March 16, 2022

INFINITESIMAL ANALYSIS 88-503 HOMEWORK SET 1

Due Date: 21 march '22

1. Consider a nonempty subset $A \subseteq \mathbb{N}$. Prove that there exists an ultrafilter F on $\mathbb{N}$ with $A \in F$.

2. Prove that an ultrafilter on a finite set is necessarily principal.

3. Let F be a nonprincipal ultrafilter on $\mathbb{N}$. Define an equivalence relation $\equiv$ on $\mathbb{R}^{\mathbb{N}}$ by $\langle r_n \rangle \equiv \langle s_n \rangle$ if and only if $\{n \in \mathbb{N} : r_n = s_n\} \in F$. Prove that

$$(1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) \not\equiv (0, 0, 0, ...).$$

4. Consider sequences $r = \langle r_n \rangle$ and $s = \langle s_n \rangle$. We will denote the set $\{n \in \mathbb{N} : r_n = s_n\}$ by $[[r = s]]$. Let $t = \langle t_n \rangle$ be a sequence. Prove that

$$[[r = s]] \cap [[s = t]] \subseteq [[r = t]].$$

5. Let ε be a sequence so that $[\varepsilon] \in {}^*\mathbb{R}$ is a positive infinitesimal. Prove that $\frac{1}{[\varepsilon]}$ is infinite.