February 22, 2011 Differential geometry 88-826 Homework 1

1. The area of a region $D \subset \mathbb{R}^2$ in polar coordinates is calculated using the area element $dA = r\,dr\,d\theta$. Thus, an integral is of the form $\int_D dA = \iint r dr d\theta$. Find the area of the following regions:

(a) $0 \leq r \leq 3$; $-\pi/2 \leq \theta \leq \pi/2$;
(b) $2 \leq r \leq 4$; $0 \leq \theta \leq \pi/4$;
(c) $0 \leq \theta \leq \pi$; $0 \leq r \leq \theta$.

2. The volume of an open region $D \subset \mathbb{R}^3$ is calculated with respect to spherical coordinates (r, θ, z) using the volume element $dV = r\,dr\,d\theta\,dz$. Namely, an integral is of the form $\int_D dV = \iiint r\,dr\,d\theta\,dz$.

(a) Find the volume of a right circular cone with height h and base a circle of radius b.
(b) evaluate the integral $\iiint_E \sqrt{x^2 + y^2} z dV$ where E is the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.
(c) Find the volume of the object filling the region above the paraboloid $z = x^2 + y^2$ and below the plane $z = 1$.

3. Spherical coordinates (ρ, θ, ϕ) range between the bounds $0 \leq \rho$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$ (note the different upper bounds for θ and ϕ). The area of a spherical region D is calculated using a volume element of the form $dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$, so that the volume of a region D is $\int_D dV = \iiint_D \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$.

(1) Find the volume of the region above the cone $\phi = \beta$ and inside the sphere of radius $\rho = c$.
(2) Find the integral $\iiint_E x^2 + y^2 + z^2 dV$, where E is the sphere $x^2 + y^2 + z^2 = b^2$.
(3) Find the integral $\iiint \frac{1}{x^2+y^2+z^2} dV$, where E is the region between two spheres: $a \leq \rho \leq b$.

4. Let $\delta^i_{\ j}$ be the Kronecker delta function on $\mathbb{R}^n$, where $i, j = 1, \ldots, n$, viewed as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Evaluate the expression

$$\delta^i_{\ j} \delta^j_{\ k} \delta^k_{\ i}.$$