

EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY
88537

MIKHAIL G. KATZ

CONTENTS

1. Ceva, Menelaus	1
2. Desargues, Pappus	2
3. Pascal's theorem and harmonic 4-tuples	3
4. Axioms of affine planes and projective planes	3
4.1. Axioms of affine plane	3
4.2. Axiom of projective plane	4
4.3. Adding points at infinity	4
5. Isomorphic models	4
5.1. Homogeneous coordinates	4
5.2. Isomorphism of two models	5
5.3. Affine neighborhoods	6
6. Cross-ratio	7
7. Geometric constr, projective transf, transitivity on triples	7
8. Projective plane over an arbitrary field	8
9. Duality, self-dual axiom systems	8
10. Cross-ratio in homogeneous coordinates	8
11. Conic sections	9
12. Polarity, reciprocity	10
13. Constructing generic point on conic through 5 points	11
14. Mobius transformations	11
15. Is it a conic?	11
16. Hyperbolic geometry	13
References	13

1. CEVA, MENELAUS

The material in this section is from the book by H. Perfect [9].

2000 *Mathematics Subject Classification.* 97A20, 97C30.

Key words and phrases. projective geometry, duality, polarity, hyperbolic geometry.

Definition 1.1. Points ABC are *collinear* if they lie on a common line.

Definition 1.2. Lines a, b, c are *concurrent* if they meet in a common point.

Suppose a point X lies on a line AB . We wish to refine the notion of length of intervals on a line so as to assign a sign to ratios of such lengths.

Definition 1.3 (signed length). *Signed length* is defined in such a way that

- (1) If X is between A and B then $\frac{AX}{XB} > 0$.
- (2) If A is between X and B then $\frac{AX}{XB} < 0$.
- (3) If B is between A and X then $\frac{AX}{XB} < 0$.

Both Ceva's theorem and Menelaus' theorem are formulated with signed length in mind.

Theorem 1.4 (Ceva's Theorem). *Straight lines AD, BE, CF passing through the vertices of triangle ABC and meeting the opposite sides in D, E, F respectively, are concurrent if and only if*

$$\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = 1.$$

Theorem 1.5 (Menelaus' Theorem). *Points D, E, F on the sides BC, CA, AB of triangle ABC are collinear if and only if*

$$\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = -1.$$

Corollary 1.6. *Concurrency of altitudes, angle bisectors, and medians in a triangle.*

2. DESARGUES, PAPPUS

Theorem 2.1 (Desargues' Theorem). *If two triangles ABC and $A'B'C'$ lying in a plane are such that lines AA', BB', CC' meet in a point O , and if the pairs of corresponding sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ meet respectively in points L, N, M , then the points L, N, M are collinear.*

There are two points of view on Desargues' theorem: the slick statement: "triangles in perspective from a point, are in perspective from a line", and a detailed statement in terms of specific intersections as above. One must insist on the explicit version, for otherwise students come away without a true understanding of Desargues' theorem.

Theorem 2.2 (Pappus' Theorem). *If A, B, C and A', B', C' are two collinear triads of points, and if BC' and $B'C$ meet in L , whereas CA' and $C'A$ meet in M , while AB' and $A'B$ meet in N , then the points L, N, M are collinear.*

3. PASCAL'S THEOREM AND HARMONIC 4-TUPLES

Theorem 3.1 (Pascal's Theorem). *If A, B, C and A', B', C' are two triads of points lying on a circle \mathcal{C} , and if BC' and $B'C$ meet in L , whereas CA' and $C'A$ meet in M , while AAB' and $A'B$ meet in N , then the points L, M, N are collinear.*

Corollary 3.2. *If a hexagon is inscribed in a circle, then the pairs of opposite sides meet in collinear points.*

Later we will deal with Brianchon's theorem, which can be viewed as the polar of Pascal's theorem.

Note that the usual statement of Brianchon's theorem in terms of the sides and diagonals of a hexagon, at least on the surface of it, is *less general* than the polar dual of Pascal.

State this duality precisely in terms of *labeled points*. The connection between the dual theorems needs to be explained in detail, otherwise the students don't learn to translate theorems to their duals/polars.

Definition 3.3. Internal and external bisectors of angles in a triangle.

