88-826 Differential Geometry, moed B
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Each of 4 problems is worth 25 points; the bonus problem is 8 points

All of the answers must be justified by providing complete explanations and proofs

1. Let M be a 12-dimensional manifold with $b_2(M) = 1$.
   (a) Give a detailed definition of $L^2_{\text{dR}}(M)$.
   (b) Give a detailed definition of the fundamental cohomology class of M.
   (c) Assuming that a cup power of a suitable class $\omega \in L^2_{\text{dR}}(M)$ is the fundamental cohomology class of M, find an upper bound for the ratio $\frac{(\text{stsys}_2(g))^6}{\text{vol}(g)}$ valid for all Riemannian metrics g on M, with proof.

2. Let $M = \mathbb{CP}^n \times \mathbb{CP}^{2n}$, where $n \geq 2$. Do all metrics g of volume 1 on M necessarily satisfy $\text{stsys}_2(g) \leq C$ for a suitable constant C independent of the metric?

3. Determine which of the following manifolds satisfy a stable systolic inequality for stsys_2 with a constant independent of the metric:
   (a) $S^1 \times S^2 \times S^1$;
   (b) $S^1 \times S^2 \times S^2$;
   (c) $S^2 \times S^2 \times \mathbb{CP}^2$.

4. Let $n \geq 2$. Determine which of the following manifolds satisfy a stable systolic inequality for stsys_2 with a constant independent of the metric:
   (1) $S^2 \times S^2 \times \mathbb{CP}^n$;
   (2) $\mathbb{CP}^2 \times S^2 \times S^n$;
   (3) $S^1 \times \mathbb{CP}^n \times S^1$.

5 (bonus). Let α be the area form of S^2, expressed away from the poles as $\alpha = \sin\phi \, d\phi \wedge d\theta$. Let β be the area form of $\mathbb{CP}^1$, expressed in an affine neighborhood as $\beta(x, y) = \frac{dx \wedge dy}{(1+x^2+y^2)^2}$. Consider the manifold $X = S^2 \times \mathbb{CP}^1$. Let $r, s \in \mathbb{R}$, and consider the form $\gamma = r\alpha + s\beta$ on X. Determine for which values of r, s the form γ is exact, with proof.

Good Luck!