①

גיאומטריה דיפרנציאלית 2 - תרגיל 2

.1

$$\langle X_{ij}, X_{kl}\rangle = \langle \Gamma_{ij}^{p}X_p + L_{ij}n,\ \Gamma_{kl}^{q}X_q + L_{kl}n\rangle$$

$$= \langle \Gamma_{ij}^{p}X_p, \Gamma_{kl}^{q}X_q\rangle + \langle \Gamma_{ij}^{p}X_p, L_{kl}n\rangle + \langle L_{ij}n, \Gamma_{kl}^{q}X_q\rangle + \langle L_{ij}n, L_{kl}n\rangle$$

$$= \Gamma_{ij}^{p}\Gamma_{kl}^{q}\underbrace{\langle X_p, X_q\rangle}_{=g_{pq}} + \Gamma_{ij}^{p}L_{kl}\underbrace{\langle X_p, n\rangle}_{=0} + L_{ij}\Gamma_{kl}^{q}\underbrace{\langle n, X_q\rangle}_{=0} + L_{ij}L_{kl}\underbrace{\langle n,n\rangle}_{=1}$$

$$= \Gamma_{ij}^{p}\Gamma_{kl}^{q}g_{pq} + L_{ij}L_{kl}.$$ ✓

א .2

(1) $k = \det(L_i^{\,j})$

(2) $k = \dfrac{\det(L_{ij})}{\det g_{ij}}$

(3) $k = \dfrac{2}{g_{11}}\left(\Gamma_{1\,[1:2]}^{2} + \Gamma_{1\,[1}^{j}\Gamma_{2]j}^{2}\right)$

(4) $k = -\Delta_{LB}(\log\sqrt{\lambda})$ ✓

ג + ב נשתמש בדרך (4), כאשר המטריקה נתונה ע"י מטריקה סקלרית כמו אצלנו אז ה- λ זה הסקלר, $\lambda = \dfrac{1}{(u^2)^2}$ ✓

$$\frac{\partial}{\partial u^1}\left(\log\sqrt{\lambda}\right) = \frac{\partial}{\partial u^1}\left(\log\frac{1}{u^2}\right) = 0 \Rightarrow \frac{\partial^2}{\partial (u^1)^2}\left(\log\sqrt{\lambda}\right) = 0$$

$$\frac{\partial^2}{\partial (u^2)^2}\left(\log\sqrt{\lambda}\right) = \frac{\partial}{\partial u^2}\left(\frac{\partial}{\partial u^2}\left(\log\frac{1}{u^2}\right)\right) = \frac{\partial}{\partial u^2}\left(-\frac{1}{(u^2)^2}\cdot\frac{1}{\frac{1}{\cancel{u^2}}}\right) =$$

$$= \frac{\partial}{\partial u^2}\left(-\frac{1}{u^2}\right) = \frac{1}{(u^2)^2}$$

$$k = -\Delta_{LB}(\log\sqrt{\lambda}) \underset{b}{=} -\frac{1}{\lambda}\left(0 + \frac{1}{(u^2)^2}\right) \overset{\lambda = \frac{1}{(u^2)^2}}{=} -(u^2)^2\cdot\frac{1}{(u^2)^2} = \boxed{-1}$$

$$\Delta_{LB}(h) = \frac{1}{\lambda}\cdot\left(\frac{\partial^2}{\partial (u^1)^2} + \frac{\partial^2}{\partial (u^2)^2}\right)(h)$$ ✓

(2)

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3. (a)

$$\langle x_{lj}, X_k\rangle (\delta_m^k) g^{ml} \underset{\downarrow}{=} \Gamma_{lj}^s \overbrace{g_{sk}\,\delta_m^k}^{g_{sm}} g^{ml}$$

$$(*)\left(\langle x_{lj}, X_k\rangle = \langle \Gamma_{lj}^s X_s + L_{lj} n, X_k\rangle = \Gamma_{lj}^s \underbrace{\langle X_s, X_k\rangle}_{g_{sk}} + 0 \right)$$

$$= \Gamma_{lj}^s g_{sm} g^{ml} = \Gamma_{lj}^s \delta_s^l = \Gamma_{lj}^l$$

כי הן הופכיות

(b)

$$\langle n_j, X_{pq}\rangle (\delta_r^j) = \langle L_j^i X_i, X_{pq}\rangle (\delta_r^j)$$

$$= L_j^i \langle X_i, X_{pq}\rangle (\delta_r^j) \underset{(*)\text{ לפי}}{=} \underbrace{L_j^i g_{si}}_{-L_{js}} \Gamma_{pq}^s \delta_r^j$$

$$= -L_{js} \Gamma_{pq}^s \delta_r^j = -L_{rs} \Gamma_{pq}^s$$

(c)

$$\langle X_{stk}, n\rangle = \langle (\Gamma_{st}^m X_m + L_{st} n)_k, n\rangle$$

$$= \langle \Gamma_{stk}^m X_m + \Gamma_{st}^m X_{mk} + L_{stk} n + L_{st} n_k, n\rangle$$

$$= \Gamma_{stk}^m \underbrace{\langle X_m, n\rangle}_{=0} + \Gamma_{st}^m \langle X_{mk}, n\rangle + L_{stk} \underbrace{\langle n, n\rangle}_{=1} + L_{st} \langle n_k, n\rangle$$

insert semicolons to avoid confusion

$$= \Gamma_{st}^m \langle \Gamma_{mk}^l X_l + L_{mk} n, n\rangle + L_{stk} + L_{st} \langle L_k^p X_p, n\rangle$$

$$= \Gamma_{st}^m \Gamma_{mk}^l \underbrace{\langle X_l, n\rangle}_{=0} + \Gamma_{st}^m L_{mk} \underbrace{\langle n, n\rangle}_{=1} + L_{stk} + L_{st} L_k^p \underbrace{\langle X_p, n\rangle}_{=0}$$

$$= \Gamma_{st}^m L_{mk} + L_{stk} \underset{\downarrow}{=} \Gamma_{st}^m L_{mk} + (-L_t^l g_{ls})_k \underset{\downarrow}{=} \Gamma_{st}^m L_{mk} - L_{tk}^l g_{ls}$$

ראינו בהרצאה

המקדמים הם סקלריים
ולכן הנגזרת = 0

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30/2

$$g_{pq}\,(\delta^q_s)\,g^{su}\,\delta^p_u = g_{ps}\,g^{sp} = \delta^p_p = n$$ d