, appeal to Lemma 3.7(1) and Lemma 3.6. \square

The preceding took care of weakly compact cardinals. Building on the work of Hayut and Lambie-Hanson from [HLH17], we can also obtain reflection at the level of successor of singulars and small inaccessibles. It is of course also possible to obtain reflection with diamond at the level of successors of regulars, but this is easier, and will be done in Corollary 3.13 below.

Corollary 3.9. (1) *If the existence of infinitely many supercompact cardinals is consistent, then it is consistent that $\aleph_{\omega+1}$ reflects with \diamond to $\aleph_{\omega+1}$;*
 (2) *If the existence of an inaccessible limit of supercompact cardinals is consistent, then it is consistent that, letting κ be the least inaccessible cardinal, κ reflects with \diamond to κ .*

Proof. Following [HLH17], we let $\text{Refl}^*(X)$ assert that whenever \mathbb{P} is a $<\kappa$ -closed-directed forcing notion of size $\leq \kappa$, $\Vdash_{\mathbb{P}} \text{Refl}(X, \kappa)$. In particular, if $\text{Refl}^*(X)$ holds and $\kappa^{<\kappa} = \kappa$, then for $\mathbb{P} := \text{Add}(\kappa, 1)$, we would have that $V^{\mathbb{P}} \models \text{Refl}(X, \kappa)$, and then by Lemma 3.6, furthermore, $V^{\mathbb{P}} \models X$ reflects with \diamond to κ .

Now, by [HLH17, Theorem 3.23], the hypothesis of Clause (1) yields the consistency of $\text{Refl}^*(\aleph_{\omega+1})$ holds. Likewise, by [HLH17, Theorem 3.24], the hypothesis

of Clause (2) yields the consistency of the statement that, there is an inaccessible cardinal and the least inaccessible cardinal κ satisfies $\text{Refl}^*(\kappa)$. \square

Another approach for adjoining diamond to reflection is as follows.

Lemma 3.10. *Suppose $\mathfrak{f}\text{-Refl}(\kappa, X, S)$ holds for stationary subsets X, S of κ . If \diamond_S^* holds, then X \mathfrak{f} -reflects with \diamond to S .*

Proof. Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ be a sequence witnessing $\mathfrak{f}\text{-Refl}(\kappa, X, S')$, so that $S' \subseteq S$. Suppose that \diamond_S^* (or just $\diamond_{S'}^*$) holds. Fix a matrix $\langle Z_\alpha^i \mid i < \alpha < \kappa \rangle$ such that, for every $Z \subseteq \kappa$, for club many $\alpha \in S'$, there is $i < \alpha$ with $Z_\alpha^i = Z \cap \alpha$. Fix a bijection $\pi : \kappa \times \kappa \leftrightarrow \kappa$. For any pair $i < \alpha < \kappa$, let $Y_\alpha^i := \{\beta < \alpha \mid \pi(\beta, i) \in Z_\alpha^i\}$.

Claim 3.10.1. *There exists $i < \kappa$ such that for every stationary $Y \subseteq X$, the set $\{\alpha \in S' \mid Y_\alpha^i = Y \cap \alpha \ \& \ Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.*

Proof. Suppose not, then for every $i < \kappa$, we may find a stationary $Y_i \subseteq X$ for which $\{\alpha \in S' \mid Y_\alpha^i = Y_i \cap \alpha \ \& \ Y_i \cap \alpha \in \mathcal{F}_\alpha^+\}$ is non-stationary. Let $Z := \{\pi(\beta, i) \mid i < \kappa, \beta \in Y_i\}$. Fix a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$:

- $\pi[\alpha \times \alpha] = \alpha$, and
- there exists $i < \alpha$ with $Z_\alpha^i = Z \cap \alpha$.

By the hypothesis, $T := \{\alpha \in S' \cap C \mid \forall i < \alpha \ (Y_i \cap \alpha \in \mathcal{F}_\alpha^+)\}$ must be stationary. By Fodor's lemma, we may fix $i^* < \kappa$ and a stationary $T' \subseteq T$ such that, for all $\alpha \in T'$, $Z_\alpha^{i^*} = Z \cap \alpha$. For all $\alpha \in T'$, we have:

- $\alpha \in T$, so that, in particular, $Y_{i^*} \cap \alpha \in \mathcal{F}_\alpha^+$;
- $\alpha \in C$, so that $Y_\alpha^{i^*} = \{\beta < \alpha \mid \pi(\beta, i^*) \in Z_\alpha^{i^*}\} = \{\beta < \alpha \mid \pi(\beta, i^*) \in Z \cap \alpha\} = Y_{i^*} \cap \alpha$.

It thus follows that $\{\alpha \in S' \mid Y_\alpha^{i^*} = Y_{i^*} \cap \alpha \ \& \ Y_{i^*} \cap \alpha \in \mathcal{F}_\alpha^+\}$ covers the stationary set T' , contradicting the choice of Y_{i^*} . \square

This completes the proof. \square

Remark 3.11. (1) Note that by the definition of \diamond_S^* , if $\kappa = \lambda^+$ is a successor cardinal, then the above argument establishes the following: If $\mathfrak{f}\text{-Refl}(\lambda, X, S)$ and \diamond_S^* both hold, then X \mathfrak{f} -reflects with \diamond to S .
(2) Note that the same proof establishes the corresponding fact for the \mathfrak{f} -free (that is, genuine) versions of reflection.

Corollary 3.12. *Suppose that λ is a regular uncountable cardinal, and $\diamond_{\lambda^+ \cap \text{cof}(\lambda)}^*$ holds. For every stationary $X \subseteq \lambda^+ \cap \text{cof}(<\lambda)$, if $\text{Refl}(\lambda, X, \lambda^+)$ holds, then X reflects with \diamond to $\lambda^+ \cap \text{cof}(\lambda)$.*

Proof. Suppose we are given $X \subseteq \lambda^+ \cap \text{cof}(<\lambda)$ for which $\mathfrak{f}\text{-Refl}(\lambda, X, \lambda^+)$ holds. By Remark 3.11, it suffices to prove that $\text{Refl}(\lambda, X, \lambda^+ \cap \text{cof}(\lambda))$ holds. For this, let $\langle Y_i \mid i < \lambda \rangle$ be some sequence of stationary subsets of X . Fix an arbitrary partition $\vec{X} = \langle X_i \mid i < \lambda \rangle$ of X into stationary sets. As $\text{Refl}(\lambda, X, \lambda^+)$ holds, the set

$$A := \{\alpha \in \lambda^+ \mid \forall i < \lambda (Y_i \cap \alpha \text{ and } X_i \cap \alpha \text{ are stationary in } \alpha)\}$$

is stationary. As the elements of \vec{X} are pairwise disjoint, it follows that $A \subseteq \lambda^+ \cap \text{cof}(\lambda)$. \square

Corollary 3.13. *Suppose that κ is a weakly compact cardinal.*

For every $\lambda \in \text{Reg}(\kappa)$, in the forcing extension by $\text{Col}(\lambda, <\kappa)$, $\lambda^+ \cap \text{cof}(<\lambda)$ reflects with \diamond to $\lambda^+ \cap \text{cof}(\lambda)$.

Proof. Let $\lambda \in \text{Reg}(\kappa)$, and work in the forcing extension by $\text{Col}(\lambda, < \kappa)$. As the proof of [BR17, Example 1.26] shows, $\diamond_{\lambda^+}^*$ holds. In addition, by Lemma 3.7(2), $\text{Refl}(\lambda^+, \lambda^+ \cap \text{cof}(< \lambda), \lambda^+)$ holds. So, by the preceding corollary, $\lambda^+ \cap \text{cof}(< \lambda)$ reflects with \diamond to $\lambda^+ \cap \text{cof}(\lambda)$. \square

Remark 3.14. It thus follows from Theorem 2.14 that in the model of the preceding, for every $\mu \in \text{Reg}(\lambda)$, $\leq^{\lambda^+ \cap \text{cof}(\mu)} \leftrightarrow_1 \subseteq^{\lambda^+ \cap \text{cof}(\lambda)}$, and hence also $=_{\lambda^+ \cap \text{cof}(\mu)} \leftrightarrow_1 =_{\lambda^+ \cap \text{cof}(\lambda)}^2$. This improves [FHK14, Theorem 55], as the result is not limited to double successors, and as we do not need to assume that our ground model is \mathbb{L} .

A variation of the proof of Lemma 3.10 yields the following:

Lemma 3.15. *Suppose Martin's Maximum (MM) holds, $\kappa \geq \aleph_2$, $X \subseteq \kappa \cap \text{cof}(\omega)$ is stationary, and $S = \kappa \cap \text{cof}(\omega_1)$. If \diamond_X holds,⁵ then X reflects with \diamond to S .*

Proof. Suppose that \diamond_X holds, as witnessed by $\langle Z_\gamma \mid \gamma \in X \rangle$. For every $Z \subseteq \kappa$, let

$$G_Z := \{\gamma \in \text{sup}(Z) \cap X \mid Z \cap \gamma = Z_\gamma\}.$$

Let $\alpha \in S$ be arbitrary. Fix a strictly increasing function $\pi_\alpha : \omega_1 \rightarrow \alpha$ whose image is a club in α , and then let

$$\mathcal{Z}_\alpha := \{Z \subseteq \alpha \mid \pi_\alpha^{-1}[G_Z] \text{ is stationary in } \omega_1\}.$$

Evidently, for all two distinct $Z, Z' \in \mathcal{Z}_\alpha$, $\pi_\alpha^{-1}[G_Z]$ and $\pi_\alpha^{-1}[G_{Z'}]$ are almost disjoint stationary subsets of ω_1 . As MM implies that NS_{ω_1} is saturated, we infer that $|\mathcal{Z}_\alpha| \leq \aleph_1$. Thus, let $\{Z_\alpha^i \mid i < \omega_1\}$ be some enumeration (possibly, with repetitions) of \mathcal{Z}_α . Fix a bijection $\pi : \kappa \times \omega_1 \leftrightarrow \kappa$. For each $\alpha \in S$ and $i < \omega_1$, let $Y_\alpha^i := \{\beta < \alpha \mid \pi(\beta, i) \in Z_\alpha^i\}$ provided that the latter set is stationary in α , and otherwise, let $Y_\alpha^i := \alpha$.

Claim 3.15.1. *There exists $i < \omega_1$ such that, for every stationary $Y \subseteq \kappa$, $\{\alpha \in S \cap \text{Tr}(Y) \mid Y \cap \alpha = Y_\alpha^i\}$ is stationary.*

Proof. Suppose not. For every $i < \omega_1$, fix a stationary $Y_i \subseteq X$ for which $\{\alpha \in S \cap \text{Tr}(Y_i) \mid Y_i \cap \alpha = Y_\alpha^i\}$ is non-stationary. Let $Z := \{\pi(\beta, i) \mid i < \omega_1, \beta \in Y_i\}$. Consider the stationary set $G := \{\gamma \in X \mid Z \cap \gamma = Z_\gamma\}$ and the club $C := \{\alpha \in \text{acc}^+(Z) \mid \pi[\alpha \times \omega_1] = \alpha\}$. As MM implies $\text{Refl}(\omega_1, X, S)$, the following set is stationary

$$S' := \{\alpha \in C \cap S \mid G \cap \alpha \text{ is stationary, and, for all } i < \omega_1 \ Y_i \cap \alpha \text{ is stationary}\}.$$

For every $\alpha \in S'$, since $\alpha \in \text{Tr}(G) \cap \text{acc}^+(Z)$, $G_{Z \cap \alpha}$ covers the stationary set $G \cap \alpha$, so there exists $i < \omega_1$ such that $Z_\alpha^i = Z \cap \alpha$. Now, fix $i^* < \omega_1$ and a stationary $T \subseteq S'$ such that, for all $\alpha \in T$, $Z_\alpha^{i^*} = Z \cap \alpha$. For all $\alpha \in T$, we have:

- $\alpha \in S'$, and hence $\alpha \in S \cap \text{Tr}(Y_{i^*})$;
- $\alpha \in C$, and hence $\{\beta < \alpha \mid \pi(\beta, i^*) \in Z_\alpha^{i^*}\} = \{\beta < \alpha \mid \pi(\beta, i^*) \in Z \cap \alpha\} = Y_{i^*} \cap \alpha$.