Theorem 3.4. *If L and L' are points on the side BC of the triangle ABC dividing the segment BC internally and externally in the same ratio, and if M, M' on CA and N, N' on AB are similarly defined, then AL, BM, CN are concurrent if and only if L', M', N' are collinear.*

Definition 3.5. A collinear 4-tuple B, C, L, L' is called a *harmonic 4-tuple* if L divides the interval BC internally in the same ratio as L' divides BC externally, so that $\frac{BL}{LC} = -\frac{BL'}{L'C}$.

4. AXIOMS OF AFFINE PLANES AND PROJECTIVE PLANES

The material is in Hartshorne [5].

4.1. Axioms of affine plane. Axiomatisation of affine planes: 3 axioms.

A1. Given two distinct points P and Q , there is one and only one line containing both P and Q .

A2. Given a line ℓ and a point P , not on ℓ , there is one and only one line m which is parallel to ℓ and which passes through P .

A3. There exist three non-collinear points.

Proofs of basic results derived from the axioms.

Note that a line by definition is parallel to itself (in previous years students protested, citing Margolis).

4.2. Axiom of projective plane. The 4 axioms of a projective plane S .

P1. Two distinct points P, Q of S lie on one and only one line.

P2. Any two lines meet in at least one point.

P3. There exist three non-collinear points.

P4. Every line contains at least three points.

4.3. Adding points at infinity. The model obtained by completing the affine line by adding points at infinity defined by pencils of parallel lines. This treatment follows Hartshorne [5].

Proof that this model satisfies the four axioms.

5. ISOMORPHIC MODELS

5.1. Homogeneous coordinates. Consider the example of projective plane, denoted S , defined as set of lines through the origin in \mathbb{R}^3 .

Recall that a (projective) line of S is the set of 1-dimensional subspaces lying in a given 2-dimensional subspace of \mathbb{R}^3 .

A point $P \in S$ is a line through $O = (0, 0, 0)$ (the origin). We will represent P by choosing any point (x_1, x_2, x_3) on the line, provided the point is different from the origin.

Definition 5.1. The numbers x_1, x_2, x_3 are the homogeneous coordinates of P .

Any other point on the line has the coordinates $(\lambda x_1, \lambda x_2, \lambda x_3)$, where $\lambda \in \mathbb{R}, \lambda \neq 0$.

Thus S is the collection of equivalence classes of triples (x_1, x_2, x_3) of real numbers, not all zero, where two triples (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) represent the same point if and only if there exists $\lambda \in \mathbb{R}$ such that

$$x'_i = \lambda x_i \text{ for each } i = 1, 2, 3.$$

Remark 5.2. If $x_3 = 0$ then the point $P = (x_1, x_2, 0)$ spans a line in the (x_1, x_2) plane of slope $m = \frac{x_2}{x_1}$. The slope is infinite, $m = \infty$, if and only if $x_1 = 0$.

Remark 5.3. Since the equation of a plane in \mathbb{R}^3 passing through O is of the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0, \quad \text{where not all } a_i \text{ are zero,} \quad (5.1)$$

we see that equation (5.1) is also the equation of a line of S in terms of the homogeneous coordinates.

5.2. Isomorphism of two models. We will prove that the completion at infinity and homogeneous coordinates give isomorphic models of the real projective plane.

Definition 5.4. Two projective planes S and S' are isomorphic if there exists a one-to-one correspondence $T: S \rightarrow S'$ which takes collinear points to collinear points.

Proposition 5.5. *The projective plane S defined by homogeneous coordinates which are real numbers, is isomorphic to the projective plane \overline{E} obtained by completing the ordinary affine plane of Euclidean geometry.*

To prove the proposition, we will use the notation (x_1, x_2, x_3) for the homogeneous coordinates in S . We will use (x, y) for the coordinates in the Euclidean plane E , whose completion by points at infinity is denoted \overline{E} . Recall that $\overline{E} = E \cup \omega$. Thus our second model is \overline{E} . The points of \overline{E} are

- points (x, y) of E , and
- ideal points at infinity of the form ℓ_∞ on the horizon ω , one ideal point for each pencil of parallel lines.

Lemma 5.6. *An ideal point ℓ_∞ is uniquely determined by its slope $m \in \mathbb{R} \cup \{\infty\}$.*

Proof. Indeed, a pencil of parallel lines $[\ell]$ is uniquely determined by its slope m . Here m may be either a real number or ∞ (when the lines of the pencil are vertical). \square

To prove Proposition 5.5, we will define a mapping $T: S \rightarrow \overline{E}$ which will exhibit an isomorphism between S and \overline{E} . Let $P = (x_1, x_2, x_3)$ be a point of S .

