88-826 Differential Geometry, moed A
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Each of 4 problems is worth 25 points; the bonus problem is 8 points

All answers must be justified by providing complete explanations and proofs

1. Let M be a closed connected n-dimensional manifold.
   (a) Consider a metric g on M. Give detailed definitions of the following three norms: the norm $\| \ \|$ in $\Lambda^2(T_p^*M)$; the norm $\| \ \|_\infty$ in $\Omega^2 M$; and the norm $\| \ \|^*$ in de Rham cohomology.
   (b) Give detailed definitions of the stable norm, $\mathrm{stsys}_2(g)$, and of the duality between the stable norm and the comass norm.
   (c) Give a detailed definition of what it means for a de Rham class $\omega \in H^2_{dR}(M)$ to be an integer class, i.e., an element of $L^2_{dR}(M)$.

2. Given a closed connected 6-dimensional Riemannian manifold (M, g), assume that $b_2(M) = 1$ and that a class $\omega \in H^2_{dR}(M)$ satisfies $\omega^{\cup 3} \neq 0$.
   (a) Let $\eta \in \omega$ be a representative differential 2-form. Estimate the integral $\int_M \eta \wedge \eta \wedge \eta$ in terms of the comass of η as well as the total volume $\mathrm{vol}(M)$ of M.
   (b) Use part (a) to provide (with proof) the best upper bound for the ratio $\mathrm{stsys}_2(g)^3/\mathrm{vol}(g)$.

3. This problem deals with de Rham cohomology.
   (a) Compute (with proof) the group $H^1_{dR}(\mathbb{R}/\mathbb{Z})$.
   (b) Let $L \subseteq \mathbb{C}$ be the Eisenstein integers. Compute (with proof) the group $H^2_{dR}(\mathbb{C}/L)$.
   (c) Compute (with proof) the group $H^3_{dR}(\mathbb{CP}^1)$.

4. Let T^2 be a torus with a Riemannian metric g, and suppose T^2 contains an annulu $A = \mathbb{R}/\mathbb{Z} \times I$ such that the class of $\mathbb{R}/\mathbb{Z}$ is nontrivial in $H_1(T^2; \mathbb{Z})$.
   (a) Define the capacity of the annulus A.
   (b) Suppose the capacity of the annulus $A \subseteq T^2$ is C. Prove an optimal inequality relating $\mathrm{sys}_1(g)$, C, and $\mathrm{area}(g)$.

5. (**bonus**) Let $M_n = S^2 \times S^n, n \geq 1$, and let g be a metric on M_n of total volume 1. Determine (with proof) for which n is there a uniform upper bound (valid for all metrics g of volume 1) for the stable 2-systole of M_n.

Good Luck!