So $\{\alpha \in S \cap \text{Tr}(Y_{i^*}) \mid Y_{i^*} \cap \alpha = Y_\alpha^{i^*}\}$ covers the stationary set T , contradicting the choice of Y_{i^*} . \square

This completes the proof. \square

Definition 3.16. A stationary subset S of κ is said to be:

- (1) *ineffable* iff for every sequence $\langle A_\alpha \mid \alpha \in S \rangle$, there exists $A \subseteq \kappa$ for which $\{\alpha \in S \mid A \cap \alpha = A_\alpha \cap \alpha\}$ is stationary.
- (2) *weakly ineffable* iff for every sequence $\langle A_\alpha \mid \alpha \in S \rangle$, there exists $A \subseteq \kappa$ for which $\{\alpha \in S \mid A \cap \alpha = A_\alpha \cap \alpha\}$ is cofinal in κ .

⁵Recall that by a theorem of Shelah [She10], if $\kappa = \kappa^{< \kappa}$ is the successor of a cardinal uncountable cofinality, then \diamond_X holds for every stationary $X \subseteq \kappa \cap \text{cof}(\omega)$.

- (3) *weakly compact* iff for every Π_1^1 -sentence ϕ and every $A \subseteq V_\kappa$ such that $\langle V_\kappa, \in, A \rangle \models \phi$, there exists $\alpha \in S$ such that $\langle V_\alpha, \in, A \cap V_\alpha \rangle \models \phi$.

Definition 3.17 (Sun, [Sun93]). For a weakly compact subset $S \subseteq \kappa$, \diamond_S^1 asserts the existence of a sequence $\langle Z_\alpha \mid \alpha \in S \rangle$ such that, for every $Z \subseteq \kappa$, the set $\{\alpha \in S \mid Z \cap \alpha = Z_\alpha\}$ is weakly compact.

The proof of [Sun93, Theorem 2.11] makes clear that for every weakly ineffable $S \subseteq \kappa$, \diamond_S^1 holds. It is also easy to see that \diamond_S^1 implies that κ reflects with \diamond to S . Therefore:

Corollary 3.18. *For every weakly ineffable $S \subseteq \kappa$, κ reflects with \diamond to S . \square*

We conclude this section by proving that \diamond -reflection is equivalent to various seemingly stronger statements. For instance, the concept of Clause (2) of the next lemma is implicit in [AHKM19], as the principle WC_κ^* from [AHKM19, Lemma 3.4] is equivalent to the instance $X := \kappa$ and $S := \text{Reg}(\kappa)$. Likewise, the question of whether Clause (1) implies Clause (3) is implicit in the statement of [AHKM19, Claim 2.11.1]. The fact that the two clauses are equivalent allows to reduce the hypothesis of “there is a Π_1^+ -indescribable cardinal” of [AHKM19, Theorem 2.11] down to “there is a weakly compact cardinal” via Corollary 3.13 above.

Lemma 3.19. *Let $X \subseteq \kappa$ and $S \subseteq \kappa \cap \text{cof}(>\omega)$. Then the following are equivalent:*

- (1) X reflects with \diamond to S ;
- (2) there exists a sequence $\langle f_\alpha \mid \alpha \in S \rangle$ such that, for every $g \in \kappa^\kappa$ and every stationary $Y \subseteq X$, the set $\{\alpha \in S \cap \text{Tr}(Y) \mid g \upharpoonright \alpha = f_\alpha\}$ is stationary;
- (3) there exists a partition $\langle S_i \mid i < \kappa \rangle$ of S such that, for all $i < \kappa$, X reflects to S_i with \diamond .

Proof. (1) \implies (2): This is a special case of the proof of Claim 2.14.1.

(2) \implies (3) Let $\langle f_\alpha \mid \alpha \in S \rangle$ be as in Clause (2). Without loss of generality, for all $\alpha \in S$, f_α is a function from α to κ . For all $i < \kappa$, let $S_i := \{\alpha \in S \mid f_\alpha(0) = i\}$, so that $\langle S_i \mid i < \kappa \rangle$ is a partition of S . For each $\alpha \in S$, let

$$Y_\alpha := \{\beta < \kappa \mid (\beta = 0 \wedge f_\alpha(1) \text{ is odd}) \vee (\beta = 1 \wedge f_\alpha(1) \geq 2) \vee (\beta > 1 \wedge f_\alpha(\beta) = 4)\}.$$

Let $i < \kappa$ be arbitrary. We claim that $\langle Y_\alpha \mid \alpha \in S_i \rangle$ witnesses that X reflects with \diamond to S_i . To see this, fix an arbitrary stationary subset Y of X . Define a function $g : \kappa \rightarrow \kappa$ as follows:

$$g(\beta) := \begin{cases} i, & \text{if } \beta = 0; \\ 1, & \text{if } \beta = 1 \text{ and } Y \cap \{0, 1\} = \{0\}; \\ 2, & \text{if } \beta = 1 \text{ and } Y \cap \{0, 1\} = \{1\}; \\ 3, & \text{if } \beta = 1 \text{ and } Y \cap \{0, 1\} = \{0, 1\}; \\ 4, & \text{if } \beta > 1 \text{ and } \beta \in Y; \\ 0, & \text{otherwise;} \end{cases}$$

Consider the stationary set $G := \{\alpha \in S \cap \text{Tr}(Y) \mid g \upharpoonright \alpha = f_\alpha\}$. Let $\alpha \in G \cap \text{acc}(\kappa)$ be arbitrary. We have $f_\alpha(0) = g(0) = i$, so that $\alpha \in S_i$. Finally, $g \upharpoonright (\kappa \setminus \{0, 1\})$ forms the characteristic function of $Y \setminus \{0, 1\}$ and $g(1)$ encodes $Y \cap \{0, 1\}$, so that $Y_\alpha = Y \cap \alpha$.

(3) \implies (1): This is trivial. \square

Question 3.20. Suppose X (resp. strongly) \mathfrak{f} -reflects to S . Must there exist a partition $\langle S_i \mid i < \kappa \rangle$ of S into stationary sets such that, for all $i < \kappa$, X (resp. strongly) \mathfrak{f} -reflects to S_i ?

4. FORCING FAKE REFLECTION

In this section, we focus on the consistency of *fake reflection*, i.e., X \mathfrak{f} -reflects to S and yet there is a stationary subset of X that does not reflect (in the classical sense) to S . By the work of Jensen [Jen72], in Gödel's constructible universe, L , stationary reflection fails at any non weakly compact cardinal, but, as we will see, filter reflection holds everywhere in L . We will also see that fake reflection is forceable.

4.1. A diamond reflecting second-order formulas. A Π_n^1 -sentence ϕ is a formula of the form $\forall X_1 \exists X_2 \cdots X_n \varphi$ where φ is a first-order sentence over a relational language \mathcal{L} as follows:

- \mathcal{L} has a predicate symbol ϵ of arity 2;
- \mathcal{L} has a predicate symbols \mathbb{X}_i , $i \leq n$, of arity $m(\mathbb{X}_i)$;
- \mathcal{L} has infinitely many predicate symbols $(\mathbb{A}_n)_{n \in \omega}$, each \mathbb{A}_m is of arity $m(\mathbb{A}_m)$.

Definition 4.1. For sets N and x , we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.

Suppose that a set N sees an ordinal α , and that $\phi = \forall X_1 \exists X_2 \cdots \varphi$ is a Π_n^1 -sentence, where φ is a first-order sentence in the above-mentioned language \mathcal{L} . For every sequence $(A_m)_{m \in \omega}$ such that, for all $m \in \omega$, $A_m \subseteq \alpha^{m(\mathbb{A}_m)}$, we write

$$\langle \alpha, \epsilon, (A_m)_{m \in \omega} \rangle \models_N \phi$$

to express that the two hold:

- (1) $(A_m)_{m \in \omega} \in N$;
- (2) $N \models (\forall X_1 \subseteq \alpha^{m(\mathbb{X}_1)})(\exists X_2 \subseteq \alpha^{m(\mathbb{X}_2)}) \cdots [\langle \alpha, \epsilon, (A_m)_{m \in \omega}, X_1, X_2, \dots \rangle \models \varphi]$, where:
 - ϵ is the interpretation of ϵ ;
 - X_i is the interpretation of \mathbb{X}_i ;
 - for all $m \in \omega$, A_m is the interpretation of \mathbb{A}_m .

Convention 4.2. We write α^+ for $|\alpha|^+$, and write $\langle \alpha, \epsilon, (A_n)_{n \in \omega} \rangle \models \phi$ for

$$\langle \alpha, \epsilon, (A_n)_{n \in \omega} \rangle \models_{H_{\alpha^+}} \phi.$$

Definition 4.3 (Fernandes-Moreno-Rinot, [FMR19]). For a stationary $S \subseteq \kappa$ and a positive integer n , $\text{DI}_S^*(\Pi_n^1)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

- (1) for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α ;
- (2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;
- (3) whenever $\langle \kappa, \epsilon, (A_m)_{m \in \omega} \rangle \models \phi$, with ϕ a Π_n^1 -sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \epsilon, (A_m \cap (\alpha^{m(\mathbb{A}_m)}))_{m \in \omega} \rangle \models_{N_\alpha} \phi$.

Remark 4.4. The principle $\text{DI}_S^+(\Pi_n^1)$ is defined by strengthening Clause (2) in the definition of $\text{DI}_S^*(\Pi_n^1)$ to require that $C \cap \alpha$ be in N_α , as well.

The Todorcevic-Väänänen principle $\diamond_S^+(\Pi_n^1)$ from [TV99] is obtained by strengthening Clause (1) in the definition of $\text{DI}_S^+(\Pi_n^1)$ to require that $|N_\alpha| = \max\{\aleph_0, |\alpha|\}$.

Lemma 4.5. *Suppose $S \subseteq \kappa$ is stationary for which $\text{DI}_S^*(\Pi_1^1)$ holds. Then:*

- (1) $\mathfrak{f}\text{-Refl}(\kappa, \kappa, S)$;
- (2) κ \mathfrak{f} -reflects with \diamond to S .

Proof. (1) Let $\langle N_\alpha \mid \alpha \in S \rangle$ witness the validity of $\text{DI}_S^*(\Pi_1^1)$. As in the proof of Lemma 3.7(1), let Φ be a Π_1^1 -sentence, such that, for every ordinal α , $\langle \alpha, \in \rangle \models \Phi$ iff $(\alpha$ is a regular cardinal).⁶ Likewise, let Ψ be a Π_1^1 -sentence such that for every ordinal α and every $A \subseteq \alpha \times \alpha$, $\langle \alpha, \in, A \rangle \models \Phi$ iff

for every $\iota < \alpha$, $\{\beta < \alpha \mid (\iota, \beta) \in A\}$ is stationary in α .

Now, let S' denote the set of all $\alpha \in S$ such that:

- (i) $\alpha > \omega$;
- (ii) $|N_\alpha| = |\alpha|$;
- (iii) $\langle \alpha, \in \rangle \models_{N_\alpha} \Phi$.

For each $\alpha \in S'$, by Clause (iii), $\mathcal{F}_\alpha := \text{CUB}(\alpha) \cap N_\alpha$ is a filter over α .

Claim 4.5.1. *Suppose $\langle Y_\iota \mid \iota < \kappa \rangle$ is a sequence of stationary subsets of κ . Then there exist stationarily many $\alpha \in S'$ such that, for all $\iota < \alpha$, $Y_\iota \cap \alpha \in \mathcal{F}_\alpha^+$.*

Proof. Set $A := \{(\iota, \beta) \mid \iota < \kappa \ \& \ \beta \in Y_\iota\}$. Clearly, $\langle \kappa, \in, A \rangle \models \Phi \wedge \Psi$. Thus, recalling Clause (3) of Definition 4.3, the set T of all $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, A \cap (\alpha \times \alpha) \rangle \models_{N_\alpha} \Phi \wedge \Psi$ is stationary. Evidently, every $\alpha \in T \setminus (\omega + 1)$ is an element of S' satisfying that, for all $\iota < \alpha$, $Y_\iota \cap \alpha \in \mathcal{F}_\alpha^+$. \square

It follows in particular that S' is stationary. Finally, recalling Clause (2) of Definition 4.3, $\vec{\mathcal{F}} := \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ captures clubs.