Remark 5.7. The idea is to cut all the lines by the plane $x_3 = 1$ in \mathbb{R}^3 . Then non-horizontal lines will correspond to finite points of \overline{E} whereas horizontal lines will correspond to ideal points at infinity of \overline{E} .

Thus, we consider the following two cases:

- (1) If $x_3 \neq 0$, we define $T(P)$ to be the point of $E \subseteq \overline{E}$ with coordinates $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$. Note that this is uniquely determined, since if we replace (x_1, x_2, x_3) by $(\lambda x_1, \lambda x_2, \lambda x_3)$, then x and y do not change. Every point of $(x, y) \in E$ can be obtained this way by using the triple $(x, y, 1)$.
- (2) If $x_3 = 0$, then we define $T(P)$ to be the ideal point of \overline{E} with slope $m = \frac{x_2}{x_1}$. This makes sense since x_1 and x_2 cannot be both zero, replacing $(x_1, x_2, 0)$ by $(\lambda x_1, \lambda x_2, 0)$ does not

change m , and each value of m occurs. Namely, if $m \neq 0$ then T sends $(1, m, 0)$ to the ideal point with slope m , and if $m = \infty$ then T sends $(0, 1, 0)$ to the ideal point with that slope.

The map $T: S \rightarrow \overline{E}$ thus defined is therefore one-to-one and onto. Therefore Proposition 5.5 results from the following theorem.

Theorem 5.8. *The map T defined above sends collinear points to collinear points.*

Recall that a line $\ell \subseteq S$ is given by equation (5.1), namely $a_1x_1 + a_2x_2 + a_3x_3 = 0$. We now consider two cases.

Case 1. Suppose a_1 and a_2 are not both zero. Then the theorem follows from the following lemma.

Lemma 5.9. *Consider the images under T of the points of ℓ . Then*

- (1) *points with nonzero x_3 remain collinear in \overline{E} , and form a line of slope $-\frac{a_1}{a_2}$;*
- (2) *the point with $x_3 = 0$ is sent under T to the ideal point with slope $m = -\frac{a_1}{a_2}$.*

Proof. If $x_3 \neq 0$ we can choose a representative with homogeneous coordinates $(x_1, x_2, 1)$. Thus we have $a_1x_1 + a_2x_2 + a_3 = 0$. Hence the line ℓ gets mapped to the line $a_1x + a_2y = -a_3$ in \overline{E} . Equivalently,

$$y = -\frac{a_1}{a_2}x - \frac{a_3}{a_2},$$

which is a line of the required slope $m = -\frac{a_1}{a_2}$.

When $x_3 = 0$, the points satisfying (5.1) are given by $x_1 = \lambda a_2$ and $x_2 = -\lambda a_1$. The map T by definition sends this point to the ideal point with slope $m = -\frac{a_1}{a_2}$. \square

Case 2. The remaining case is $a_1 = a_2 = 0$. In this case, the line ℓ in S is defined by the equation $x_3 = 0$. Any point of S with $x_3 = 0$ goes to an ideal point of \overline{E} , and these points form precisely the horizon line $\omega \subseteq \overline{E}$.

5.3. Affine neighborhoods.

Definition 5.10. The real projective plane $\mathbb{R}P^2$ is the model of the axioms $P1$ through $P4$ obtained either via homogeneous coordinates or via completion.

From now on when we mention the real projective plane we will refer to its isomorphism type, regardless of the construction chosen.

Definition 5.11. An *affine neighborhood* in the real projective plane $\mathbb{R}P^2$ is obtained by deleting a projective line.

Theorem 5.12. *Each affine plane of $\mathbb{R}P^2$ is isomorphic to the Euclidean plane.*

Proof. Deleting the projective line $\omega \subseteq \overline{E}$ produces the Euclidean plane E . The claim follows from the fact that any two planes in \mathbb{R}^3 differ by a linear transformation that preserves 1-dimensional subspaces and 2-dimensional subspaces. \square

6. CROSS-RATIO

The material on cross-ratios is in Adler [1].

Definition 6.1. Cross-ratio¹ of four collinear points A, B, C, D is

$$R(A, B, C, D) = \frac{AC/CB}{AD/DB}.$$

Definition 6.2. Perspectivity from a point.

Prove invariance under perspectivity, using areas.

Definition 6.3. Cross ratio of a pencil of lines.

The 6 cross-ratios: $\lambda, 1 - \lambda, \frac{1}{\lambda}$, etc.

Role of the symmetric group on 4 letters and of the Klein 4-group.

