88-826 Differential Geometry, moed A
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Each of 4 problems is worth 25 points; the bonus problem is 8 points
All answers must be justified by providing complete explanations and proofs

1. Let M be an 8-dimensional manifold with $b_2(M) = 1$, with an integer de Rham class $\omega \in L^2_{\mathrm{dR}}(M)$ such that $\omega^{\cup 4}$ is the fundamental cohomology class of M. Find an upper bound for the ratio $\frac{(\mathrm{stsys}_2(g))^4}{\mathrm{vol}(g)}$ valid for all Riemannian metrics g on M, with proof.

2. Let $M = \mathbb{CP}^2 \times \mathbb{CP}^3$. Prove that all metrics g of volume 1 on M satisfy $\mathrm{stsys}_2(g) \leq C$ for a suitable constant C independent of the metric.

3. Determine which of the following 12-dimensional manifolds satisfy a stable systolic inequality for stsys_2 with a constant independent of the metric:
   (1) $S^2 \times S^{10}$;
   (2) $S^6 \times \mathbb{CP}^3$;
   (3) $S^2 \times S^4 \times S^6$.

4. Determine which of the following manifolds satisfy a stable systolic inequality for stsys_2 with a constant independent of the metric:
   (1) $S^2 \times S^2 \times \mathbb{CP}^n$;
   (2) $\mathbb{CP}^2 \times S^n$;
   (3) $\mathbb{CP}^n \times T^2$.

5. (Bonus) Let $d(x, y)$ be the distance on $\mathbb{CP}^2$ defined by $d(x, y) = \arccos |H(\tilde{x}, \tilde{y})|$ where $\tilde{x}$, $\tilde{y}$ are unit vectors in $\mathbb{C}^3$ representing x and y. Let g be the associated metric on $\mathbb{CP}^2$. Define $\alpha \in \Omega^2(\mathbb{CP}^2)$ by setting $\alpha(X, Y) = g(JX, Y)$. Evaluate $\int_{\mathbb{CP}^2} \alpha \wedge \alpha$.

Good Luck!