(2) Continuing the proof of Clause (1), we see that $\langle N_\alpha \cap \mathcal{P}(\alpha) \mid \alpha \in S' \rangle$ is a $\diamond_{S'}^*$ -sequence, and $\vec{\mathcal{F}}$ witnesses that $\mathfrak{f}\text{-Refl}(\kappa, \kappa, S')$ holds. The conclusion now follows from Lemma 3.10. \square

In [FMR19], we proved that $\text{DI}_S^*(\Pi_2^1)$ holds in \mathbb{L} for any stationary subset S of any regular uncountable cardinal κ . Therefore:

Corollary 4.6. *Suppose $\mathbb{V} = \mathbb{L}$. Then, for every stationary $S \subseteq \kappa$, κ \mathfrak{f} -reflects with \diamond to S . In particular, fake reflection holds at any non weakly compact cardinal. \square*

Furthermore, in [FMR19], we proved that $\text{DI}_S^*(\Pi_2^1)$ follows from a forceable condensation principle called ‘‘Local Club Condensation’’ (LCC). In effect, \mathfrak{f} -reflection is forceable (without assuming any large cardinals). In this short section, we shall present an alternative and simpler poset for forcing $\text{DI}_S^*(\Pi_2^1)$ to hold. The idea is to connect the latter with the following strong form of diamond due to Sakai.

Definition 4.7 (Sakai, [Sak11a]). \diamond^{++} asserts the existence of a sequence $\langle K_\alpha \mid \alpha < \omega_1 \rangle$ satisfying the following:

- (1) for every $\alpha < \omega_1$, K_α is a countable set;
- (2) for every $X \subseteq \omega_1$, there exists a club $C \subseteq \omega_1$ such that, for all $\alpha \in C$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;
- (3) the following set is stationary in $[H_{\omega_2}]^\omega$:

$$\{M \in [H_{\omega_2}]^\omega \mid M \cap \omega_1 \in \omega_1 \wedge \text{clps}(M, \in) = (K_{M \cap \omega_1}, \in)\}.$$

First, we generalize Sakai’s principle in the obvious way.

Definition 4.8. For a stationary $S \subseteq \kappa$, \diamond_S^{++} asserts the existence of a sequence $\langle K_\alpha \mid \alpha \in S \rangle$ satisfying the following:

- (1) for every infinite $\alpha \in S$, K_α is a set of size $|\alpha|$;
- (2) for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;

⁶Recall Convention 4.2.

(3) the following set is stationary in $[H_{\kappa^+}]^{<\kappa}$:

$$\{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \& \ \text{clps}(M, \in) = (K_{M \cap \kappa}, \in)\}.$$

Remark 4.9. For a structure \mathfrak{M} , $\text{clps}(\mathfrak{M})$ denotes its Mostowski collapse. Hereafter, ZF^- denotes ZF without the powerset axiom.

Lemma 4.10. *For every stationary $S \subseteq \kappa$, \diamond_S^{++} implies $\diamond_S^+(\Pi_2^1)$.*

Proof. Suppose $\langle K_\alpha \mid \alpha \in S \rangle$ is a \diamond_S^{++} -sequence. Define a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ by letting N_α be the p.r.-closure of $K_\alpha \cup (\alpha + 1)$. By the way the sequence \vec{N} was constructed, N_α sees α for all $\alpha \in S$, and by Clause (1) of Definition 4.8, for every infinite $\alpha \in S$, $|N_\alpha| = |\alpha|$. In addition, for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that $C \cap \alpha, X \cap \alpha \in K_\alpha \subseteq N_\alpha$ for all $\alpha \in C \cap S$.

Let us show that \vec{N} satisfies Clause (3) of Definition 4.3 with $n = 2$. To this end, let $\phi = \forall X \exists Y \varphi$ be a Π_2^1 -sentence and $(A_m)_{m \in \omega}$ be such that $\langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \phi$. Given an arbitrary club $C \subseteq \kappa$, we consider the following set

$$\mathcal{C} := \{M \prec H_{\kappa^+} \mid M \cap \kappa \in C \ \& \ (A_m)_{m \in \omega} \in M\}.$$

Claim 4.10.1. *\mathcal{C} is a club in $[H_{\kappa^+}]^{<\kappa}$.*

Proof. By the Löwenheim–Skolem theorem, for every $B \in [H_{\kappa^+}]^{<\kappa}$, we know that

$$\mathcal{M}_B = \{M \in [H_{\kappa^+}]^{<\kappa} \mid B \subseteq M \prec H_{\kappa^+} \ \& \ (A_m)_{m \in \omega} \in M\}$$

is a club in $[H_{\kappa^+}]^{<\kappa}$, so that $\kappa \cap \{M \cap \kappa \mid M \in \mathcal{M}_B\}$ is a club in κ . Consequently, \mathcal{C} is cofinal in $[H_{\kappa^+}]^{<\kappa}$.

To see that \mathcal{C} is closed, assume we are given a chain $M_0 \subseteq M_1 \subseteq \dots$ of length $\alpha < \kappa$ of elements of \mathcal{C} . As this is a chain of elementary submodels of H_{κ^+} of size smaller than κ , $M^* := \bigcup_{i < \alpha} M_i$ is an elementary submodel of H_{κ^+} with $M^* \cap \kappa \in \mathcal{C}$, so that $M^* \in \mathcal{C}$. \square

Since \vec{K} a \diamond_S^{++} -sequence, we may now pick M in the following intersection

$$\mathcal{C} \cap \{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \& \ \text{clps}(M, \in) = (K_{M \cap \kappa}, \in)\}.$$

So, $M \prec H_{\kappa^+}$, $(A_m)_{m \in \omega} \in M$, $\alpha := M \cap \kappa$ is in $S \cap C$, and $\text{clps}(M, \in) = (K_\alpha, \in)$.

As $M \cap (\kappa + 1) = \alpha \cup \{\kappa\}$, $\alpha \cup \{\alpha\}$ is a subset of the collapse of M , so that K_α sees α and $N_\alpha = K_\alpha$. Let $\pi : M \rightarrow N_\alpha$ denote the transitive collapsing map. Note that

- (i) $\pi \upharpoonright \alpha = \text{id}_\alpha$,
- (ii) $\pi(\kappa) = \alpha$, and
- (iii) $\pi(A_m) = A_m \cap \alpha$, for all $m \in \omega$.

Since $\langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \phi$, by definition, $\langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models_{H_{\kappa^+}} \forall X \exists Y \varphi$. That is,

$$H_{\kappa^+} \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})} \exists Y \subseteq \kappa^{m(\mathbb{Y})} \langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \varphi\text{”}.$$

By elementarity and the fact that “ $\forall X \subseteq \kappa^{m(\mathbb{X})} \exists Y \subseteq \kappa^{m(\mathbb{Y})} (\langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \varphi)$ ” is equivalent to

$$\begin{aligned} \forall X ((\forall x (x \in X \rightarrow x \in \kappa^{m(\mathbb{X})})) \rightarrow \\ (\exists Y ((\forall y (y \in Y \rightarrow y \in \kappa^{m(\mathbb{Y})})) \wedge \langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \varphi))), \end{aligned}$$

which is a first-order formula in the parameters $m(\mathbb{X})$, $m(\mathbb{Y})$, κ , $\langle \kappa, \in, \vec{A} \rangle$ and φ , we have

$$M \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})} \exists Y \subseteq \kappa^{m(\mathbb{Y})} (\langle \kappa, \in, (A_m)_{m \in \omega} \rangle \models \varphi)\text{”}.$$

Since π is an elementary embedding,

$$\pi[M] \models \text{“}\forall X \subseteq \pi(\kappa^{m(\mathbb{X})}) \exists Y \subseteq \pi(\kappa^{m(\mathbb{Y})}) (\langle \pi(\kappa), \in, (\pi(A_m))_{m \in \omega} \rangle \models \varphi)\text{”}.$$

By the properties (i),(ii) and (iii) of π it follows that

$$N_\alpha \models \text{“}\forall X \subseteq \alpha^{m(\mathbb{X})} \exists Y \subseteq \alpha^{m(\mathbb{Y})} (\langle \alpha, \in, (A_m \cap (\alpha^{m(A_m)}))_{m \in \omega} \rangle \models \varphi)\text{”}.$$

We conclude $\langle \alpha, \in, (A_n \cap (\alpha^{m(A_n)}))_{n \in \omega} \rangle \models_{N_\alpha} \phi$, as sought. \square

Remark 4.11. An obvious tweaking of the above proof shows that $\langle N_\alpha \mid \alpha \in S \rangle$ in fact witnesses $\diamond_S^+(\Pi_n^1)$ for every positive integer n .

The following answers a question of Thilo Weinert:

Corollary 4.12. *It is consistent that \diamond_S^+ holds, but \diamond_S^{++} fails.*

Proof. In [FMR19, §4], we identified a model in which \diamond_S^+ holds for $S := \omega_2 \cap \text{cof}(\omega)$ but $\text{DI}_S^*(\Pi_2^1)$ fails. By Lemma 4.10, \diamond_S^{++} fails in this model, as well. \square

In [Sak11a, Definition 3.1], Sakai presented a poset for forcing \diamond^{++} to hold. The following is an obvious generalization (and a minor simplification) of Sakai’s poset.

Definition 4.13. Let \mathbb{S} be the poset of all pairs (k, \mathcal{B}) with the following properties:

- (1) k is a function such that $\text{dom}(k) < \kappa$;
- (2) for each $\alpha \in \text{dom}(k)$, $k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \upharpoonright \alpha \in k(\alpha)$;
- (3) \mathcal{B} is a subset of $\mathcal{P}(\kappa)$ of size $\leq \text{dom}(k)$;

$(k', \mathcal{B}') \leq (k, \mathcal{B})$ in \mathbb{P} if the following holds:

- (i) $k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;
- (ii) for any $B \in \mathcal{B}$ and any $\alpha \in \text{dom}(k') \setminus \text{dom}(k)$, $B \cap \alpha \in k'(\alpha)$.

It is clear that \mathbb{S} is $< \kappa$ -closed. Also, since we assume $\kappa^{< \kappa} = \kappa$, \mathbb{S} has the κ^+ -cc. Finally, Sakai’s proof of [Sak11a, Lemma 3.4] makes clear that the following holds.

Proposition 4.14. *For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models \diamond_S^{++}$.* \square

Note that while Sakai’s forcing is considerably simpler than the poset to force LCC to hold, it only gives κ \mathfrak{f} -reflects to S for stationary subsets $S \subseteq \kappa$ from the ground model, whereas, LCC gives κ \mathfrak{f} -reflects to S for any stationary subset $S \subseteq \kappa$.

Remark 4.15. For stationary subsets X, S of κ , if X \mathfrak{f} -reflects to S , then for every notion of forcing \mathbb{P} of size $< \kappa$, $V^{\mathbb{P}} \models X$ \mathfrak{f} -reflects to S . It takes a little more effort, but it can be shown that if X \mathfrak{f} -reflects with \diamond to S , then for every notion of forcing \mathbb{P} of size $< \kappa$, $V^{\mathbb{P}} \models X$ \mathfrak{f} -reflects with \diamond to S .

In this section and in the previous one, we have collected a long list of sufficient conditions for filter reflection to hold. We have seen it is compatible with large cardinals, strong forcing axioms, but also with inner models like \mathbb{L} , in which anti-reflection principles like \square_λ hold. This suggests it is not trivial to destroy filter reflection. The next section is dedicated to demonstrating it is nevertheless possible.

5. KILLING FAKE REFLECTION

Definition 5.1. Let $X \subseteq \kappa$. We define a collection $I[\kappa - X]$, as follows.

A set Y is in $I[\kappa - X]$ iff $Y \subseteq \kappa$ and there exists a sequence $\langle a_\beta \mid \beta < \kappa \rangle$ of elements of $[\kappa]^{< \kappa}$ along with a club $C \subseteq \kappa$ such that, for every $\delta \in Y \cap C$, there is a cofinal subset $A \subseteq \delta$ of order-type $\text{cf}(\delta)$ such that

- (1) $\{A \cap \gamma \mid \gamma < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}$, and
- (2) $\text{acc}^+(A) \cap X = \emptyset$.

Remark 5.2. Note that $I[\kappa - X]$ is an ideal, and that $X \subseteq X'$ entails $I[\kappa - X] \supseteq I[\kappa - X']$. Shelah’s *approachability ideal* $I[\kappa]$ is equal to $I[\kappa - \emptyset] \upharpoonright \text{Sing}$ (cf. [Eis10]). In particular, for every $\mu \in \text{Reg}(\kappa)$, $I[\kappa] \upharpoonright \text{cof}(\mu)$ coincides with $I[\kappa - \emptyset] \upharpoonright \text{cof}(\mu)$.