Exceptional case: the 3 cross-ratios.

Remark 6.4. Over the complex numbers: an additional exceptional case of only 2 distinct values, when

$$\lambda = e^{\pm i\pi/3}.$$

The cross-ratio of 4 points on a circle.

Relation to polarity (which has not been treated formally yet): work with tangent lines to a circle instead of points on a circle.

Then a variable tangent line t meets a 4-tuple of fixed tangents in a 4-tuple of points whose cross-ratio is independent of t .

7. GEOMETRIC CONSTR, PROJECTIVE TRANSF, TRANSITIVITY ON TRIPLES

Exceptional values $0, 1, \infty$ of the cross-ratio when some of the points collide.

Theorem 7.1. $R(\infty, 0, 1, \lambda) = \lambda$.

¹yachas hakaful.

Constructions in projective geometry.

An explicit geometric construction of the 4th harmonic point, using Ceva and Menelaus.

Recall the notion of a perspectivity.

Definition 7.2. A projectivity is a transformation preserving the cross-ratio.

The notation: a wedge under the equality sign.

Theorem 7.3. *Transitivity of projective transformations on triples of collinear points.*

Proof by composition of suitable perspectivities.

Corollary 7.4. *On every line in projective plane, given a triple of points, a fourth point is uniquely determined by the cross-ratio.*

More axiomatics: prove from the 4 axioms the following:

Theorem 7.5. *There is a 1-1 correspondence between points on a line ℓ and lines through a point A not on ℓ .*

8. PROJECTIVE PLANE OVER AN ARBITRARY FIELD

Construction of the projective plane over an arbitrary field.
Formulas for numbers of points in finite planes.

9. DUALITY, SELF-DUAL AXIOM SYSTEMS

Duality.

The four axioms of projective geometry give rise to a self-dual system, i.e. the dual of each axiom can be proved from the original list of four.

Discussion the construction in homogeneous coordinates over any field, using a generalisation of the vector product.

Counting points in a projective plane, discuss in a bit more detail the notion of an affine neighborhood (to break the idea that the affine plane is "special").

Detailed discussion of the case over the field F_2 , writing out the homogeneous coordinates of all the points, and explicit equations of some of the lines.

10. CROSS-RATIO IN HOMOGENEOUS COORDINATES

The definition of cross-ratio in homogeneous coordinates follows the book by Kaplansky [8].

Here if $A, B, C, D \in \mathbb{RP}^1$ we view A, B, C, D as 1-dimensional subspaces in \mathbb{R}^2 . We choose representative nonzero vectors $\alpha \in A$, $\beta \in$

$B, \gamma \in C$, and $\delta \in D$. We show that the vectors can be picked in such a way as to satisfy the relations

$$\gamma = \alpha + \beta$$

and

$$\delta = k\alpha + \beta, \quad (10.1)$$

where $k \in \mathbb{R}$ is suitably chosen. Then the coefficient k in (10.1) is the cross-ratio of A, B, C, D :

Theorem 10.1. *The coefficient k is independent of choices made and satisfies $R(A, B, C, D) = k$.*

11. CONIC SECTIONS

Conic sections: intersection of cone in \mathbb{R}^3 and plane.

Ellipse, parabola, hyperbola and number of points at infinity: 0, 1, 2.

Theorem 11.1. *Every nondegenerate nonempty real conic section is projectively equivalent to the circle.*

Example 11.2. To transform a circle into a parabola by a projective transformation, consider the equation of the circle

$$+x_1^2 + x_2^2 - x_3^2 = 0. \quad (11.1)$$

Here in the affine neighborhood $x_3 \neq 0$ we obtain the usual circle equation

$$x^2 + y^2 = 1 \quad (11.2)$$

where $x = \frac{x_1}{x_3}$ and $y = \frac{x_2}{x_3}$. We would like to transform this into a parabola

$$X_2X_3 = X_1^2. \quad (11.3)$$

Here in the affine neighborhood $X_3 \neq 0$ this becomes the usual equation of a parabola $Y = X^2$, where $X = \frac{X_1}{X_3}$ and $Y = \frac{X_2}{X_3}$. We rewrite (11.3) as

$$(X_2 + X_3)^2 - (X_2 - X_3)^2 = (2X_1)^2,$$

or

$$+(2X_1)^2 + (X_2 - X_3)^2 - (X_2 + X_3)^2 = 0. \quad (11.4)$$

Note that the signs $+, +, -$ in equations (11.1) and (11.4) are compatible. Therefore we exploit the transformation

$$x_1 = 2X_1, \quad x_2 = X_2 - X_3, \quad x_3 = X_2 + X_3.$$

This is a linear transformation in homogeneous coordinates and therefore defines a projective transformation on the projective planes.

