June 8, 2014

DIFFERENTIAL GEOMETRY   88-826-01   HOMEWORK SET 3

1. Consider the real sequence $u = \langle u_n : n \in \mathbb{N} \rangle$, where each term $u_n \in \mathbb{R}$. Let $[u] \in {}^*\mathbb{R}$ denote the equivalence class of u in the ultrapower construction of ${}^*\mathbb{R}$ exploiting a nonprincipal ultrafilter $\mathcal{F}$.

  (a) Consider the sequence with general term $u_n = \frac{n+1}{n}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

  (b) Consider the sequence with general term $u_n = \frac{n+1}{n^2}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

  (c) Consider the sequence with general term $u_n = \frac{n^2+1}{n}$. Does there exist a real number r infinitely close to $[u]$? If so, find r, with proof.

2. If $A \subset \mathbb{R}$ is finite, show that ${}^*A = A$.

3. Show that $\approx$ (being infinitely close) is an equivalence relation on the field ${}^*\mathbb{R}$.

4. Show that $\mathrm{hal}(b) = \{b + \epsilon : \epsilon \in \mathrm{hal}(0)\}$, where "hal" denotes the halo.

5. Show taht if $x \approx y$ and b is finite then $bx \approx by$. Show that the result can fail for infinite b.

6. Use transfer of the usual properties of trigonometric functions, show the following:

  (a) If ϵ is infinitesimal then $\sin \epsilon \approx 0$.
  (b) If ϵ is infinitesimal then $\cos \epsilon \approx 1$.
  (c) If ϵ is infinitesimal then $\frac{\sin \epsilon}{\epsilon} \approx 1$.

7. Show that every hyperreal is infinitely close to some hyperrational number.

8. Let $\mathbb{L}$ denote the ring of finite hyperreals, and $\mathbb{I}$ the ring of infinitesimals. Show that $\mathbb{R}$ is isomorphic to the ring of finite hyperrationals ${}^*\mathbb{Q} \cap \mathbb{L}$ factored by its ideal ${}^*\mathbb{Q} \cap \mathbb{I}$ of hyperrational infinitesimals.