Fact 5.3 (folklore). *Every separative $<\kappa$ -closed notion of forcing of size κ is forcing equivalent to $\text{Add}(\kappa, 1)$.*

Theorem 5.4. *Suppose X, S are disjoint stationary subsets of κ , with $S \in I[\kappa - X]$. For every $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$, $V^{\text{Add}(\kappa, 1)} \models X$ does not $\vec{\mathcal{F}}$ -reflect to S .*

Proof. Towards a contradiction, suppose that $\vec{\mathcal{F}}$ is a counterexample. As $\text{Add}(\kappa, 1)$ is almost homogeneous and $X, S, \vec{\mathcal{F}}$ live in the ground model, it follows that, in fact, $V^{\text{Add}(\kappa, 1)} \models X$ $\vec{\mathcal{F}}$ -reflects to S .

Let R denote the set of all pairs $(p, q) \in 2^{<\kappa} \times 2^{<\kappa}$ such that:

- $\text{dom}(p) = \text{dom}(q)$ is in $\text{nacc}(\kappa)$;
- $\{\alpha \in \text{dom}(p) \mid p(\alpha) = q(\alpha) = 1\}$ is disjoint from X ;
- $\{\alpha \in \text{dom}(q) \mid q(\alpha) = 1\}$ is a closed set of ordinals.

We let $\mathbb{R} := (R, \leq)$ where $(p', q') \leq (p, q)$ iff $p' \supseteq p$ and $q' \supseteq q$.

Claim 5.4.1. \mathbb{R} is $<\kappa$ -closed.

Proof. Given $\theta \in \text{acc}(\kappa)$ and a strictly decreasing sequence $\langle (p_i, q_i) \mid i < \theta \rangle$ of conditions in \mathbb{R} , let $p := (\bigcup_{i < \theta} p_i) \frown 0$ and $q := (\bigcup_{i < \theta} q_i) \frown 1$. Clearly, (p, q) is a legitimate condition extending (p_i, q_i) for all $i < \theta$. \square

It thus follows from Fact 5.3 that \mathbb{R} is forcing equivalent to $\text{Add}(\kappa, 1)$. Also, let $P := \{p \mid \exists q (p, q) \in R\}$. It is easy to see that $\mathbb{P} := (P, \supseteq)$ is $<\kappa$ -closed, so that \mathbb{P} is, as well, forcing equivalent to $\text{Add}(\kappa, 1)$. Next, let G be \mathbb{R} -generic over V . Let G_0 denote the projection of G to the first coordinate, so that G_0 is \mathbb{P} -generic over V . In $V[G_0]$, let $Q := \{q \in 2^{<\kappa} \mid \exists p \in G_0 (p, q) \in R\}$. Clearly, $\mathbb{Q} := (Q, \supseteq)$ is isomorphic to the quotient forcing \mathbb{R}/G_0 . It follows that, in $V[G]$, we may read a \mathbb{Q} -generic set G_1 over $V[G_0]$ such that, in particular, $V[G] = V[G_0][G_1]$.

Denote $\eta := \bigcup G_0$ and let $Y := \{\alpha \in X \mid \eta(\alpha) = 1\}$.

Claim 5.4.2. In $V[G_0]$, Y is stationary.

Proof. We run a density argument for \mathbb{P} in V . Let \dot{Y} be the \mathbb{P} -name for Y , that is,

$$\dot{Y} := \{(\check{\alpha}, p) \mid p \in P, \alpha \in X \cap \text{dom}(p), p(\alpha) = 1\}.$$

Let p be an arbitrary condition that \mathbb{P} -forces that some \dot{D} is a \mathbb{P} -name for a club in κ ; we shall find $p^\bullet \supseteq p$ such that $p^\bullet \Vdash_{\mathbb{P}} \dot{D} \cap \dot{Y} \neq \emptyset$.

Recursively define a sequence $\langle (p_i, \alpha_i) \mid i < \kappa \rangle$ as follows:

- Let (p_0, α_0) be such that $p_0 \supseteq p$ and $p_0 \Vdash_{\mathbb{P}} \check{\alpha}_0 \in \dot{D}$.
- Suppose that $i < \kappa$ for which $\langle (p_j, \alpha_j) \mid j \leq i \rangle$ has already been defined. Set $\varepsilon_i := \max\{\alpha_i, \text{dom}(p_i)\} + 1$. Then pick $p_{i+1} \supseteq p_i$ and $\alpha_{i+1} < \kappa$ such that $\varepsilon_i \in \text{dom}(p_{i+1})$ and $p_{i+1} \Vdash_{\mathbb{P}} \check{\alpha}_{i+1} \in \dot{D} \setminus \check{\varepsilon}_i$.
- Suppose that $i \in \text{acc}(\kappa)$ and that $\langle (p_j, \alpha_j) \mid j < i \rangle$ has already been defined. Evidently,

$$\sup_{j < i} \varepsilon_j = \sup_{j < i} (\text{dom}(p_j)) = \sup_{j < i} \alpha_j,$$

so we let α_i denote the above common value.

Finally, set $p_i := (\bigcup_{j < i} p_j) \frown 1$, so that p_i is a legitimate condition satisfying $\text{dom}(p_i) = \alpha_i + 1$ and $p_i(\alpha_i) = 1$.

This completes the recursive construction. Evidently, $E := \{\alpha_i \mid i < \kappa\}$ is a club, so as X is stationary, we may pick $\beta \in X$ such that $\alpha_\beta = \beta$. Then $p_\beta \Vdash_{\mathbb{P}} \check{\beta} \in \dot{D} \cap \dot{X}$, so that, from $p_\beta(\beta) = 1$, we infer that $p_\beta \Vdash_{\mathbb{P}} \dot{D} \cap \dot{Y} \neq \emptyset$. \square

Work in $V[G_0]$. Since X $\vec{\mathcal{F}}$ -reflects to S , $T := \{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Claim 5.4.3. In $V[G_0][G_1]$, T is stationary.

Proof. Fix \vec{a}, C in V that witness together that S is in $I[\kappa - X]$. As \mathbb{P} is cofinality-preserving, in $V[G_0]$, the above two still witness together that S is in $I[\kappa - X]$. Work in $V[G_0]$. As T is a subset of S , \vec{a}, C also witness together that T is in $I[\kappa - X]$.

We now run a density argument for \mathbb{Q} in $V[G_0]$. Let q be an arbitrary condition that \mathbb{Q} -forces that some \dot{D} is a \mathbb{Q} -name for a club in κ ; we shall find $q^\bullet \supseteq q$ such that $q^\bullet \Vdash_{\mathbb{Q}} \dot{D} \cap \check{T} \neq \emptyset$.

Fix a large enough regular cardinal Θ and some well-ordering $<_\Theta$ of H_Θ . By Claim 5.4.2, T is stationary, so we may find an elementary submodel $N \prec (H_\Theta, <_\Theta)$ such that $\vec{a}, C, \mathbb{Q}, q, \dot{D} \in N$ and $\delta := N \cap \kappa$ is in T .

As $C \in N$, we altogether have $\delta \in C \cap T$. Thus, we may pick a cofinal subset $A \subseteq \delta$ with $\text{otp}(A) = \text{cf}(\delta)$ and $\text{acc}^+(A) \cap X = \emptyset$ such that:

$$\{A \cap \gamma \mid \gamma < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}.$$

In particular, any proper initial segment of A is in N .

Let $\langle \delta_i \mid i < \text{cf}(\delta) \rangle$ be the increasing enumeration of A . For every initial segment a of A , we recursively define the following sequence $\langle (q_i, \alpha_i) \mid i \leq \sigma(a) \rangle$, where $\sigma(a)$ will be the length of the recursion (see the second case below).

- Let q_0 be the $<_\Theta$ -least condition in \mathbb{Q} extending q for which there is $\alpha < \kappa$ such that $q_0 \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D}$. Now, let α_0 be the $<_\Theta$ -least ordinal α such that $q_0 \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D}$.
- Suppose that $\langle (q_j, \alpha_j) \mid j \leq i \rangle$ has already been defined. If $a \setminus \max\{\alpha_i, \text{dom}(q_i), \delta_i\}$ is empty, then we terminate the recursion, and set $\sigma(a) := i$. Otherwise, let ε_i be the $<_\Theta$ -least element of $a \setminus \max\{\alpha_i, \text{dom}(q_i), \delta_i\}$, and then let q_{i+1} be the $<_\Theta$ -least condition in \mathbb{Q} extending q_i satisfying $\varepsilon_i \in \text{dom}(q_{i+1})$ and satisfying that there is $\alpha < \kappa$ such that $q_{i+1} \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D} \setminus \varepsilon_i$. Now, let α_{i+1} be the $<_\Theta$ -least ordinal α such that $q_{i+1} \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D} \setminus \varepsilon_i$.
- Suppose that i is a limit ordinal and that $\langle (q_j, \alpha_j) \mid j < i \rangle$ has already been defined. Evidently,

$$\sup_{j < i} \varepsilon_j = \sup_{j < i} (\text{dom}(q_j)) = \sup_{j < i} \alpha_j,$$

so we let α_i denote the above common value. As $\{\varepsilon_j \mid j < i\} \subseteq a \subseteq A$ and as $\text{acc}^+(A) \cap X = \emptyset$, we infer that $\alpha_i \notin X$. So, $q_i := (\bigcup_{j < i} q_j)^\frown 1$ is a legitimate condition satisfying $\text{dom}(q_i) = \alpha_i + 1$ and $q_i(\alpha_i) = 1$.

This completes the recursive construction. Since every proper initial segment of A is in N , for every $\gamma < \text{cf}(\delta)$, $\langle (q_i, \alpha_i) \mid i \leq \sigma(A \cap \gamma) \rangle$ is in N , so that $\sigma(A) = \text{cf}(\delta)$ and $\alpha_{\text{cf}(\delta)} = \delta$. Altogether, $q_{\text{cf}(\delta)} \Vdash_{\mathbb{Q}} \check{\delta} \in \dot{D}$. Recalling that $\delta \in T$, our proof is complete. \square

Claim 5.4.4. *In $V[G_0][G_1]$, Y is nonstationary.*

Proof. It is clear that $C := \{\alpha < \kappa \mid \exists q \in G_1 (q(\alpha) = 1)\}$ is a closed subset of κ which is disjoint from Y . Thus, we are left with proving that C is unbounded in κ . To this end, we run a density argument for \mathbb{Q} in $V[G_0]$. For every condition q in \mathbb{Q} , find $\delta \in S$ above $\text{dom}(q)$, and then define $q^\bullet : \delta + 1 \rightarrow 2$ via:

$$q^\bullet(\alpha) := \begin{cases} q(\alpha), & \text{if } \alpha \in \text{dom}(q); \\ 1, & \text{if } \alpha = \delta; \\ 0, & \text{otherwise.} \end{cases}$$

As $X \cap S = \emptyset$, q^\bullet is a legitimate condition extending q , and, in addition, $\{\alpha < \kappa \mid q^\bullet(\alpha) = 1\}$ is a proper end-extension of $\{\alpha < \kappa \mid q(\alpha) = 1\}$. \square

Work in $V[G_0][G_1]$. Fix a club C disjoint from Y . Since X $\vec{\mathcal{F}}$ -reflects to S , in particular, $\vec{\mathcal{F}}$ capture clubs, so that $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary. Recalling that T is stationary, we now fix $\alpha \in T$ such that $C \cap \alpha \in \mathcal{F}_\alpha$. By definition of T , we also have $Y \cap \alpha \in \mathcal{F}_\alpha^+$, so that $(C \cap \alpha) \cap (Y \cap \alpha)$ is nonempty, contradicting the fact that C is disjoint from Y . \square

Corollary 5.5. *Suppose X, S are disjoint stationary subsets of κ , with $S \in I[\kappa - X]$. After forcing with $\text{Add}(\kappa, \kappa^+)$, X does not \mathfrak{f} -reflect to S .*

Proof. Let G be $\text{Add}(\kappa, \kappa^+)$ -generic over V . For every $\iota \leq \kappa^+$, let G_ι denote the projection of G into the ι^{th} stage.