Next, this can be expressed in an affine neighborhood by noting that

$$\frac{x_1}{x_3} = \frac{2X_1}{X_2 + X_3} = \frac{2\frac{X_1}{X_3}}{\frac{X_2}{X_3} + 1}$$

and

$$\frac{x_2}{x_3} = \frac{X_2 - X_3}{X_2 + X_3} = \frac{\frac{X_2}{X_3} - 1}{\frac{X_2}{X_3} + 1}.$$

In affine coordinates, we obtain

$$x = \frac{X}{Y + 1}, \quad y = \frac{Y - 1}{Y + 1}. \quad (11.5)$$

Substituting (11.5) into the circle equation (11.2) we obtain the equation of parabola $Y = X^2$.

Example 11.3. Transform parabola into hyperbola.

Example 11.4. Transform ellipse $x^2 + xy + y^2 = 1$ into parabola $Y = X^2$.

12. POLARITY, RECIPROCITY

Definition of polar line.

Metric characterisation of polar lines.

Axioms of Fano, Desargues, and Pappus.

Discussion of relation between algebraic properties and geometric axioms:

Theorem 12.1. *Suppose a projective plane π satisfies the axioms $P1, \dots, P4$ as well as Desargues' axiom. Then there exists a division ring D such that $\pi = DP^2$.*

Theorem 12.2. *Suppose in addition to the hypotheses above, π satisfies Fano's axiom (the diagonal points of a complete quadrilateral are not collinear). Then $\text{char } D \neq 2$.*

Theorem 12.3. *Suppose in addition to the hypotheses above, π satisfies Pappus' axiom. Then $\pi = DP^2$ where D is a field.*

This point of view may be found in the book by Kadison and Kromann [7, chapter 8]. It originates with Hilbert's book [6], see chapter 5 there, particularly paragraph 24: "Introduction of an algebra of segments based upon Desargues's theorem and independence of the axioms of congruence", starting on page 79. Hilbert mentions that this was also discussed by Moore.

1. proof of the reciprocity theorem: if Q is on p, then P is on q.

2. proof of the fact that polarity is a projective transformation, in two stages. First one proves it for 4 points lying on a tangent to the conic. Then one proves it for an arbitrary collinear 4-tuple.

3. A nice application is the theorem that every conic defines a projective transformation from points on a tangent, to points on another tangent. Namely, a point B on a tangent t is sent to a point B' on tangent t' if and only if the line BB' is tangent to the conic.

4. Present another example of a construction in projective geometry. So far the only construction we had is the construction of the fourth harmonic point, using Menelaus theorem.

5. Using the result that polarity is a projective transformation, construct a conic from 5 pieces of data. The 5 pieces are points L and L', the corresponding tangent lines l and l' through them, and an additional tangent line a". One constructs the map as in item 3 above, as the composition of two perspectivities.

Geometric constructions using projective theorems is an important topic in projective geometry that we have barely touched upon.

13. CONSTRUCTING GENERIC POINT ON CONIC THROUGH 5 POINTS

Construction of a generic point on a conic passing through 5 given points, using Pascal's theorem.

Translating it to a polar statement, so as to construct the polar pencil of parallel lines to the conic.

Finding a projective map between a pair of pencils of lines through a pair of points on a conic.

14. MOBIUS TRANSFORMATIONS

Every projective map from P^1 to itself is of the form

$$x \rightarrow \frac{ax + b}{cx + d}.$$

I already mentioned the fact that projective transformations correspond to linear maps when you write them in homogeneous coordinates. The fractional-linear presentation is a consequence of this.

More material on axioms of Fano, Desargues, Pappus, related material on the polar line, perhaps a proof of Desargues assuming existence of imbedding in projective 3-space.

15. IS IT A CONIC?

Given collinear points A, B, C , a variable line through C meets a conic at P and Q . Why does $AP \cap BQ$ trace another conic?

Sometimes one obtains a straight line segment rather than a conic (in a special case). Assume the conic is the standard unit circle. We can place the line ABC at infinity, and assume that C is the horizontal direction, whereas A and B are the directions of $e^{\pi i/3}$ and $e^{2\pi i/3}$. The intersection points P and Q are symmetric with respect to the y -axis, and therefore the intersection point $AP \cap BQ$ will always lie on the y -axis. This is of course a degenerate case.

