INFINITESIMAL ANALYSIS 88-503 MOED B 16 SEP '24

Each problem is worth 22 points. All answers must be fully justified by giving complete proofs. **The writing in the exam book must be fully legible and neat.**

1. This question deals with saturation (revaya).
   (a) Give a formulation of the property of countable saturation.
   (b) Give a definition of a negative infinite member of ${}^*\mathbb{R}$.
   (c) Use countable saturation to deduce the existence of negative infinite members of ${}^*\mathbb{R}$.

2. This question deals with the LSEQ transformation.
   (a) Give a definition of the LSEQ transformation when applied to a formula $\phi$.
   (b) Apply LSEQ to $\Psi_1$ and determine whether the new formula is true, where $\Psi_1$ is the formula
   $$(\forall r \in \mathbb{R})(\exists x \in \mathbb{R})\ \ln r < x.$$

3. Apply the LSEQ transformation to $\Psi_i$ in
   (a) the formula $(\forall \epsilon \in \mathbb{R}^+)\,\Psi_2(\epsilon)$ and determine whether the new formula is true, where $\Psi_2(\epsilon)$ is the formula
   $$(\forall x \in \mathbb{R})(\exists \delta \in \mathbb{R}^+)(\forall y \in \mathbb{R})\ \left(|x-y| < \delta\ \to\ x^{\frac{2}{3}} - y^{\frac{2}{3}} < \epsilon\right).$$
   (b) the formula $(\forall \epsilon \in \mathbb{R}^+)\,\Psi_3(\epsilon)$ and determine whether the new formula is true, where $I = \left(0, \frac{\pi}{2}\right)$ and $\Psi_3(\epsilon)$ is the formula
   $$(\forall x \in I)\,(\exists \delta \in \mathbb{R}^+)\,(\forall y \in I)\ \left(|x-y| < \delta\ \to\ \cot x - \cot y < \epsilon\right).$$

4. Let ${}^*\mathbb{Q}$ be the $*$-transform of $\mathbb{Q}$.
   (a) Use transfer so show that every hyperrational in ${}^*\mathbb{Q}$ is a ratio of hyperintegers.
   (b) Let $L$ denote the set of finite hyperrationals, and let $I$ denote the set of infinitesimal hyperrationals. Determine the quotient $L/I$.

5. This question deals with internal sets.
   (a) Give a detailed definition of the concept of an internal set $X$.
   (b) Show that if an internal subset $X \subseteq {}^*\mathbb{R}$ contains all points that are infinitely close to $b \in {}^*\mathbb{R}$ then there is a real $\epsilon > 0$ such that $X$ contains all points that are within $\epsilon$ of $b$.

# Good Luck!