Work in $V[G_{\kappa^+}]$. Towards a contradiction, suppose that S' is a stationary subset of S , and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ is a sequence such that X $\vec{\mathcal{F}}$ -reflects to S' . As $\text{Add}(\kappa, \kappa^+)$ does not add bounded subsets of κ , $\vec{\mathcal{F}} \subseteq (H_\kappa)^V$. In addition, $\text{Add}(\kappa, \kappa^+)$ has the κ^+ -cc, so that, altogether, $\vec{\mathcal{F}}$ admits a nice name of size κ . It follows that we may find a large enough $\iota < \kappa^+$ such that $\vec{\mathcal{F}}$ is in $V[G_\iota]$. Now, by Theorem 5.4, $V[G_{\iota+1}] \models$ “ X does not $\vec{\mathcal{F}}$ -reflect to S' ”. Recalling that $V[G_{\kappa^+}] \models$ “ X $\vec{\mathcal{F}}$ -reflect to S' ”, it must be the case that there exists a stationary subset of κ in $V[G_{\iota+1}]$ that ceases to be stationary in $V[G_{\kappa^+}]$. However, the quotient forcing $\text{Add}(\kappa, \kappa^+)/G_{\iota+1}$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$ and the latter preserves stationary subsets of κ . This is a contradiction. \square

Lemma 5.6. *Suppose that κ is strongly inaccessible or $\kappa = \lambda^+$ with $\lambda^{<\lambda} = \lambda$. For every stationary $X, Y \subseteq \kappa$ such that $\text{Tr}(X) \cap Y$ is nonstationary, $Y \in I[\kappa - X]$.*

Proof. Let $\theta := \text{sup}(\text{Reg}(\kappa))$, so that $\kappa \in \{\theta, \theta^+\}$, and, for every $\gamma < \kappa$, $|\gamma|^{<\theta} < \kappa$. Let $\langle a_\beta \mid \beta < \kappa \rangle$ be some enumeration of $[\kappa]^{<\theta}$, and then define a function $f : \kappa \rightarrow \kappa$ via:

$$f(\gamma) := \min\{\tau < \kappa \mid [\gamma]^{<\theta} \subseteq \{a_\beta \mid \beta < \tau\}\}.$$

Now, fix arbitrary $X, Y \subseteq \kappa$ for which $\text{Tr}(X) \cap Y$ is nonstationary. To see that $Y \in I[\kappa - X]$, fix a subclub C of $\{\delta < \kappa \mid f[\delta] \subseteq \delta\}$ disjoint from $\text{Tr}(X) \cap Y$. Let $\delta \in Y \cap C$ be arbitrary. Fix a club A in δ of order-type $\text{cf}(\delta)$ such that $A \cap X = \emptyset$. By definition of θ , we have $\text{cf}(\delta) \leq \theta$, so that, for every $\gamma < \delta$, we have $\text{otp}(A \cap \gamma) < \theta$ and $f(\gamma) < \delta$, and hence there must exist some $\beta < \delta$ with $A \cap \gamma = a_\beta$. \square

We are now ready to derive Theorem C:

Corollary 5.7. *If κ is strongly inaccessible, then in the forcing extension by $\text{Add}(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, S of κ , the following are equivalent:*

- (1) X \mathfrak{f} -reflects to S ;
- (2) every stationary subset of X reflects in S .

Proof. The implication (2) \implies (1) holds true in any model, since if X reflects in S , then $S' := S \setminus \text{cof}(\omega)$ must be stationary, and $X \langle \text{CUB}(\alpha) \mid \alpha \in S' \rangle$ -reflects to S' . In particular, X \mathfrak{f} -reflects to S . Thus, we shall focus on the other implication.

Let G be $\text{Add}(\kappa, \kappa^+)$ -generic over V . For every $\iota \leq \kappa^+$, we let G_ι denote the projection of G into the ι^{th} stage. Work in $V[G]$. We verify that $\neg(2) \implies \neg(1)$.

Suppose that X and S are disjoint stationary subsets of κ such that X admits a stationary subset $Z \subseteq X$ that does not reflect in S . As Z and S are elements of H_{κ^+} and as $\text{Add}(\kappa, \kappa^+)$ has the κ^+ -cc, we may find a large enough $\iota < \kappa^+$ such that Z and S are in $V[G_\iota]$. As the quotient forcing $\text{Add}(\kappa, \kappa^+)/G_\iota$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$ and the latter does not add bounded subsets of κ and does preserve stationary subsets of κ , also, in $V[G_\iota]$, Z does not reflect in S . Now, by Lemma 5.6, $S \in I[\kappa - Z]$. So, since $\text{Add}(\kappa, \kappa^+)/G_\iota$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$, Corollary 5.5 implies

that Z does not \mathfrak{f} -reflect to S in $V[G]$. But $Z \subseteq X$, contradicting Monotonicity Lemma 2.4. \square

Definition 5.8. Let $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ be some sequence, with $\Gamma \subseteq \text{OR}$.

- \vec{C} is said to be a *C-sequence over Γ* iff, for every $\alpha \in \Gamma$, C_α is a closed subset of α with $\text{sup}(C_\alpha) = \text{sup}(\alpha)$;
- \vec{C} is said to be *coherent* iff, for all $\alpha \in \Gamma$ and $\bar{\alpha} \in \text{acc}(C_\alpha)$, $\bar{\alpha} \in \Gamma$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- \vec{C} is said to be *regressive* iff $\text{otp}(C_\alpha) < \alpha$ for all $\alpha \in \Gamma$.

- Remark 5.9.*
- (1) Jensen proved [Jen72] that if $V = L$, then there exists a coherent regressive C -sequence over Sing (the class of singular ordinals).
 - (2) Jensen's principle \square_λ is equivalent to the assertion that there exists a coherent regressive C -sequence over a club in λ^+ .
 - (3) By [She91], for every regular uncountable cardinal λ , there exists a sequence $\langle \Gamma_i \mid i < \lambda \rangle$ such that $\bigcup_{i < \lambda} \Gamma_i = \text{acc}(\lambda^+ \setminus \lambda) \cap \text{cof}(< \lambda)$ and, for all $i < \lambda$, there exists a coherent regressive C -sequence over Γ_i .
 - (4) By [Sak11b], MM implies the existence of a coherent regressive C -sequence over some $\Gamma \subseteq \omega_2$ for which $\Gamma \cap \text{cof}(\omega_1)$ is stationary.

Lemma 5.10. *Let $\Gamma \subseteq \text{Sing}(\kappa)$ be stationary. If there exists a coherent regressive C -sequence over Γ , then, for every stationary $X \subseteq \Gamma$, there exists a stationary $Z \subseteq X$ with $\Gamma \in I[\kappa - Z]$.*

Proof. Suppose that $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ is a coherent regressive C -sequence. For every $\epsilon < \kappa$, let $\Gamma_\epsilon := \{\alpha \in \Gamma \mid \text{otp}(C_\alpha) = \epsilon\}$. By Fodor's lemma, for every stationary $X \subseteq \Gamma$, there must exist some $\epsilon < \kappa$ such that $X \cap \Gamma_\epsilon$ is stationary. Thus, we shall focus on proving that $\Gamma \in I[\kappa - \Gamma_\epsilon]$ for all $\epsilon < \kappa$.

For all $\eta \in \kappa \cap \text{cof}(> \omega)$ and $\epsilon < \kappa$, fix a subclub $d_{\eta, \epsilon}$ of $\text{acc}(\eta)$ of order-type $\text{cf}(\eta)$ such that $\epsilon \notin d_{\eta, \epsilon}$; then, for every $\alpha \in \Gamma$, let

$$C_\alpha^{\eta, \epsilon} := \{\zeta \in C_\alpha \mid \text{otp}(C_\alpha \cap \zeta) \in d_{\eta, \epsilon}\}.$$

To help the reader digest the above definition, we mention that each such a set $C_\alpha^{\eta, \epsilon}$ is a closed (possibly empty) subset of $\text{acc}(C_\alpha)$ of order-type $\leq \text{otp}(d_{\eta, \epsilon})$.

Next, let $\vec{a} = \langle a_\beta \mid \beta < \kappa \rangle$ be some enumeration of

$$[\kappa]^{< \omega} \cup \{C_\alpha^{\eta, \epsilon} \mid \alpha \in \Gamma, \eta \in \kappa \cap \text{cof}(> \omega), \epsilon < \kappa\}.$$

Fix a club D in κ such that, for all $\delta < \kappa$:

- $[\delta]^{< \omega} \subseteq \{a_\beta \mid \beta < \delta\}$, and
- $\{C_\alpha^{\eta, \epsilon} \mid \alpha \in \Gamma \cap \delta, \eta \in \delta \cap \text{cof}(> \omega), \epsilon < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}$.

Let $\epsilon < \kappa$. We claim that \vec{a} and $D \setminus (\epsilon + 1)$ witness together that $\Gamma \in I[\kappa - \Gamma_\epsilon]$. To this end, let $\delta \in \Gamma \cap D \setminus (\epsilon + 1)$ be arbitrary. There are two cases to consider:

► If $\text{cf}(\delta) = \omega$, then let A be an arbitrary cofinal subset of δ of order-type ω . Clearly, $\text{acc}^+(A) = \emptyset$. In addition, by $\delta \in D$, any proper initial segment of A is indeed listed in $\{a_\beta \mid \beta < \delta\}$.

► If $\text{cf}(\delta) > \omega$, then let $\eta := \text{otp}(C_\delta)$, so that $\eta < \delta$. As C_δ is a club in δ , $\text{cf}(\eta) = \text{cf}(\delta)$. As $d_{\eta, \epsilon}$ is a subclub of $\text{acc}(\eta)$ of order-type $\text{cf}(\eta) = \text{cf}(\delta)$, $A := C_\delta^{\eta, \epsilon}$ is a subclub of $\text{acc}(C_\delta)$ of order-type $\text{cf}(\delta)$. Now, if $\alpha \in A \cap \Gamma_\epsilon$, then $\alpha \in \text{acc}(C_\delta)$, so that $\alpha \in \Gamma$ and $C_\delta \cap \alpha = C_\alpha$, and also $\alpha \in \Gamma_\epsilon$ so that $\text{otp}(C_\delta \cap \alpha) = \text{otp}(C_\alpha) = \epsilon$. Recalling that $\alpha \in A = C_\delta^{\eta, \epsilon} = \{\zeta \in C_\delta \mid \text{otp}(C_\delta \cap \zeta) \in d_{\eta, \epsilon}\}$, this means that $\epsilon \in d_{\eta, \epsilon}$, contradicting the choice of $d_{\eta, \epsilon}$. It follows in particular that $\text{acc}^+(A) \cap \Gamma_\epsilon = \emptyset$.

Finally, let $\gamma < \delta$, and we shall show that $A \cap \gamma \in \{a_\beta \mid \beta < \delta\}$. Put $\alpha := \min(A \setminus \gamma)$, so that $A \cap \gamma = A \cap \alpha$. As observed earlier, the fact that $\alpha \in A$ entails $C_\alpha = C_\delta \cap \alpha$. But then $C_\alpha^{\eta, \epsilon} = C_\delta^{\eta, \epsilon} \cap \alpha = A \cap \gamma$. Recalling that $\delta \in D$ and

$\max\{\alpha, \eta, \epsilon\} < \delta$, we infer that the proper initial segment $A \cap \gamma$ is indeed listed in $\{a_\beta \mid \beta < \delta\}$. \square

Corollary 5.11. *Suppose that there exists a coherent regressive C -sequence over a stationary subset Γ of κ . After forcing with $\text{Add}(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, S of Γ , X does not \mathfrak{f} -reflect to S .*

Proof. Let G be $\text{Add}(\kappa, \kappa^+)$ -generic over V . As in the proof of Corollary 5.5, for every $\iota \leq \kappa^+$, we let G_ι denote the projection of G into the ι^{th} stage.

