88201 Analytic and Differential Geometry, Prof. Katz, moed C, 30 october 2025. Each problem is worth 27 points.

You must write legibly and provide explanation and justification for all answers.

1. For each of the following curves, determine whether it is degenerate and determine its type.
   (a) $-x^2 + 2xy - 3y^2 - 1 = 0$.
   (b) $x^2 - 8x + 3y^2 + 16 = 0$.
   (c) $-x^2 + 6xy - 3y^2 = 0$.
   (d) $2x^2 - 4xy + y + 8y^2 = 0$.

2. This problem deals with surfaces of revolution around the z-axis in $\mathbb{R}^3$.
   (a) Let $c > 0$, and consider the curve in the (x, z) plane defined by $(z - c)^2 + x^2 = 9$. Let $M_c \subseteq \mathbb{R}^3$ be the corresponding surface of revolution. Determine the mean curvature of the surface M_c.
   (b) Let $d > 1$, and consider the curve in the (x, z) plane defined by $z^2 + (x - d)^2 = 1$. Let $M_d \subseteq \mathbb{R}^3$ be the corresponding surface of revolution. Determine the Gaussian curvature of the surface M_d at points with maximal z-coordinate.
   (c) For the surface M_d of item (b), determine the Gaussian curvature at points with maximal x-coordinate.

3. The following expressions use the Einstein summation convention. All of the expressions need to be expressed in terms of the coefficients of the first and second fundamental forms, and the Gamma coefficients.
   (a) Which indices in expression $\langle \delta^i{}_j\, x_k, g^{jk}\, n_\ell \rangle$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
   (b) Which indices in expression $\langle x_i, \delta^i{}_\ell\, x_{jk} \rangle$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
   (c) Which indices in expression $\langle x_{ij}, \delta^i{}_k\, x_{\ell m} \rangle$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.
   (d) All indices run from 1 to 2. Which indices in $\delta^i{}_k\, g_{j\ell}\, L^\ell{}_m\, \delta^k{}_i$ are free? Simplify the expression as much as possible, and determine which indices in the final expression are free.

4. Find point or points of maximum (if they exist) of curvature for the following curves in the (x, y)-plane:

(a) $x^2 + 2y + 1 = 0$.
(b) $xy - 2 = 0,\ x > 0$.
(c) $x - \sqrt{2}\ln y = 1$.

Good luck!

List of formulas:

$$D_B(F) = F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2$$

$$k_C = \frac{|D_B(F)|}{|\nabla F|^3}$$

$$\Gamma_{ij}^k = \frac{1}{2}(g_{i\ell,j} - g_{ij,\ell} + g_{j\ell,i})g^{\ell k}$$

$$K = \frac{2}{g_{11}}\left(\Gamma^2_{1[1,2]} + \Gamma^j_{1[1}\Gamma^2_{2]j}\right)$$