June 16, 2024

INFINITESIMAL ANALYSIS 88-503 HOMEWORK SET 4

**Due Date: 30 june '24**

1. Prove that there exists a hyperinteger $H$ divisible by all standard integers $n \in \mathbb{N}$.

2. Show that if a sequence converges in $\mathbb{R}$ then it has exactly one cluster point (nekudat hitztabrut).

3. Suppose that $a_i \geq 0$ for all $i \in \mathbb{N}$. Prove that the series $\sum_1^\infty a_i$ converges if and only if $\sum_1^n a_i$ is finite for *all* infinite $n$, and that this holds if and only if $\sum_1^n a_i$ is finite for *some* infinite $n$.

4. Use the hyperreal characterisation of uniform continuity (see Section 6.8 of the class notes) to show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

5. Consider the LSEQ transformation (see Section 6.9 of the class notes). Apply LSEQ to $\Psi_i$ in the following formulas and determine whether the new formula is true:

(1) $\Psi_1$ is the formula $(\forall r \in \mathbb{R})(\exists n \in \mathbb{N})\ r < n$;
(2) $\Psi_2$ is the formula $(\forall q \in \mathbb{Q})(\exists n, m \in \mathbb{Z})\ q = \frac{n}{m}$;
(3) $(\forall \epsilon \in \mathbb{R}^+)\Psi_3(\epsilon)$, where $\Psi_3(\epsilon)$ is the formula

$$(\forall x \in \mathbb{R})(\exists \delta \in \mathbb{R}^+)(\forall y \in \mathbb{R})\ \big(|x - y| < \delta\ \rightarrow\ x^2 - y^2 < \epsilon\big);$$

(4) $(\forall \epsilon \in \mathbb{R}^+)\Psi_4(\epsilon)$, where where $\Psi_4(\epsilon)$ is the formula

$$(\forall x \in \mathbb{R})(\exists \delta \in \mathbb{R}^+)(\forall y \in \mathbb{R})\ \big(|x - y| < \delta\ \rightarrow\ \sin x - \sin y < \epsilon\big).$$