Work in $V[G]$. Suppose that X and S are disjoint stationary subsets of Γ . As X and S are elements of H_{κ^+} and as $\text{Add}(\kappa, \kappa^+)$ has the κ^+ -cc, we may find a large enough $\iota < \kappa^+$ such that X and S are in $V[G_\iota]$. Now, in V , and hence also in $V[G_\iota]$, there exists a coherent regressive C -sequence over Γ . So, by Lemma 5.10, there is a stationary $Z \subseteq X$ such that $Y \in I[\kappa - Z]$. Finally, since $\text{Add}(\kappa, \kappa^+)/G_\iota$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$, Corollary 5.5 implies that, in $V[G]$, Z does not \mathfrak{f} -reflect to S . As $Z \subseteq X$, it follows from Monotonicity Lemma 2.4 that, in $V[G]$, X does not \mathfrak{f} -reflect to S . \square

We can now derive Theorem B:

Corollary 5.12 (Dense non-reflection). *There exists a cofinality-preserving forcing extension in which:*

- (1) *For all stationary subsets X, S of κ , there exist stationary subsets $X' \subseteq X$ and $S' \subseteq S$ such that X' does not \mathfrak{f} -reflect to S' ;*
- (2) *There exists an injection $h : \mathcal{P}(\kappa) \rightarrow \text{NS}_\kappa^+$ such that, for all $X, S \in \mathcal{P}(\kappa)$, $X \subseteq S$ iff $h(X)$ \mathfrak{f} -reflect to $h(S)$;*
- (3) *For all two disjoint stationary subsets X, S of κ , X does not \mathfrak{f} -reflect to S .*

Proof. A moment reflection makes it clear that Clauses (1) and (2) both follow from Clause (3), so we focus on proving the latter.

Suppose first that κ is a successor cardinal, say, $\kappa = \lambda^+$. Let \mathbb{S} denote the standard cofinality-preserving notion of forcing for adding a \square_λ -sequence (see [CFM01, §6.1]). Now, work in $V[H][G]$, where $H * G$ is $\mathbb{S} * \text{Add}(\kappa, \kappa^+)$ -generic over V . As, in $V[H]$, there exists a coherent regressive C -sequence over a club in κ , Corollary 5.11 entails that, in $V[H][G]$, for all two disjoint stationary subsets X, S of κ , X does not \mathfrak{f} -reflect to S .

Next, suppose that κ is inaccessible. Let $\mathbb{S}_0 := (S_0, \leq_0)$, where $c \in S_0$ iff c is closed bounded subset of κ disjoint from $\text{Reg}(\kappa)$, and $d \leq_0 c$ iff d end-extends c . It is easy to see that \mathbb{S}_0 is $<\kappa$ -distributive,⁷ of size $\kappa^{<\kappa} = \kappa$, and shoots a club through $\text{Sing}(\kappa)$, so that, \mathbb{S}_0 is cofinality-preserving, and, in $V^{\mathbb{S}_0}$, κ is not Mahlo. Now, let $\dot{\mathbb{S}}_1$ be the \mathbb{S}_0 -name for the poset from [CS02, §6] for adding a coherent regressive C -sequence over $\text{Sing}(\kappa)$. Then $\mathbb{S} := \mathbb{S}_0 * \dot{\mathbb{S}}_1$ is a cofinality-preserving notion of forcing that adds a coherent regressive C -sequence over a club in κ . So, as in the previous case, any forcing extension by $\mathbb{S} * \text{Add}(\kappa, \kappa^+)$ gives the desired model. \square

Lemma 5.13. *Let $X \subseteq \kappa$ be stationary, and $\mu \in \text{Reg}(\kappa)$.*

- (1) $\kappa \cap \text{cof}(\omega) \in I[\kappa - X]$;
- (2) *If $X \subseteq \text{cof}(\geq \mu)$, then $I[\kappa - \emptyset] \upharpoonright \text{cof}(\leq \mu) = I[\kappa - X] \upharpoonright \text{cof}(\leq \mu)$;*
- (3) *If $\mu^{<\mu} < \kappa$, then $S \in I[\kappa - X] \upharpoonright \text{cof}(\leq \mu)$ iff $S \in I[\kappa - \emptyset] \upharpoonright \text{cof}(\leq \mu)$ and $\text{Tr}(X) \cap S$ is nonstationary.*
- (4) *If $2^{<\lambda} = \lambda$ and X is non-reflecting, then $I[\lambda^+ - \emptyset] = I[\lambda^+ - X]$.*

⁷For every condition c , and every $\theta \in \text{Reg}(\kappa)$, $d := c \cup \{\text{sup}(c) + \theta\}$ is a condition extending c such that $\mathbb{S}_0 \downarrow d$ is θ^+ -closed.

Proof. (1) Let $\langle a_\beta \mid \beta < \kappa \rangle$ be any enumeration of all finite subsets of κ , and consider the club $C := \{\delta < \kappa \mid [\delta]^{<\omega} = \{a_\beta \mid \beta < \delta\}\}$.

(2) The witness is the same.

(3) The forward implication is clear, so we focus on the converse. Suppose $S \in I[\kappa - \emptyset] \upharpoonright \text{cof}(\leq \mu)$ and $\text{Tr}(X) \cap S$ is nonstationary. Fix a list $\langle a_\beta \mid \beta < \kappa \rangle$ and a club C witnessing together that $S \in I[\kappa - \emptyset]$. By shrinking C , we may assume that $\text{Tr}(X) \cap S \cap C = \emptyset$. As $S \subseteq \text{cof}(\leq \mu)$, we may also assume that $|a_\beta| < \mu$ for all $\beta < \kappa$. Thus, assuming $\mu^{<\mu} < \kappa$, we may let $\langle a_\beta^\bullet \mid \beta < \kappa \rangle$ be some enumeration of $\bigcup \{\mathcal{P}(\text{cl}(a_\beta)) \mid \beta < \kappa\}$. Now, define a function $f : \kappa \rightarrow \kappa$ by letting for all $\beta < \kappa$:

$$f(\beta) := \min\{\alpha < \kappa \mid \mathcal{P}(\text{cl}(a_\beta)) \subseteq \{a_\gamma^\bullet \mid \gamma < \alpha\}\}.$$

Put $D := \{\delta \in C \mid f[\delta] \subseteq \delta\}$. To see that $\langle a_\beta^\bullet \mid \beta < \kappa \rangle$ and D witness together that $S \in I[\kappa - X]$, let $\delta \in S \cap D$ be arbitrary. In particular, $\delta \in S \cap C$, and we may fix a cofinal subset $A \subseteq \delta$ of order-type $\text{cf}(\delta)$ such that any proper initial segment of A is listed in $\{a_\beta \mid \beta < \delta\}$. As $\delta \in S \cap C$, $\delta \notin \text{Tr}(X)$, so we may fix a subclub A^\bullet of $\text{acc}^+(A)$ which is disjoint from X . Now, let $\gamma < \delta$ be arbitrary. Put $\alpha := \min(A \setminus \gamma)$ and then find $\beta < \delta$ such that $A \cap \alpha = a_\beta$. It follows that $A^\bullet \cap \gamma \subseteq \text{cl}(a_\beta)$. So, as $f[\delta] \subseteq \delta$, we infer that $A^\bullet \cap \gamma$ is indeed listed in $\{a_\beta^\bullet \mid \beta < \delta\}$.

(4) This follows from (3). \square

Corollary 5.14. *Suppose $X \subseteq \kappa$ and $S \subseteq \kappa \cap \text{cof}(\omega)$ are stationary sets, for which $X \setminus S$ is stationary. After forcing with $\text{Add}(\kappa, \kappa^+)$, X does not \mathfrak{f} -reflect to S .*

Proof. Consider the stationary set $Z := X \setminus S$. By Lemma 5.13(1), $\kappa \cap \text{cof}(\omega) \in I[\kappa - Z]$, in particular, $S \in I[\kappa - Z]$. By Corollary 5.5, we conclude that after forcing with $\text{Add}(\kappa, \kappa^+)$, Z does not \mathfrak{f} -reflect to S . Now, Monotonicity Lemma 2.4 finishes the proof. \square

Thus, we get a slight improvement of Corollary 4.12:

Corollary 5.15. *It is consistent that for $S := \aleph_2 \cap \text{cof}(\omega)$ and $X := \aleph_2 \cap \text{cof}(\omega_1)$, \diamond_S^+ holds, but X does not \mathfrak{f} -reflect to S .*

Proof. Suppose $\kappa = \aleph_2 = 2^{2^{\aleph_0}}$, and work in the forcing extension by $\text{Add}(\kappa, \kappa^+)$. By [Gre76, Lemma 2.1], \diamond_S^+ holds. By Corollary 5.14, X does not \mathfrak{f} -reflect to S . \square

6. DENSE NON-REDUCTION

In the previous section, we showed how adding κ^+ many Cohen subsets of κ could ensure the failure of instances of \mathfrak{f} -refl at the level of κ . In this section, motivated by Lemma 2.8, we shall derive a stronger conclusion.

This section builds heavily on the ideas of Chapter 4, Section 4 of [FHK14], where, given X, Y stationary subsets of κ , it is forced under certain hypothesis that $=_X^2 \not\dot{\simeq}_B =_Y^2$. Here we adapt their arguments to get $=_X^2 \not\dot{\simeq}_{BM} =_Y^2$, let alone $=_X \not\dot{\simeq}_B =_Y$ or $=_X^2 \not\dot{\simeq}_B =_Y^2$. In addition, our proof takes advantage of the ideal $I[\kappa - X]$ from the previous section, hence, the findings here are applicable also for κ successor of singular in which Y concentrates on points of cofinality above the cofinality of the singular.

Convention 6.1. We denote elements of $\kappa^{<\kappa}$ by English letters (e.g., p and q), and elements of κ^κ by Greek letters (e.g., η and ξ). Subsets of κ will be denoted by X, Y, Z and S . For all $\eta \in \kappa^\kappa$ and $A \subseteq \kappa$, we denote $A_\eta := \{\alpha \in A \mid \eta(\alpha) \neq 0\}$. We also let $\vec{0}$ denote the constant κ -sequence with value 0.

Definition 6.2. A function $F : 2^{<\kappa} \times \kappa^{<\kappa} \rightarrow 2$ is said to *encode a map from 2^κ to κ^κ* iff for all $p \in 2^{<\kappa}$ and $q, q' \in \kappa^{<\kappa}$, $F(p, q) = F(p, q') = 1$ entails that $q \cup q' \in \kappa^{<\kappa}$.

The interpretation of F is the function $F^* : 2^\kappa \rightarrow \kappa^\kappa$ defined as follows. Given $\eta \in 2^\kappa$, if there exists $\xi \in \kappa^\kappa$ satisfying that, for all $\varepsilon < \kappa$, there is a tail of $\delta < \kappa$, such that $F(\eta \upharpoonright \delta, \xi \upharpoonright \varepsilon) = 1$, then ξ is unique and we let $F^*(\eta) := \xi$. Otherwise, we let $F^*(\eta) := \bar{0}$.

Definition 6.3. Let $\theta \in [2, \kappa]$. A basic open set in the space θ^κ is a set of the form $N_p := \{\eta \in \theta^\kappa \mid p \subseteq \eta\}$ for some $p \in \theta^{<\kappa}$. A binary relation $A \subseteq \theta^\kappa \times \theta^\kappa$ is said to be *analytic* iff there is a closed subset F of the product space $\theta^\kappa \times \theta^\kappa \times \theta^\kappa$ such that A is equal to the projection $\text{pr}(F) := \{(\eta, \xi) \in \theta^\kappa \times \theta^\kappa \mid \exists \zeta \in \theta^\kappa (\eta, \xi, \zeta) \in F\}$.

Definition 6.4. A subset $D \subseteq 2^\kappa$ is said to be *comeager* if $D \supseteq \bigcap \mathcal{D}$ for some nonempty family \mathcal{D} of at most κ -many dense open subsets of 2^κ .

A subset of 2^κ is said to be *meager* iff its complement is comeager.

Fact 6.5 (Generalized Baire Category Theorem [Vää11, Theorem 9.87]). *For any nonempty family \mathcal{D} of at most κ -many dense open subsets of 2^κ , $\bigcap \mathcal{D}$ is dense. In particular, the collection \mathcal{M}_κ of all meager subsets of 2^κ forms a κ^+ -additive proper ideal, and the intersection of a comeager set with a set of the form N_p (as in Definition 6.3) is nonempty.*

Definition 6.6. A function $H : \kappa \times \kappa \rightarrow 2^{<\kappa}$ is said to *encode a comeager set* iff, for every $i < \kappa$, the fiber $H[\{i\} \times \kappa]$ is cofinal in $(2^{<\kappa}, \subseteq)$.

The interpretation of H is the set:

$$H^* := \bigcap_{i < \kappa} \bigcup_{j < \kappa} N_{H(i,j)}.$$

The following is obvious.