In general, one can argue as follows.

- (1) Use a projective transformation to send the line ABC to the line at infinity.
- (2) Use an affine transformation to send the ellipse to the unit circle.
- (3) By a rotation we can assume that C corresponds to the pencil of horizontal lines (including the x -axis in the plane).

Then the points A and B correspond to pencils of parallel lines of slopes $m_1, m_2 \neq 0$.

A horizontal line cuts the unit circle in (at most) two points $(s, \sqrt{1-s^2})$ and $(-s, \sqrt{1-s^2})$. We need to show that the intersection point (x, y) of a pair of lines of slopes m_1 and m_2 passing through such a pair of points satisfies a quadratic equation. Then by definition it will be a conic section (at least in the nondegenerate case).

The equations of the two lines are

$$y = \sqrt{1-s^2} + m_1(x+s)$$

passing through the point $(-s, \sqrt{1-s^2})$, and

$$y = \sqrt{1-s^2} + m_2(x-s),$$

passing through the point $(s, \sqrt{1-s^2})$.

It follows that $x = \frac{m_1+m_2}{m_2-m_1}s$ or $s = mx$ where $m = \frac{m_2-m_1}{m_1+m_2}$. From the equation of the first line we obtain

$$(y - m_1(x + mx))^2 = 1 - (mx)^2.$$

This is a quadratic equation in x and y , as required.

This argument works in the case when the conic and the line ABC have no common points (either finite or infinite).

If the conic K and the line ABC have a single intersection point, one can apply a similar argument.

- (1) We move the line ABC to the line at infinity.

- (2) We move C to the pencil of horizontal lines and the point $ABC \cap K$ to the pencil of vertical lines.
- (3) Apply a translation to make sure that the axis of symmetry of the parabola K is the y -axis and the apex of the parabola is at the origin.
- (4) Afterwards apply a scaling to make sure the parabola is the standard parabola $y = x^2$.

A horizontal line cuts the parabola at $(\pm s, s^2)$. The equations of the two lines of slopes m_1 and m_2 are then $y = s^2 + m_1(x + s)$ and $y = s^2 + m_2(x - s)$. We obtain a similar relation $s = mx$ (for the same m as above). In this case also we obtain a quadratic relation between x and y , of the form $y = (mx)^2 + m(x + mx)$, as required.

16. HYPERBOLIC GEOMETRY

We introduce the Poincaré disk model following Greenberg [3].

Here a point is represented by the interior of a Euclidean circle γ .

A line is represented by one of the following:

- (1) an open diameter of the disk bounded by γ , or
- (2) an arc of circle δ contained in the disk and perpendicular to γ .

The notion of hyperbolic distance is defined via the cross-ratio.

REFERENCES

- [1] Adler, Claire Fisher. Modern geometry. An integrated first course. McGraw-Hill Book Company, Inc., New York-Toronto-London 1958.
- [2] Bair, J.; Henry, V.: Implicit Differentiation with Microscopes. *The Mathematical Intelligencer* **32** (2010), no. 1, 53–55.
- [3] Greenberg, M. *Euclidean and non-Euclidean geometries. Development and history. Third edition.* W. H. Freeman and Company, New York, 1993.
- [4] Halbeisen, L.; Hungerbühler, N. “A Simple Proof of Poncelet’s Theorem (on the Occasion of Its Bicentennial).” *Amer. Math. Monthly* **122** (2015), no. 6, 537–551.
- [5] Hartshorne, Robin: Foundations of projective geometry. Lecture Notes, Harvard University, 1966/67 W. A. Benjamin, Inc., New York 1967
- [6] Hilbert, D.: Foundations of Geometry.
- [7] Kadison, Lars; Kromann, Matthias T.: Projective geometry and modern algebra. Birkhäuser Boston, Inc., Boston, MA, 1996. (51A05 KAD p at Bar Ilan math library) See also <http://books.google.com/books?id=jo2g-sM8aCYC>
- [8] Kaplansky, Irving: Linear algebra and geometry. A second course. Reprint of the 1974 revised edition. Dover Publications, Inc., Mineola, NY, 2003.
- [9] Perfect, Hazel: Topics in geometry. Pergamon Press, Oxford-London-Paris-Frankfurt; The Macmillan Co., New York, 1963.

DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN 52900
ISRAEL

Email address: `katzmik@math.biu.ac.il`