March 9, 2015

DIFFERENTIAL GEOMETRY 88-826-01 HOMEWORK SET 1

1. Let $\mathcal{F} \subset \mathcal{P}(I)$ be an ultrafilter on a set I. Let $\{A_1, \ldots, A_n\}$ be a finite collection of pairwise disjoint ($A_i \cap A_j = \emptyset$) sets such that $A_1 \cup \ldots \cup A_n \in \mathcal{F}$. Then $A_i \in \mathcal{F}$ for exactly one i such that $1 \leq i \leq n$. [Hint: use induction combined with one of the defining properties of the ultrafilter.]

2. Consider the real sequence $u = \langle u_n : n \in \mathbb{N} \rangle$, where each term $u_n \in \mathbb{R}$. Let $[u] \in {}^*\mathbb{R}$ denote the equivalence class of u in the ultrapower construction of ${}^*\mathbb{R}$ exploiting a nonprincipal ultrafilter $\mathcal{F}$.

(a) Consider the sequence with general term $u_n = \frac{n+1}{n}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

(b) Consider the sequence with general term $u_n = \frac{n+1}{n^2}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

(c) Consider the sequence with general term $u_n = \frac{n^2+1}{n}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

3. If $A \subset \mathbb{R}$ is finite, show that ${}^*A = A$.

4. Show that $\approx$ (being infinitely close) is an equivalence relation on the field ${}^*\mathbb{R}$.

5. Show that if $x \approx y$ and b is finite then $bx \approx by$. Show that the result can fail for infinite b.

6. Use transfer of the usual properties of trigonometric functions, show the following:

(a) If ϵ is infinitesimal then $\sin \epsilon \approx 0$.

(b) If ϵ is infinitesimal then $\cos \epsilon \approx 1$.

(c) If ϵ is infinitesimal then $\frac{\sin \epsilon}{\epsilon} \approx 1$.

7. Show that every hyperreal is infinitely close to some hyperrational number.

8. Let $\mathbb{L}$ denote the ring of finite hyperreals, and $\mathbb{I}$ the ring of infinitesimals. Show that $\mathbb{R}$ is isomorphic to the ring of finite hyperrationals ${}^*\mathbb{Q} \cap \mathbb{L}$ factored by its ideal ${}^*\mathbb{Q} \cap \mathbb{I}$ of hyperrational infinitesimals.