Lemma 6.7. *For any comeager set $D \subseteq 2^\kappa$, there is a function $H : \kappa \times \kappa \rightarrow 2^{<\kappa}$ encoding a comeager subset H^* of D .* \square

Definition 6.8. A subset $B \subseteq 2^\kappa$ is said to have *the Baire property* iff there is an open set $U \subseteq 2^\kappa$ for which the symmetric difference $U \triangle B$ is meager.

The collection of all subsets of 2^κ having the Baire property is denoted by \mathcal{B}_κ .

Proposition 6.9. \mathcal{B}_κ is an algebra which is closed under unions of length $\leq \kappa$.

Proof. Given $B \in \mathcal{B}_\kappa$, to see that $B^c := 2^\kappa \setminus B$ is in \mathcal{B}_κ , fix an open set U such that $U \triangle B$ is meager, and let V denote the interior of $U^c := 2^\kappa \setminus U$; then $B^c \triangle V \subseteq (B^c \triangle U^c) \cup (U^c \setminus V) = (B \triangle U) \cup (U^c \setminus V)$ which is the union of two meager sets.

To see that \mathcal{B}_κ is closed under union of length $\leq \kappa$, note that since \mathcal{M}_κ forms a κ^+ -additive ideal, this follows from the fact that, for any sequence of pairs $\langle (B_i, U_i) \mid i < \kappa \rangle$, $\bigcup \{B_i \mid i < \kappa\} \triangle \bigcup \{U_i \mid i < \kappa\} \subseteq \bigcup \{B_i \triangle U_i \mid i < \kappa\}$. \square

Definition 6.10. A function $f : 2^\kappa \rightarrow \kappa^\kappa$ is said to be *Baire measurable* iff for any open set $U \subseteq 2^\kappa$, $f^{-1}[U]$ has the Baire property. The existence of a Baire measurable function reduction of R_0 to R_1 is denoted by $R_0 \hookrightarrow_{BM} R_1$.

Lemma 6.11. *Baire measurable functions are continuous on a comeager set.*

Proof. Let $f : 2^\kappa \rightarrow \kappa^\kappa$ be a Baire measurable function. For any $p \in \kappa^{<\kappa}$, choose an open set $U_p \subseteq 2^\kappa$ such that $U_p \triangle f^{-1}[N_p]$ is meager. As $\kappa^{<\kappa} = \kappa$, the following set is comeager:

$$D := \kappa^\kappa \setminus \bigcup_{p \in \kappa^{<\kappa}} (U_p \triangle f^{-1}[N_p]).$$

Now, to see that $f \upharpoonright D$ is continuous, it suffices to show that, for every $p \in \kappa^{<\kappa}$, $D \cap f^{-1}[N_p]$ is an open set in D . But this is clear, since $D \cap f^{-1}[N_p] = D \cap U_p$. \square

Definition 6.12. For stationary subsets X, Y of κ , we say that (F, H) is an (X, Y) -pair iff all of the following hold:

- (1) F encodes a map from 2^κ to κ^κ ;
- (2) H encodes a comeager set;
- (3) For every $\eta \in H^*$,

$$X_\eta \text{ is stationary} \iff Y_{F^*(\eta)} \text{ is stationary.}$$

Our next task is to show that if $=_X^2 \hookrightarrow_{BM} =_Y$, then there exists an (X, Y) -pair. For this, we first introduce the notion of a *positivity* of a reduction and prove a lemma about it.

Definition 6.13. For every function $f : 2^\kappa \rightarrow \kappa^\kappa$, we define the *positivity* of f , $f^+ : 2^\kappa \rightarrow \kappa^\kappa$, via:

$$f^+(\eta)(\alpha) := \begin{cases} 0, & \text{if } f(\eta)(\alpha) = f(\vec{0})(\alpha); \\ f(\eta)(\alpha), & \text{if } f(\eta)(\alpha) \neq f(\vec{0})(\alpha) \text{ and } f(\eta)(\alpha) \neq 0; \\ f(\vec{0})(\alpha), & \text{if } f(\eta)(\alpha) \neq f(\vec{0})(\alpha) \text{ and } f(\eta)(\alpha) = 0. \end{cases}$$

Lemma 6.14. *Suppose $f : 2^\kappa \rightarrow \kappa^\kappa$ is a Baire measurable function. Then so is f^+ .*

Proof. Since every open set is the union of at most $\kappa^{<\kappa} = \kappa$ many sets of the form N_p , and since \mathcal{B}_κ is closed under unions of length $\leq \kappa$, it suffices to verify that for every $p \in \kappa^{<\kappa}$, $(f^+)^{-1}[N_p]$ has the Baire property. Given $p \in \kappa^{<\kappa}$, if there are no $\eta \in 2^\kappa$ and $\beta < \kappa$ such that $p = f^+(\eta) \upharpoonright \beta$, then $(f^+)^{-1}[N_p]$ is empty, and we are done. Thus, it suffices to consider p 's of the form $f^+(\eta) \upharpoonright \beta$ for $\eta \in 2^\kappa$ and $\beta < \kappa$.

Let $\eta \in 2^\kappa$ and $\beta < \kappa$ be arbitrary. Denote $\xi := f^+(\eta)$. In order to compute $(f^+)^{-1}[N_{\xi \upharpoonright \beta}]$, we consider the following sets defined to handle each case in the definition of f^+ :

- a) $B_0 := \{\alpha < \beta \mid f(\eta)(\alpha) = f(\vec{0})(\alpha)\}$;
- b) $B_1 := \{\alpha < \beta \mid f(\eta)(\alpha) \neq f(\vec{0})(\alpha) \text{ and } f(\eta)(\alpha) \neq 0\}$;
- c) $B_2 := \{\alpha < \beta \mid f(\eta)(\alpha) \neq f(\vec{0})(\alpha) \text{ and } f(\eta)(\alpha) = 0\}$.

Next, consider the following sets:

- A) $Q_0 := \{q \in \kappa^\beta \mid q \upharpoonright B_0 = f(\vec{0}) \upharpoonright B_0\}$;
- B) $Q_1 := \{q \in \kappa^\beta \mid q \upharpoonright B_1 = f(\eta) \upharpoonright B_1\}$;
- C) $Q_2 := \{q \in \kappa^\beta \mid q \upharpoonright B_2 = f(\eta) \upharpoonright B_2\}$.

As f is Baire measurable, for each $i < 3$,

$$W_i := \bigcup \{f^{-1}[N_q] \mid q \in Q_i\}$$

is the union of $\leq \kappa^{<\kappa} = \kappa$ many sets, each having the Baire property. So, by Proposition 6.9, $W_0 \cap W_1 \cap W_2$ has the Baire property. Thus, the next claim finishes the proof.

Claim 6.14.1. $(f^+)^{-1}[N_{\xi \upharpoonright \beta}] = W_0 \cap W_1 \cap W_2$.

Proof. (\implies) Suppose $\zeta \in W_0 \cap W_1 \cap W_2$. Then $f(\zeta) \upharpoonright B_0 = f(\vec{0}) \upharpoonright B_0 = f(\eta) \upharpoonright B_0$, $f(\zeta) \upharpoonright B_1 = f(\eta) \upharpoonright B_1$ and $f(\zeta) \upharpoonright B_2 = f(\eta) \upharpoonright B_2$, which implies that $f^+(\zeta) \in N_{\xi \upharpoonright \beta}$.

(\impliedby) Suppose $\zeta \in 2^\kappa$ with $f^+(\zeta) \in N_{\xi \upharpoonright \beta}$. Then

$$f^+(\zeta) \upharpoonright \beta = \xi \upharpoonright \beta = f^+(\eta) \upharpoonright \beta = (\vec{0} \upharpoonright B_0) \cup (f(\eta) \upharpoonright B_1) \cup (f(\vec{0}) \upharpoonright B_2).$$

This implies that

- $f(\zeta) \upharpoonright B_0 = f(\vec{0}) \upharpoonright B_0$,
- $f(\zeta) \upharpoonright B_1 = f(\eta) \upharpoonright B_1$, and
- $f(\zeta) \upharpoonright B_2 = f(\eta) \upharpoonright B_2 = \vec{0} \upharpoonright B_2$.

Thus $f(\zeta) \in \bigcap_{i < 3} \bigcup_{q \in Q_i} N_q$, so that $\zeta \in W_0 \cap W_1 \cap W_2$. \square

\square

Lemma 6.15. *If $\mathbb{R} \stackrel{2}{\dashv} \dashv_{BM} =_Y$, then there exists an (X, Y) -pair.*

Proof. Suppose $f : 2^\kappa \rightarrow \kappa^\kappa$ is a Baire measurable reduction from $\mathbb{R} \stackrel{2}{\dashv} \dashv_X$ into $=_Y$. Let f^+ be the positivity of f , so that, by Lemma 6.14, f^+ is Baire measurable.

By Lemma 6.11, Baire measurable functions are continuous on a comeager set, so we may choose a function $H : \kappa \times \kappa \rightarrow 2^{<\kappa}$ encoding a comeager set H^* and satisfying that $f^+ \upharpoonright H^*$ is continuous. Now, define a function $F : 2^{<\kappa} \times \kappa^{<\kappa} \rightarrow 2$ via $F(p, q) := 1$ iff $f^+[N_p \cap H^*] \subseteq N_q$.

Claim 6.15.1. *(F, H) is an (X, Y) -pair.*

Proof. We go over the clauses of Definition 6.12:

- (1) Suppose $p \in 2^{<\kappa}$, $q, q' \in \kappa^{<\kappa}$ and $F(p, q) = 1 = F(p, q')$. Then $f^+[N_p \cap H^*] \subseteq N_q$ and $f^+[N_p \cap H^*] \subseteq N_{q'}$. Hence $f^+[N_p \cap H^*] \subseteq N_q \cap N_{q'}$. By Remark 6.5, the former is nonempty, so the latter is nonempty and $q \cup q' \in \kappa^{<\kappa}$. Altogether, F encodes a map from 2^κ to κ^κ .

Let us show that $F^* \upharpoonright H^* = f^+ \upharpoonright H^*$. For this, let $\eta \in H^*$ be arbitrary. Set $\xi := f^+(\eta)$. By the continuity of $f^+ \upharpoonright H^*$, for every $\varepsilon < \kappa$, there is a tail of $\delta < \kappa$ such that

$$N_{\eta \upharpoonright \delta} \cap H^* \subseteq (f^+)^{-1}[N_{\xi \upharpoonright \varepsilon}],$$

so $f^+[N_{\eta \upharpoonright \delta} \cap H^*] \subseteq N_{\xi \upharpoonright \varepsilon}$. Thus $F(\eta \upharpoonright \delta, \xi \upharpoonright \varepsilon) = 1$ for a tail of $\delta < \kappa$, and it follows from Definition 6.2 that $F^*(\eta) = \xi$.

- (2) By its very choice, H encodes the comeager set H^* .
- (3) Since $F^* \upharpoonright H^* = f^+ \upharpoonright H^*$, we fix an arbitrary $\eta \in H^*$ and prove that X_η is stationary iff $Y_{f^+(\eta)}$ is stationary. The following are equivalent:
 - (a) X_η is stationary;
 - (b) $\{\alpha \in X \mid \eta(\alpha) \neq \vec{0}(\alpha)\}$ is stationary;
 - (c) $\neg(\eta \stackrel{2}{\dashv} \vec{0})$;
 - (d) $\neg(f(\eta) =_Y f(\vec{0}))$;
 - (e) $\{\alpha \in Y \mid f(\eta)(\alpha) \neq f(\vec{0})(\alpha)\}$ is stationary;
 - (f) $\{\alpha \in Y \mid f^+(\eta)(\alpha) \neq 0\}$ is stationary.
 - (g) $Y_{f^+(\eta)}$ is stationary.

The equivalence of (c) and (d) follows from the fact that f reduces $\mathbb{R} \stackrel{2}{\dashv} \dashv_X$ to $=_Y$. The equivalence of (e) and (f) follows from Definition 6.13 \square

This completes the proof. \square

Theorem 6.16. *Suppose X, Y are disjoint stationary subsets of κ , with $Y \in I[\kappa - X]$. For every pair (F, H) , $V^{\text{Add}(\kappa, 1)} \models (F, H)$ is not an (X, Y) -pair.*

Proof. Towards a contradiction, suppose that (F, H) is a counterexample. As $\text{Add}(\kappa, 1)$ is almost homogeneous and X, Y, F, H live in the ground model, it follows that, in fact, $V^{\text{Add}(\kappa, 1)} \models (F, H)$ is an (X, Y) -pair.

