Differential geometry   88-826-01   homework set 1

Due Date: 26 april '26

1. Let $\mathrm{Mat}_{n,n}(\mathbb{R})$ be the set of square matrices with real coefficients. Consider the subset $S \subseteq \mathrm{Mat}_{n,n}(\mathbb{R})$ consisting of all matrices X such that $\mathrm{Tr}(X) \neq 0$ (matrices with nonzero trace). Determine whether S is an open submanifold, with explanation.

2. Let $X = \mathbb{C}^2 \setminus \{0\}$ be the collection of pairs $x = (x^0, x^1)$ distinct from the origin. Define an equivalence relation $\sim$ between $x, y \in X$ by setting $x \sim y$ if and only if there is a complex number $t \neq 0$ such that $y = tx$, i.e.,

$$y^i = tx^i, \quad i = 0, 1 \quad \text{where} \quad t \in \mathbb{C} \setminus \{0\}.$$

Denote by $[x]$ the equivalence class of $x \in X$. Define the complex projective line, $\mathbb{CP}^1$, as the collection of equivalence classes $[x]$, i.e., $\mathbb{CP}^1 = \{[x] \colon x \in X\}$.

(1) Prove that $\mathbb{CP}^1$ is a smooth manifold by exhibiting charts and the transition function ϕ;
(2) check the metrizability condition;
(3) determine the real dimension of $\mathbb{CP}^1$.

3. Let A and B be copies of $\mathbb{R}^3$, with coordinates $u = (u^1, u^2, u^3)$ in A and $v = (v^1, v^2, v^3)$ in B, and with transition function $u = \phi(v) = \frac{v}{v \cdot v}$ whenever $v \in \mathbb{R}^3 \setminus \{0\}$. Prove that the resulting manifold M with coordinate patches A and B is metrizable by exhibiting a specific metric.

4. Let $M = T^3$ be the 3-torus. Prove that the tangent bundle of M is diffeomorphic to the product $T^3 \times \mathbb{R}^3$.