Let $\mathbb{R}, \mathbb{P}, G, G_0, \mathbb{Q}$, and G_1 be all defined as in the proof of Theorem 5.4. Denote $\eta := \bigcup G_0$.

Claim 6.16.1. *In $V[G_0]$, $\eta \in H^*$.*

Proof. Let $i < \kappa$. The set $H[\{i\} \times \kappa]$ is cofinal in $(2^{<\kappa}, \subseteq)$, so, in particular, $D_i := \{p \cap 0 \mid p \in H[\{i\} \times \kappa]\}$ is dense in \mathbb{P} . Pick $p \in G_0 \cap D_i$. Then $\eta \in N_p \subseteq N_{p \upharpoonright \max(p)}$ and the latter is equal to $N_{H(i, j)}$ for some $j < \kappa$. It thus follows that $\eta \in \bigcap_{i < \kappa} \bigcup_{j < \kappa} N_{H(i, j)} = H^*$. \square

Claim 6.16.2. *In $V[G_0]$, $Y_{F^*(\eta)}$ is stationary.*

Proof. In $V[G_0]$, (F, H) is an (X, Y) -pair, thus, to prove that $Y_{F^*(\eta)}$ is stationary, it suffices to prove that X_η is stationary. But the latter is precisely what is proved in Claim 5.4.2. \square

Claim 6.16.3. *In $V[G_0][G_1]$, $Y_{F^*(\eta)}$ is stationary.*

Proof. This is the content of Claim 5.4.3. \square

Claim 6.16.4. *In $V[G_0][G_1]$, X_η is nonstationary.*

Proof. This is the content of Claim 5.4.4. \square

By Claims 6.16.1, 6.16.3 and 6.16.4 we infer that, in $V[G]$, (F, H) is not an (X, Y) -pair, contradicting the choice of (F, H) . \square

Corollary 6.17. *Suppose X, Y are disjoint stationary subsets of κ , with $Y \in I[\kappa - X]$. After forcing with $\text{Add}(\kappa, \kappa^+)$, $=_X^2 \not\rightarrow_{BM} =_Y$.*

Proof. Let G be $\text{Add}(\kappa, \kappa^+)$ -generic over V . For every $\iota \leq \kappa^+$, let G_ι denote the projection of G into the ι^{th} stage.

Work in $V[G_{\kappa^+}]$. Towards a contradiction, suppose that $f : 2^\kappa \rightarrow \kappa^\kappa$ is a Baire measurable reduction from $=_X^2$ to $=_Y$. By Lemma 6.15, we may fix an (X, Y) -pair, say (F, H) . As F and H are elements of H_{κ^+} and as $\text{Add}(\kappa, \kappa^+)$ has the κ^+ -cc, we may find a large enough $\iota < \kappa^+$ such that (F, H) is in $V[G_\iota]$. Now, by Theorem 6.16, $V[G_{\iota+1}] \models \text{“}(F, H) \text{ is not an } (X, Y)\text{-pair”}$. As V and $V[G_{\kappa^+}]$ have the same bounded subsets of κ , the fact that (F, H) is an (X, Y) -pair in $V[G_{\kappa^+}]$, but not in $V[G_{\iota+1}]$ must mean that Clause (3) of Definition 6.12 fails in $V[G_{\iota+1}]$. It thus follows that there exists a stationary subset of κ in $V[G_{\iota+1}]$ that ceases to be stationary in $V[G_{\kappa^+}]$. However, the quotient forcing $\text{Add}(\kappa, \kappa^+)/G_{\iota+1}$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$ and the latter preserves stationary subsets of κ . This is a contradiction. \square

Corollary 6.18. *Suppose that there exists a coherent regressive C -sequence over a stationary subset Γ of κ . After forcing with $\text{Add}(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, Y of Γ , $=_X^2 \not\rightarrow_{BM} =_Y$.*

Proof. Let G be $\text{Add}(\kappa, \kappa^+)$ -generic over V . As in the proof of Corollary 6.17, for every $\iota \leq \kappa^+$, we let G_ι denote the projection of G into the ι^{th} stage.

Work in $V[G]$. Suppose that X and Y are disjoint stationary subsets of Γ . As X and Y are elements of H_{κ^+} and as $\text{Add}(\kappa, \kappa^+)$ has the κ^+ -cc, we may find a large enough $\iota < \kappa^+$ such that X and Y are in $V[G_\iota]$. Now, in V , and hence also in $V[G_\iota]$, there exists a coherent regressive C -sequence over Γ . So, by Lemma 5.10, there is a stationary $Z \subseteq X$ such that $Y \in I[\kappa - Z]$. Finally, since $\text{Add}(\kappa, \kappa^+)/G_\iota$ is isomorphic to $\text{Add}(\kappa, \kappa^+)$, Corollary 6.17 implies that, in $V[G]$, $=_Z^2 \not\rightarrow_{BM} =_Y$. As $Z \subseteq X$, it follows from Monotonicity Lemma 2.7 that, in $V[G]$, $=_X^2 \not\rightarrow_{BM} =_Y$. \square

Corollary 6.19 (Dense non-reduction). *There exists a cofinality-preserving forcing extension in which:*

- (1) *For all stationary subsets X, Y of κ , there exist stationary subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that $=_{X'}^2 \not\rightarrow_{BM} =_{Y'}$;*
- (2) *There exists an injection $h : \mathcal{P}(\kappa) \rightarrow \text{NS}_\kappa^+$ such that, for all $X, Y \in \mathcal{P}(\kappa)$, $X \subseteq Y$ iff $=_{h(X)}^2 \rightarrow_{BM} =_{h(Y)}$;*
- (3) *For all two disjoint stationary subsets X, Y of κ , $=_X^2 \not\rightarrow_{BM} =_Y$.*

Proof. This is the same model of Corollary 5.12. That is, we force to add a coherent regressive C -sequence over a club in κ , and then add κ^+ many Cohen subsets of κ . The only difference is that this time we appeal to Corollary 6.18 instead of to Corollary 5.11. \square

Corollary 6.20. *Suppose $X \subseteq \kappa$ and $S \subseteq \kappa \cap \text{cof}(\omega)$ are stationary sets, for which $X \setminus S$ is stationary. After forcing with $\text{Add}(\kappa, \kappa^+)$, $=_X^2 \not\dot{\dashv} \dot{\dashv}_{BM} = S$.*

Proof. Consider the stationary set $Z := X \setminus S$. By Lemma 5.13(1), $\kappa \cap \text{cof}(\omega) \in I[\kappa - Z]$, in particular, $S \in I[\kappa - Z]$. By Corollary 6.17, we conclude that after forcing with $\text{Add}(\kappa, \kappa^+)$, $=_Z^2 \not\dot{\dashv} \dot{\dashv}_{BM} = S$. Now, Monotonicity Lemma 2.7 finishes the proof. \square

Based on the work done thus far, we are now able to answer a few questions from the literature.

Definition 6.21. An equivalence relation R is said to be Σ_1^1 -complete iff it is analytic and, for every analytic equivalence relation E , $E \dot{\dashv} \dot{\dashv}_B R$.

Question 6.22 (Aspero-Hyttinen-Kulikov-Moreno, [AHKM19, Question 4.3]). Is it consistent that κ is inaccessible and $=_S^2$ is not Σ_1^1 -complete for some stationary $S \subseteq \kappa$?

We answer the preceding in the affirmative:

Theorem 6.23. *If κ is an inaccessible cardinal, then there exists a cofinality-preserving forcing extension in which (κ is inaccessible, and) for every stationary co-stationary $S \subseteq \kappa$, $=_S^2$ is not a Σ_1^1 -complete equivalence relation.*

Proof. This is the forcing extension of Corollary 6.19. In this model, for all two disjoint stationary subsets X, S of κ , $=_X^2 \not\dot{\dashv} \dot{\dashv}_{BM} = S$. In particular $=_X^2 \not\dot{\dashv} \dot{\dashv}_{BM} =_S^2$, so that $=_S^2$ is not Σ_1^1 -complete. \square

Question 6.24 (Moreno, [Mor17, Question 4.16]). Is it consistent that

$$=_{\kappa \cap \text{cof}(\mu)} \not\dot{\dashv} \dot{\dashv}_B =_{\kappa \cap \text{cof}(\nu)}$$

holds for all infinite regular cardinals $\mu \neq \nu$ below κ ?

We answer the preceding in the affirmative:

Theorem 6.25. *There is a cofinality-preserving forcing extension, in which, for all infinite regular cardinals $\mu \neq \nu$ below κ , $=_{\kappa \cap \text{cof}(\mu)} \not\dot{\dashv} \dot{\dashv}_{BM} =_{\kappa \cap \text{cof}(\nu)}$.*

Proof. This is the forcing extension of Corollary 6.19. For all infinite regular cardinals $\mu \neq \nu$ below κ , $\kappa \cap \text{cof}(\mu)$ and $\kappa \cap \text{cof}(\nu)$ are disjoint stationary subsets of κ , and hence $=_{\kappa \cap \text{cof}(\mu)}^2 \not\dot{\dashv} \dot{\dashv}_{BM} =_{\kappa \cap \text{cof}(\nu)}$. In particular, $=_{\kappa \cap \text{cof}(\mu)} \not\dot{\dashv} \dot{\dashv}_{BM} =_{\kappa \cap \text{cof}(\nu)}$. \square

Question 6.26 (Aspero-Hyttinen-Kulikov-Moreno, [AHKM19, Question 2.12]). Is it consistent that, for all infinite regular $\mu < \nu < \kappa$, the following hold?

$$=_{\kappa \cap \text{cof}(\mu)} \dot{\dashv} \dot{\dashv}_B =_{\kappa \cap \text{cof}(\nu)}^2 \ \& \ =_{\kappa \cap \text{cof}(\nu)}^2 \not\dot{\dashv} \dot{\dashv}_B =_{\kappa \cap \text{cof}(\mu)}.$$

We answer the preceding in the affirmative, along the way, proving Theorem E:

Theorem 6.27. *Suppose MM holds. After forcing with $\text{Add}(\omega_2, \omega_3)$, MM still holds, and so are all of the following:⁸*

- (1) $=_{S_0^2} \dot{\dashv} \dot{\dashv}_1 =_{S_1^2}^2$;
- (2) For every stationary $X \subseteq S_1^2$, $=_X^2 \not\dot{\dashv} \dot{\dashv}_{BM} =_{S_0^2}$;
- (3) There are stationary subsets $X \subseteq S_0^2$ and $Y \subseteq S_1^2$ such that $=_X^2 \not\dot{\dashv} \dot{\dashv}_{BM} =_Y$;

⁸As in Remark 3.5, we denote $S_m^n := \aleph_n \cap \text{cof}(\aleph_m)$.

- (4) *There is a stationary $Y \subseteq S_1^2$ such that $=_{S_1^2}^2 \not\curvearrowright_{BM} =_Y$;*
 (5) *$=_{S_0^2} \hookrightarrow_1 =_{S_1^2}^2$ and $=_{S_1^2}^2 \not\curvearrowright_{BM} =_{S_0^2}$.*

Proof. We start with a model of $V \models \text{MM}$, and pass to $V[G]$, where G is $\text{Add}(\omega_2, \omega_3)$ -generic over V . By [Lar00, Theorem 4.3], MM is preserved by $<\omega_2$ -directed-closed forcing, so that $V[G] \models \text{MM}$.

(1) MM implies that $2^{\aleph_1} = \aleph_2$, which, by a Theorem of Shelah [She10], implies that $\diamond_{S_0^2}$ holds. Now appeal to Lemma 3.15.

(2) This is Corollary 6.20.

(3) Work in V . By [Sak11b], MM implies the existence of a coherent regressive C -sequence over some $\Gamma \subseteq \omega_2$ for which $Y := \Gamma \cap \text{cof}(\omega_1)$ is stationary. A moment reflection makes it clear that $X := \Gamma \cap \text{cof}(\omega)$ must be stationary, as well. Now, by Corollary 6.18, in $V[G]$, $=_X^2 \not\curvearrowright_{BM} =_Y$.

(4) By Clauses (1) and (3).

(5) By Clauses (1) and (2). □

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